Moyal Formulation of String Field Theory

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based on

Collaboration with I.Bars and Y.Matsuo

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Introduction

- Witten's String Field Theory (1986)

$$S = -\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi, Q_B\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right]$$

It was used recently:

Noncommutative boom, Sen's conjecture,

VSFT conjecture,…

- Moyal approach

[Bars(2001),Bars-Matsuo(2002)]

Witten's $*$ → Moyal ★

<u>and</u>

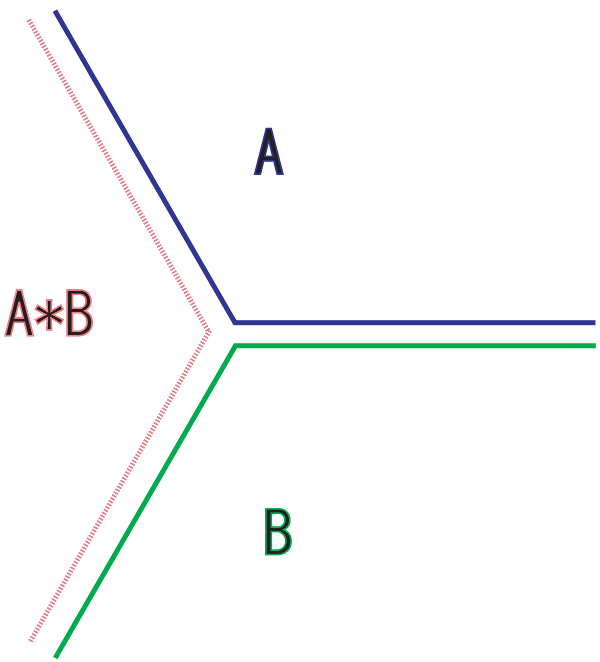
regularization fixed!

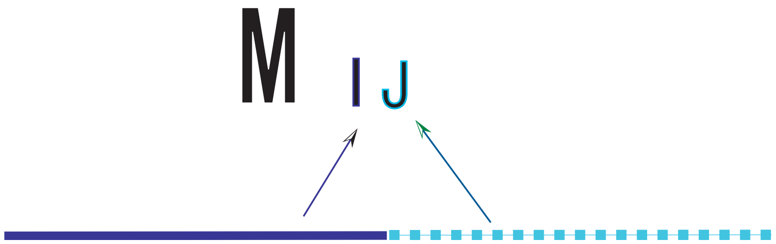
$$\star = \exp\left(\frac{i\theta}{2} \sum_e \left(\frac{\overleftarrow{\partial}}{\partial x_e} \frac{\overrightarrow{\partial}}{\partial p_e} - \frac{\overleftarrow{\partial}}{\partial p_e} \frac{\overrightarrow{\partial}}{\partial x_e} \right) \right)$$

Half-string formulation

- Witten's $*$ product $\sim$ Matrix product

[RSZ, Gross-Taylor, Kawano-Okuyama]





$$(A*B)_{ij} = \sum_k A_{ik} B_{kj}$$

Half-string formulation

- Mode expansion → Splitting

$$\Psi[X(\sigma)] \Rightarrow \hat{\Psi}\left[\bar{x}, l_o, r_o\right],$$

$$\widehat{\Psi * \Psi'}\left[\bar{x}, l_o, r_o\right] = \int dk_o \, \hat{\Psi}\left[\bar{x}, l_o, k_o\right] \hat{\Psi}'\left[\bar{x}, k_o, r_o\right]$$

- Moyal formulation

By Fourier transformation, the above product can be rewritten using Moyal ★ product [I.Bars] :

$$A\left[\bar{x}, x_e, p_e\right] = \det{}^{d/2}(2T) \int dx_o \exp\left(-i\frac{2}{\theta} p_e T_{eo} x_o\right) \hat{\Psi}\left[\bar{x}, x_o + R_{oe} x_e, -x_o + R_{oe} x_e\right]$$

$$= \det{}^{d/2}(2T) \int dx_o \exp\left(-i\frac{2}{\theta} p_e T_{eo} x_o\right) \Psi\left[\bar{x} + w_e x_e, x_e, x_o\right],$$

Witten' $*$ product becomes Moyal product: $\star = \exp\left(\frac{i\theta}{2}\sum_e\left(\frac{\overleftarrow{\partial}}{\partial x_e}\frac{\overrightarrow{\partial}}{\partial p_e} - \frac{\overleftarrow{\partial}}{\partial p_e}\frac{\overrightarrow{\partial}}{\partial x_e}\right)\right)$

Here

$$T_{eo}=\frac{4}{\pi}\int_0^{\pi/2}d\sigma\cos(e\sigma)\cos(o\sigma),\quad w_e=-\sqrt{2}\cos\frac{e\pi}{2}$$

$$\sum_e R_{oe}T_{eo'}=\delta_{oo'},\sum_o T_{eo}R_{oe'}=\delta_{ee'},$$ more explicitly,

$$T_{eo}=\frac{4oi^{o-e+1}}{\pi\left(e^2-o^2\right)},R_{oe}=\frac{4e^2i^{o-e+1}}{\pi o\left(e^2-o^2\right)},w_e=\sqrt{2}i^{-e+2},v_o=\frac{2\sqrt{2}i^{o-1}}{\pi o}.$$

These matrices and vectors satisfy

$$R_{oe}=o^{-2}T_{eo}e^2,R_{oe}=T_{eo}+v_ow_e,v_o=\sum_e T_{eo}w_e,w_e=\sum_o R_{oe}v_o.$$

- However, there is a subtlety: associativity anomaly of $\infty\times\infty$ matrices. [Bars-Matsuo]

$$Tv=0,\qquad T\bar{T}=1,\qquad \bar{T}T=1-v\bar{v}.$$

This situation causes ambiguity in computation, for example,

$$R(Tv)=R\cdot 0=0 \qquad \text{v.s.} \qquad (RT)v=1\cdot v=v.$$

We need appropriate regularization!

Moyal formulation of String Field Theory

- Setup:

For arbitrary

$$\kappa_e, \kappa_o, N \quad (e = 2, 4, \ldots, 2N, o = 1, 3, \ldots, 2N-1),$$

define matrices *R,T* and vectors *w,v* :

$$R = \kappa_o^{-2} \bar{T} \kappa_e^2, \quad R = \bar{T} + v\bar{w}, \quad v = \bar{T} w, \quad w = \bar{R} v.$$

In fact, we can solve them explicitly:

$$T_{eo} = \frac{w_e v_o \kappa_o^2}{\kappa_e^2 - \kappa_o^2}, \quad R_{oe} = \frac{w_e v_o \kappa_e^2}{\kappa_e^2 - \kappa_o^2}, \quad w_e^2 = \frac{\prod_{o'} \left(\kappa_e^2 / \kappa_{o'}^2 - 1 \right)}{\prod_{e' \neq e} \left(\kappa_e^2 / \kappa_{e'}^2 - 1 \right)}, \quad v_o^2 = \frac{\prod_{e'} \left(1 - \kappa_o^2 / \kappa_{e'}^2 \right)}{\prod_{o' \neq o} \left(1 - \kappa_o^2 / \kappa_{o'}^2 \right)}.$$

- Some relations:

$$T R=1, R T=1, \bar{R} R=1+w\bar{w}, \bar{T} T=1-v\bar{v},$$

$$T\bar{T}=1-\frac{w\bar{w}}{1+\bar{w}w}, T v=\frac{w}{1+\bar{w}w}, \bar{v}v=\frac{\bar{w}w}{1+\bar{w}w},$$

$$R w=v\left(1+\bar{w}w\right), R\bar{R}=1+v\bar{v}\left(1+\bar{w}w\right),$$

Modified!

$$1+\bar{w}w=\frac{\prod_{e}\kappa_{e}^{2}}{\prod_{o}\kappa_{o}^{2}}.$$

※In the case of $\kappa_e=e, \kappa_o=o, N=\infty$ (then $\bar{w}w=\infty$)
these quantities reproduce original ones of Witten's SFT.
We should take this limit at the last stage of computation to avoid subtlety of infinite matrices.

Using the transformation from conventional string field,

$$A\left[\bar{x}, x_e, p_e\right] = \det{}^{d/2}(2T)\int dx_o \exp\left(-i\frac{2}{\theta}p_e T_{eo} x_o\right)\Psi\left[\bar{x}+w_e x_e, x_e, x_o\right],$$

we have regularized gauge fixed action:

$$S = \int d^d\bar{x}\,\mathrm{Tr}\left(\frac{1}{2}A\bigstar(L_0-1)A+\frac{1}{3}A\bigstar A\bigstar A\right),$$

$$\mathrm{Tr}\left(A\left(\bar{x},\xi\right)\right) := \det{}^{-d/2}(2\pi\sigma)\int d\xi A\left(\bar{x},\xi\right), \quad \sigma = i\theta\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \bigstar = \exp\left(\frac{1}{2}\overleftarrow{\partial_\xi}\sigma\overrightarrow{\partial_\xi}\right).$$

$$L_0 = \frac{1}{2}\beta_0^2 - \frac{d}{2}Tr\left(\tilde{\kappa}\right) - \frac{1}{4}\bar{D}_\xi M_0^{-1}\tilde{\kappa}D_\xi + \bar{\xi}\tilde{\kappa}M_0\xi,$$

where $$D_\xi = \begin{pmatrix} \partial_{x_e} - i\beta_0 l_s^{-1} w_e \\ \partial_{p_e} \end{pmatrix}, \qquad \tilde{\kappa} = \begin{pmatrix} \kappa_e & \\ & T\kappa_o R \end{pmatrix}$$

We can represent perturbative vacuum as a gaussian:

$$|\Omega\rangle \leftrightarrow A_0(\xi) = N\exp\left(-\bar{\xi}M_0\xi\right), \qquad \xi := \begin{pmatrix} x_e \\ p_e \end{pmatrix}, \qquad M_0 = \begin{pmatrix} \frac{\kappa_e}{2l_s^2} & \\ & \frac{2l_s^2}{\theta^2}T\kappa_e^{-1}\bar{T} \end{pmatrix}.$$

More generally, external states are given by using gaussian:

$$A_p(\xi) = Ne^{-\bar{\xi}M\xi-\bar{\xi}\lambda}e^{ip\bar{x}}.$$

These form Monoid:

- Closed under ★ product
- ★product is associative
- 1 is included
- Every element doesn't have inverse

  ← ∃ projectors: sliver, butterfly,...

Computing Feynman Graphs with Monoid

- ξ -basis

$$e^{-\tau L_0}\left(N e^{-\bar{\xi}M\xi-\bar{\xi}\lambda}e^{ip\bar{x}}\right)=N(\tau)\,e^{-\bar{\xi}M(\tau)\xi-\bar{\xi}\lambda(\tau)}e^{ip\bar{x}},$$

where

$$M(\tau)=\left[\sinh\tau\tilde{\kappa}+\left(\sinh\tau\tilde{\kappa}+M_0M^{-1}\cosh\tau\tilde{\kappa}\right)^{-1}\right]\left(\cosh\tau\tilde{\kappa}\right)^{-1}M_0,$$

$$\lambda(\tau)=\left[\left(\cosh\tau\tilde{\kappa}+MM_0^{-1}\sinh\tau\tilde{\kappa}\right)^{-1}(\lambda+iwp)\right]-iwp,$$

$$N(\tau)=\frac{Ne^{-\frac{1}{2}l_s^2p^2\tau}\exp\left[\frac{1}{4}\left(\bar{\lambda}+i\bar{w}p\right)\left(M+\coth\tau\tilde{\kappa}M_0\right)^{-1}(\lambda+iwp)\right]}{\det^{d/2}\left(\frac{1}{2}\left(1+MM_0^{-1}\right)+\frac{1}{2}\left(1-MM_0^{-1}\right)e^{-2\tau\tilde{\kappa}}\right)}.$$

- Example

$$ {}_1^2 > - <_4^3 = \int d^d \overline{x} \, \mathrm{Tr}\left(e^{-\tau L_0} \left(A_1 \bigstar A_2 \right) \bigstar A_3 \bigstar A_4 \right). $$

We can evaluate such quantities using formula:

$$ N_1 e^{-\overline{\xi} M_1 \xi - \overline{\xi}\lambda_1} e^{ip_1 \overline{x}} \bigstar N_2 e^{-\overline{\xi} M_2 \xi - \overline{\xi}\lambda_2} e^{ip_2 \overline{x}} = N_{12} e^{-\overline{\xi} M_{12} \xi - \overline{\xi}\lambda_{12}} e^{i\left(p_1 + p_2\right)\overline{x}}, $$

$$ m_1 := M_1 \sigma, m_2 := M_2 \sigma, m_{12} := M_{12} \sigma, $$

$$ m_{12} = (1+m_2)m_1(1+m_2 m_1)^{-1} + (1-m_1)m_2(1+m_1 m_2)^{-1}, $$

$$ \lambda_{12} = (1-m_1)(1+m_2 m_1)^{-1}\lambda_2 + (1+m_2)(1+m_1 m_2)^{-1}\lambda_1, $$

$$ N_{12} = \frac{N_1 N_2}{\det^{d/2}\left(1+m_2 m_1\right)} e^{\frac{1}{4}\left(\left(\overline{\lambda_1}+\overline{\lambda_2}\right)\sigma\left(m_1+m_2\right)^{-1}\left(\lambda_1+\lambda_2\right)-\overline{\lambda_{12}}\sigma m_{12}^{-1}\lambda_{12}\right)}, $$

$$ \mathrm{Tr}\left(N e^{-\overline{\xi} M \xi - \overline{\xi}\lambda} e^{ip\overline{x}} \right) = \frac{N e^{ip\overline{x}} e^{\frac{1}{4}\overline{\lambda} M^{-1} \lambda}}{\det^{d/2}\left(2M\sigma\right)}. $$

- Momentum (Fourier transformed) basis:

$$\widetilde{A}(\eta) = \int \frac{d\xi}{2\pi} \; e^{-i\bar{\eta}\xi} A(\xi)$$

Vertex

$$\mathrm{Tr}\left(e^{i\bar{\eta}_1\xi} \star \cdots \star e^{i\bar{\eta}_1\xi}\right) = \det{}^{-d/2}\left(\frac{\sigma}{2\pi}\right) \exp\left(-\frac{1}{2}\sum_{i<j} \overline{\eta_i} \sigma \eta_j\right) \delta\left(\eta_1 + \cdots + \eta_n\right)$$

Simpler than conventional vertex using Neumann coefficients.

The same form as ordinary noncommutative field theory.

External state

From gaussian to gaussian:

$$A_p(\xi) = N e^{-\bar{\xi} M \xi - \bar{\xi}\lambda} e^{ip\bar{x}} \rightarrow \widetilde{A_p}(\eta) = \widetilde{N} e^{-\frac{1}{4}\bar{\eta} M^{-1}\eta + \frac{i}{2}\bar{\eta} M^{-1}\lambda} e^{ip\bar{x}}.$$

Propagator

Complicated compared to ordinary noncommutative field theory.

$$\Delta(\eta,\eta',\tau,p)=\int\frac{(d\xi)}{(2\pi)^{2Nd}}e^{-i\bar{\eta}\xi}e^{-\tau L_0}e^{i\bar{\eta}'\xi}$$

$$=g(\tau,p)\exp\left(-\bar{\eta}F(\tau)\eta-\bar{\eta}'F(\tau)\eta'+2\bar{\eta}G(\tau)\eta'+(\bar{\eta}+\bar{\eta}')H(\tau,p)\right),$$

where

$$g(\tau,p)=\left(\frac{\theta}{2\pi}\right)^{Nd}(1+\bar{w}w)^{d/4}\left(\prod_{e>0}\left(1-e^{-2\tau\kappa_e}\right)\prod_{o>0}\left(1-e^{-2\tau\kappa_o}\right)\right)^{-d/2}e^{-\left(\tau/2+\bar{w}\kappa_e^{-1}\tanh(\tau\kappa_e/2)w\right)l_s^2p^2},$$

$$F(\tau)=\frac{1}{4}M_0^{-1}\left(\tanh\tau\tilde{\kappa}\right)^{-1},\quad G(\tau)=\frac{1}{4}M_0^{-1}\left(\sinh\tau\tilde{\kappa}\right)^{-1},\quad H(\tau,p)=\kappa_e^{-1}\tanh\left(\frac{\tau\kappa_e}{2}\right)wl_s^2p^2.$$

- 1-loop vacuum amplitude

$$\int d^d p \int d\eta \Delta(\eta, \eta, \tau, p) = (2\pi)^{d/2} l_s^{-d} \tau^{-d/2} \prod_e \left(1 - e^{-\tau\kappa_e}\right)^{-d} \prod_o \left(1 - e^{-\tau\kappa_o}\right)^{-d}.$$

Correct spectrum!

Note: If we take naïve limit $\kappa_e = e, \kappa_o = o, N = \infty \Rightarrow w\bar{w} = \infty$

in L_0 first, we have wrong result because

The kinetic term is written by $L_0 = \frac{1}{2}\beta_0^2 - \frac{d}{2}Tr\left(\tilde{\kappa}\right) - \frac{1}{4}\bar{D}_\xi M_0^{-1}\tilde{\kappa}D_\xi + \bar{\xi}\tilde{\kappa}M_0\xi.$

$$M_0^{-1}\tilde{\kappa} = \begin{pmatrix} 2l_s^2 & \\ & \frac{\theta^2}{2l_s^2}\kappa_e^2 \end{pmatrix}, \qquad \tilde{\kappa}M_0 = \begin{pmatrix} \frac{\kappa_e^2}{2l_s^2} & \\ & \frac{2l_s^2}{\theta^2}T\bar{T} \end{pmatrix}, \qquad T\bar{T} = 1 - \frac{w\bar{w}}{1 + w\bar{w}} \rightarrow 1.$$

bc-ghost sector

We have constructed Moyal formulation of bc-ghost sector in the same way. [Erler]

Half-string $\longrightarrow$ *Moyal formulation*

There are some differences:

- $b(\sigma)$ is expanded by *sine* mode in full-string.
- $c(\sigma)$ satisfies *anti*-overlapping condition.
- Moyal ★ is defined by Grassmann *odd* variables : non-*anti*commutative product.

Neumann coefficients

From identification with oscillator approach:

$$\int d^d \bar{x} d\xi_0^{(1)} d\xi_0^{(2)} d\xi_0^{(3)} \mathrm{Tr}\left(A_{c1}\left[\xi_0^{(1)}\right] \star A_{c2}\left[\xi_0^{(2)}\right] \star A_{c3}\left[\xi_0^{(3)}\right]\right)$$

$$\sim \quad \langle \Psi_{c1} | \langle \Psi_{c2} | \langle \Psi_{c3} | | V_3 \rangle$$

where $\mathrm{Tr}(A) := \int dx_e dp_e dx_o dp_o dy_o dq_o A\left[\bar{x}, x_e, p_e, \xi_0, x_o, p_o, y_o, q_o\right]$,

and $|\Psi c\rangle$: coherent state, we define Neumann coefficients in MSFT

Then Gross-Jevicki's Neumann coefficients relations are all satisfied for arbitrary κ_e, κ_o, N.

$$M^{11} + M^{12} + M^{13} = 1, \cdots, \widetilde{M}^{11} + \widetilde{M}^{12} + \widetilde{M}^{13} = 1, \cdots$$

Summary and Discussion

- We developed the method to compute Feynman diagrams in MSFT:

(a) Monoid algebra in noncommutative ξ -space

(b) non-diagonal propagator + phase in momentum-space

All computations are gaussian integration and can be applied to any frequencies and **finite** N : Well-defined!

- We also formulated MSFT in bc ghost sector where we use Grassmann odd variables version of Moyal product.
- We reproduced Gross-Jevicki relations among Neumann coefficients for arbitrary κ_e, κ_o, N and correct spectrum in 1-loop vacuum amplitude.

Problems

Can we reproduce the Veneziano amplitude in MSFT?

Gauge symmetry ?

Definition of BRST charge?

Restriction to κ_e, κ_o, N, d ?

Non-perturbative vacuum?

Toward tachyon vacuum in the Siegel gauge

- The equation of motion (p=0,Siegel gauge) can be rewritten:

$$\mathcal{L}_0 \star A + A \star \mathcal{L}_0 + (\gamma - 1) A + A \star A = 0,$$

$$\mathcal{L}_0 = 2\sum_{e>0} \beta_{-e} \star \beta_e + 2\sum_{o>0} \kappa_o \left(\beta^b_{-o} \star \beta^c_o + \beta^c_{-o} \star \beta^b_o \right) + \frac{d+2}{4} \sum (\kappa_e - \kappa_o),$$

$$\gamma = -\frac{1}{1+\bar{w}w} \frac{2l_s^2}{\theta^2} \left(\bar{w} p_e \right)^2 + \frac{4i}{g^2} \left(1 + \bar{w}w \right) \bar{v} \kappa_o p_o \bar{v} q_o .$$

- butterfly-like state:

$$\beta_e \star A_b = A_b \star \beta_{-e} = \beta^c_o \star A_b = A_b \star \beta^c_{-o} = \beta^b_o \star A_b = A_b \star \beta^b_{-o} = 0, \quad A_b \star A_b = A_b \text{ : a projector}$$

more expricitly, $$A_b = \xi_0 \exp\left(-\bar{x}_e \frac{\kappa_e}{2l_s^2} x_e - \bar{p}_e \frac{2l_s^2}{\theta^2 \kappa_e} p_e - i\bar{x}_o y_o - \frac{4i}{g^2} \bar{p}_o q_o \right).$$

In $\kappa_e = \kappa_o$ case, butterfly-like state A_b is an EXACT solution!

However, in usual openstring case, $\kappa_e - \kappa_o = 1$.