Idempotency Equation and Boundary States in Closed String Field Theory

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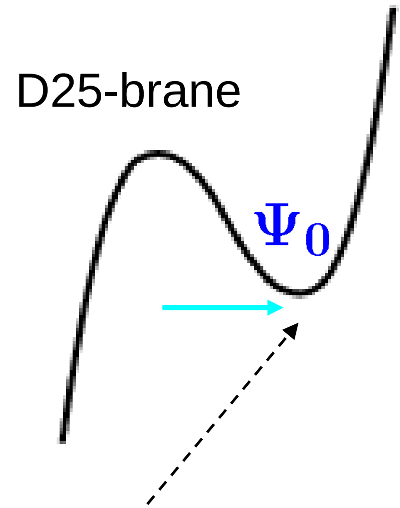
Introduction and motivation

- Sen's conjecture :

Witten's open SFT

$$S = \frac{1}{2}\Psi \cdot Q_B \Psi + \frac{1}{3}\Psi \cdot \Psi * \Psi$$

$\exists$ tachyon vacuum Ψ_0



- Vacuum String Field Theory (VSFT)

[(Gaiotto)-Rastelli-Sen-Zwiebach(2000/2001)]

$$S = \frac{1}{2}\Psi \cdot Q \Psi + \frac{1}{3}\Psi \cdot \Psi * \Psi$$

$Q = c(\pi/2)$: Pure ghost BRST operator

VSFT:

D-brane

~ classical solution of

$$Q|\Psi_0\rangle + |\Psi_0\rangle * |\Psi_0\rangle = 0$$

~ Projector with respect to Witten's $*$ product in the matter sector:

Sliver, Butterfly,… are constructed explicitly.

$$|\Xi\rangle * |\Xi\rangle = |\Xi\rangle$$

Essentially, they are the same as noncommutative solitons because Witten's $*$ can be expressed as the Moyal product.

Around the sliver solution, we can solve

$$\Xi * \delta\Psi + \delta\Psi * \Xi = \delta\Psi.$$

As solutions for $\delta\Psi$, open string spectrum is included and D-brane tension is reproduced: [Okawa(2002)]

$$-S|_{\Xi}/V_{26} = T_{25},$$

$$\delta\Psi_T \sim \int dt e^{ikX(t)}\Xi.$$

However, there are more solutions which give different tension. [Hata-Kawano(2001)]

$$\delta|\Psi_T\rangle_{\rm HK} = e^{kt_n\alpha_{-n}}|\Xi\rangle.$$

On the other hand,

D-brane $\sim$ Boundary state $\leftarrow$ closed string

Closed SFT description is more natural (!?)

$$S = \frac{1}{2}\Phi \cdot Q\Phi + \frac{1}{3}\Phi \cdot \Phi * \Phi \ (+\cdots)$$

HIKKO cubic closed SFT (Nonpolynomial closed SFT)

$$|B\rangle * |B\rangle = |B\rangle \ (?)$$

Contents

Star product in closed SFT

$*$ product is defined by 3-string vertex:

$$|\Phi_1 * \Phi_2\rangle_3 = {}_1\langle\Phi_1|_2\langle\Phi_2|V(1,2,3)\rangle$$

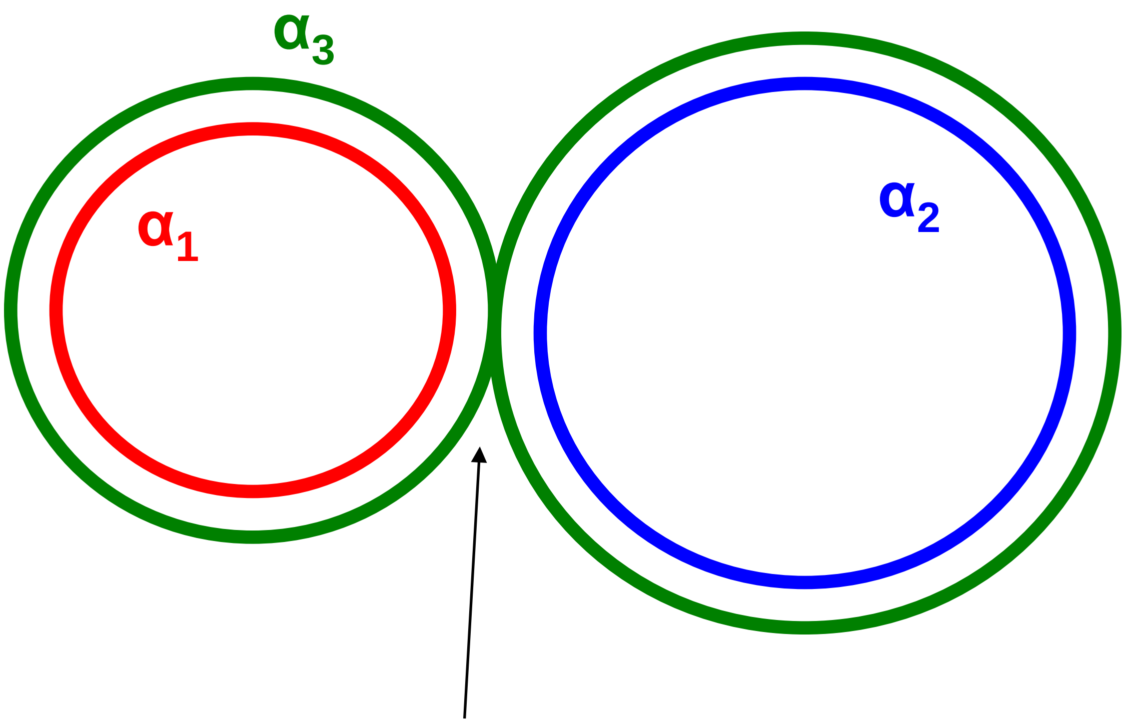
- HIKKO (Hata-Itoh-Kugo-Kunitomo-Ogawa) type

$$(X^{(3)} - \Theta_1 X^{(1)} - \Theta_2 X^{(2)})|V_0(1,2,3)\rangle = 0$$

and ghost sector (to be compatible with BRST invariance)
with projection:

$$|V(1,2,3)\rangle = \wp_1\wp_2\wp_3|V_0(1,2,3)\rangle, \quad \wp_r := \oint \frac{d\theta}{2\pi} e^{i\theta(L_0^{(r)} - \tilde{L}_0^{(r)})}$$

Overlapping condition for 3 closed strings



Interaction point

- Explicit representation of the 3-string vertex:

solution to overlapping condition [HIKKO]

$$|V(1,2,3)\rangle = \int \delta(1,2,3)[\mu(1,2,3)]^2 \wp_1\wp_2\wp_3 \frac{\alpha_1\alpha_2}{\alpha_3} \Pi_c\, \delta\Big(\sum_{r=1}^{3} \alpha_r^{-1}\pi_c^{0(r)}\Big)$$
$$\times \prod_{r=1}^{3}\Big[1+2^{-\frac{1}{2}} w_I^{(r)} \bar{c}_0^{(r)}\Big] e^{F(1,2,3)} |p_1,\alpha_1\rangle_1 |p_2,\alpha_2\rangle_2 |p_3,\alpha_3\rangle_3$$

$$F(1,2,3) = \sum_{r,s=1}^{3} \sum_{m,n\geq 1} \tilde{N}^{rs}_{mn}\Big[\frac{1}{2} a_m^{(r)\dagger} a_n^{(s)\dagger} + \sqrt{m}\alpha_r c_{-m}^{(r)} (\sqrt{n}\alpha_s)^{-1} b_{-n}^{(s)}$$
$$+\frac{1}{2}\tilde{a}_m^{(r)\dagger}\tilde{a}_n^{(s)\dagger} + \sqrt{m}\alpha_r \tilde{c}_{-m}^{(r)} (\sqrt{n}\alpha_s)^{-1} \tilde{b}_{-n}^{(s)}\Big]$$
$$+\frac{1}{2}\sum_{r=1}^{3}\sum_{n\geq 1} \tilde{N}^r_n (a_n^{(r)\dagger} + \tilde{a}_n^{(r)\dagger})\mathrm{P} - \frac{\tau_0}{4\alpha_1\alpha_2\alpha_3}\mathrm{P}^2$$

(Gaussian !)

$\tilde{N}^{rs}_{mn},\ \tilde{N}^{r}_{n}$: Neumann coefficients of light-cone type

$$\tilde{N}^{rs}_{mn} = \frac{mn\alpha_1\alpha_2\alpha_3}{\alpha_r n + \alpha_s m}\tilde{N}^r_m \tilde{N}^s_n,$$

$$\tilde{N}^r_m = \frac{\sqrt{m}}{\alpha_r m!}\frac{\Gamma(-m\alpha_{r+1}/\alpha_r)}{\Gamma(1+m\alpha_{r-1}/\alpha_r)}e^{\frac{m\tau_0}{\alpha_r}}, \qquad \tau_0 = \sum_{r=1}^{3}\alpha_r \log|\alpha_r|$$

We can prove various relations. [Mandelstam, Green-Schwarz,…]
In particular, Yoneya formulae are essential to computation of B ∗ B.

$$\sum_{t=1}^{3}\sum_{p=1}^{\infty}\tilde{N}^{rt}_{mp}\tilde{N}^{ts}_{pn} = \delta_{r,s}\delta_{m,n}, \quad \sum_{t=1}^{3}\sum_{p=1}^{\infty}\tilde{N}^{rt}_{mp}\tilde{N}^{t}_{p} = -\tilde{N}^r_m,$$

$$\sum_{t=1}^{3}\sum_{p=1}^{\infty}\tilde{N}^{t}_{p}\tilde{N}^{t}_{p} = \frac{2\tau_0}{\alpha_1\alpha_2\alpha_3}.$$

Star product of boundary state

The boundary state for Dp-brane with constant flux:

$$|B(x^\perp)\rangle = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\mathcal{O}\tilde{\alpha}_{-n} + \sum_{n=1}^{\infty}(c_{-n}\tilde{b}_{-n} + \tilde{c}_{-n}b_{-n})\right) \times c_0^+ c_1\tilde{c}_1|p^{\|} = 0, x^\perp\rangle \otimes |0\rangle_{gh},$$

$$\mathcal{O}^\mu{}_\nu = \left[(1+F)^{-1}(1-F)\right]^\mu{}_\nu, \quad \mu,\nu = 0,1,\cdots,p, \qquad \text{(Neumann)}$$

$$\mathcal{O}^i{}_j = -\delta^i_j, \qquad i,j = p+1,\cdots,d-1. \qquad \text{(Dirichlet)}$$

↓

We *define* the string field $\Phi_B(x^\perp,\alpha)$:

$$|\Phi_B(x^\perp,\alpha)\rangle = c_0^- b_0^+ |B(x^\perp)\rangle \otimes |\alpha\rangle$$

• Comment on the ghost sector

The ghost sector of conventional boundary state:

$$(b_n - \tilde{b}_{-n})|B\rangle = (c_n + \tilde{c}_{-n})|B\rangle = 0.$$

↓

$$Q_B|\ \rangle_{\rm mat} \otimes |B\rangle_{\rm gh} = \sum_n c_{-n}(L_n^{\rm mat} - \tilde{L}_{-n}^{\rm mat})|\ \rangle_{\rm mat} \otimes |B\rangle_{\rm gh}$$

$$\Rightarrow \qquad Q_B|B(x^\perp)\rangle = 0.$$

Note 1: $|B(x^\perp)\rangle * |B(y^\perp)\rangle = 0,$

which follows from $b_0^-|B(x^\perp)\rangle = 0.$

Note 2: $|\Phi\rangle = c_0^-|\phi\rangle + c_0^- c_0^+|\psi\rangle + |\chi\rangle + c_0^+|\eta\rangle$

φ: "physical sector" i.e.,

$$\frac{1}{2}\Phi \cdot Q_B\Phi = \frac{1}{2}\langle I[\phi](L_0 + \tilde{L}_0 - 2)\phi\rangle + \cdots .$$

$|\Phi_B(x^\perp,\alpha)\rangle$ and $|V(1,2,3)\rangle$ are "Gaussian." $\mathcal{O}$ is orthogonal.
Using *Yoneya formula* for Neumann matrices, we have obtained

$$|\Phi_B(x^\perp,\alpha_1)\rangle * |\Phi_B(y^\perp,\alpha_2)\rangle = \delta(x^\perp - y^\perp)\mathcal{C}c_0^+|\Phi_B(x^\perp,\alpha_1+\alpha_2)\rangle$$

"idempotency equation"

$\mathcal{C}$ is given by

$$\mathcal{C} = [\mu(1,2,3)]^2[\det(1-(\tilde{N}^{33})^2)]^{-\frac{d-2}{2}}$$

where $\mu(1,2,3) = e^{-\tau_0\sum_{r=1}^{3}\alpha_r^{-1}}$.

$\mathcal{C}$ is divergent because $\tilde{N}^{33}_{mn}$ is $\infty \times \infty$ matrix.

However, by *regularizing* with parameter T:

$$\tilde{N}^{33}_{mn} \to \tilde{N}^{33}_{mn} e^{-(m+n)\frac{T}{|\alpha_3|}}$$

$\mathcal{C}$ can be simplified drastically for $d = 26$.

We use *Cremmer-Gervais identity* to evaluate the regularized $\mathcal{C}$.

By algebraic calculation, we obtain the differential equation:

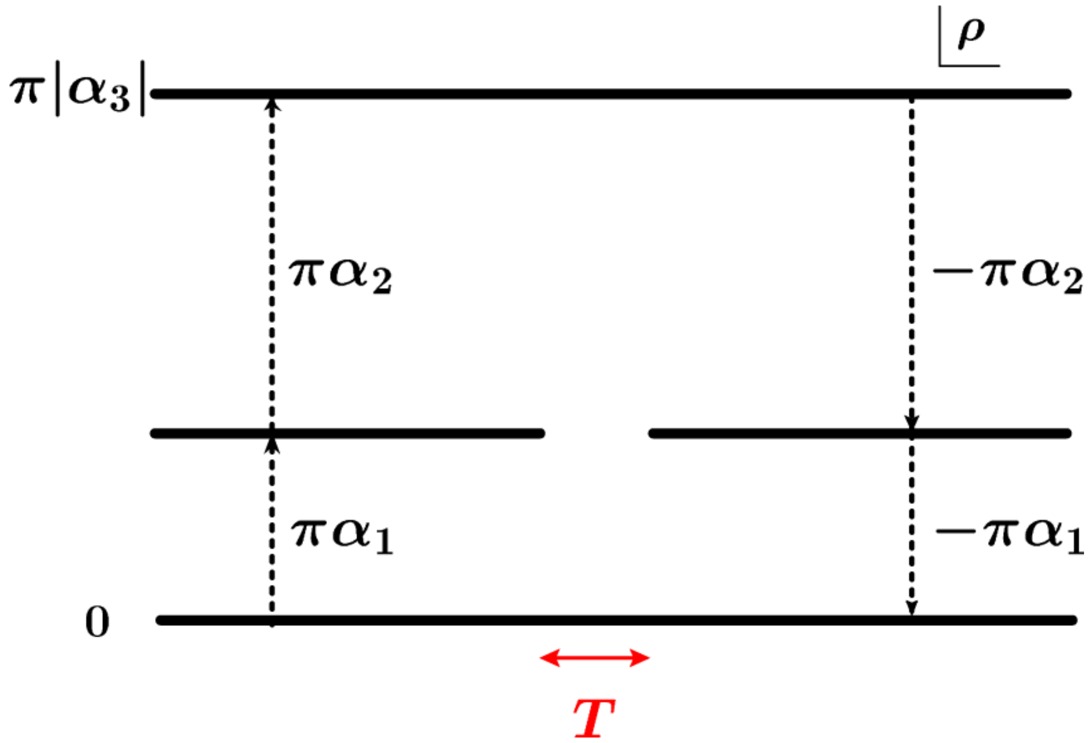
$$\frac{\partial^2}{\partial T^2} \log\det\left(1 - \tilde{N}^{66}\tilde{N}^{55}_T\right) = -\frac{1}{4}\left[\frac{\partial_T^2 a}{\partial_T b}\right]^2$$

where

$$\begin{aligned} a &= \alpha_1\alpha_2\tilde{N}^6_n\left[\tilde{N}^{55}_T(1-\tilde{N}^{66}\tilde{N}^{55}_T)^{-1}\right]_{nm}\tilde{N}^6_m \\ b &= \tilde{N}^5_{Tn}\left[(1-\tilde{N}^{66}\tilde{N}^{55}_T)^{-1}\right]_{nm}\tilde{N}^6_m. \end{aligned}$$

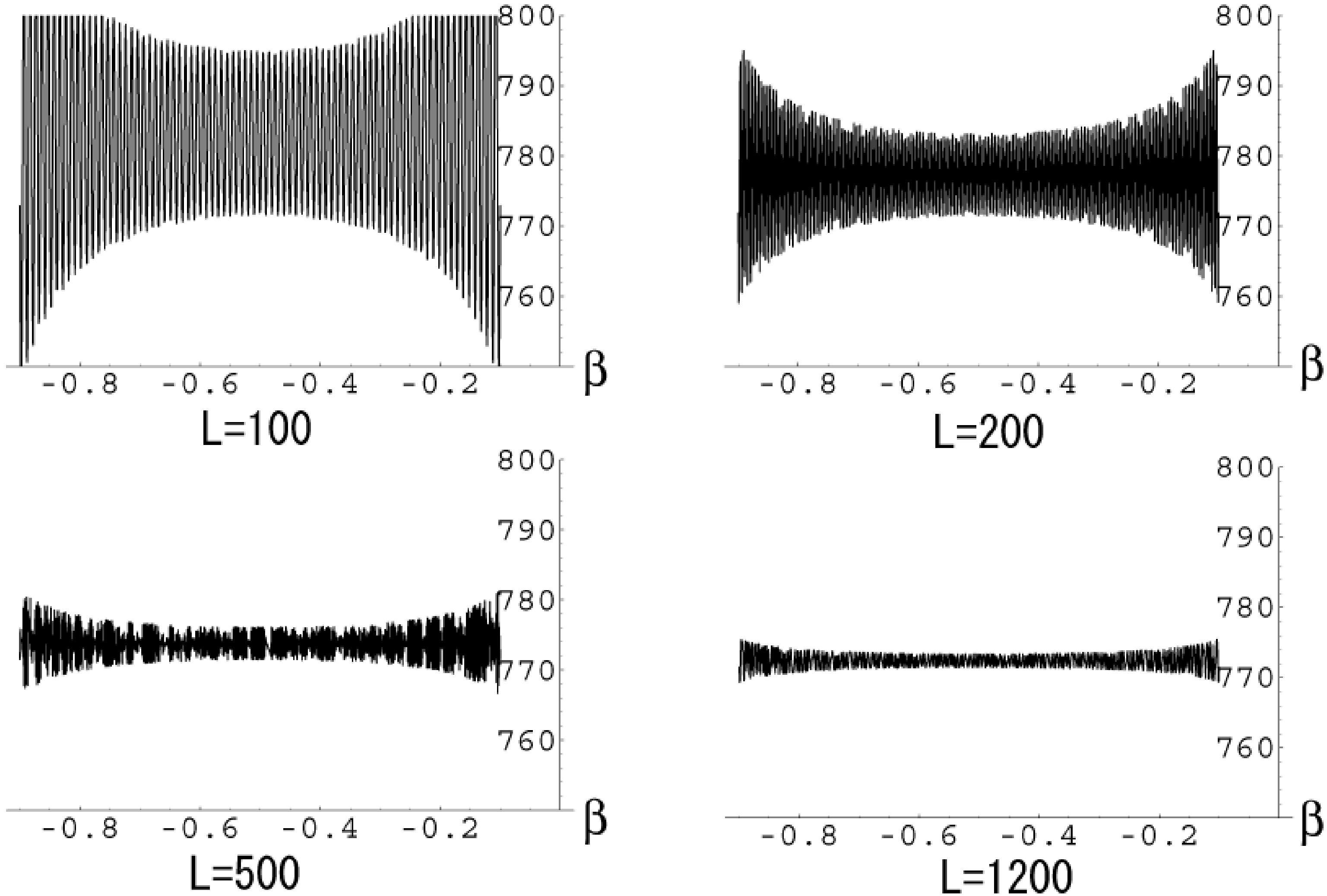
These are evaluated by identifying Neumann coefficients for zero modes in 4-string vertex:

$$\langle R(5,6)|e^{-\frac{T}{\alpha_5}(L_0^{(5)}+\tilde{L}_0^{(5)})}|V(1,2,6)\rangle|V(5,3,4)\rangle \sim |V_T(1,2,3,4)\rangle.$$



The equation can be integrated and the result is
$\mathcal{C} = 2^5 T^{-3}|\alpha_1\alpha_2\alpha_3|$ for $T \to +0$ (and $d = 26$).

On the other hand, we have computed $\mathcal{C}$ numerically by truncating the size of $\tilde{N}^{33}_{mn}$ to L. We have observed $\mathcal{C} \sim L^3|(\alpha_1/\alpha_3)(\alpha_2/\alpha_3)|$, therefore, $T \sim |\alpha_3|/L$.



Plots of $\mathcal{C}/[L^3(-\beta(1+\beta)]$ by level truncation $(\beta := -\alpha_1/(\alpha_1+\alpha_2))$ using *Mathematica5.*

Idempotency equation

$$|\Phi(\alpha_1)\rangle * |\Phi(\alpha_2)\rangle = K^3 \hat{\alpha}^2 c_0^+ |\Phi(\alpha_1+\alpha_2)\rangle$$

where $c_0^+ = \frac{1}{2}(c_0 + \tilde{c}_0)$,
$K(\sim T^{-1} \to \infty)$: constant and $\alpha_1\alpha_2 > 0$

$\hat{\alpha}^2 c_0^+$ is a "pure ghost" BRST operator which is nilpotent, partial integrable and derivation with respect to $*$ product.

The boundary state which corresponds to Dp-brane is a solution to this equation *in the following sense*.

- Boundary state as an “idempotent” :

$$|\Phi_f(\alpha)\rangle = \int d^{d-p-1}x^{\perp} f(x^{\perp}) |\Phi_B(x^{\perp},\alpha)\rangle/\alpha$$

$f(x^{\perp})$ is a solution to $f(x^{\perp})^2 = f(x^{\perp})$.

Namely, “commutative soliton” $f(x^{\perp}) = \begin{cases} 1 & (x^{\perp} \in \Sigma) \\ 0 & (\text{otherwise}) \end{cases}$

for some subset Σ of R^{d-p-1}.

↓

$$|\Phi_f(\alpha_1)\rangle * |\Phi_f(\alpha_2)\rangle = K^3 \hat{\alpha}^2 c_0^{+} |\Phi_f(\alpha_1+\alpha_2)\rangle$$

Fluctuations

Infinitesimal deformation of "idempotency equation" around $\Phi_B(x^\perp, \alpha)$:

$$\delta\Phi_B(x^\perp, \alpha_1) * \Phi_B(y^\perp, \alpha_2) + \Phi_B(x^\perp, \alpha_1) * \delta\Phi_B(y^\perp, \alpha_2)$$
$$= \delta^{d-p-1}(x^\perp - y^\perp)\mathcal{C}c_0^+ \delta\Phi_B(x^\perp, \alpha_1 + \alpha_2).$$

Ansatz: $\delta\Phi_B(x^\perp, \alpha) = \oint \frac{d\sigma}{2\pi} V(\sigma)\Phi_B(x^\perp, \alpha)$

By *straightforward computation in oscillator language*, we found scalar and vector type "solutions":

$$V_S(\sigma) =: e^{ik_\mu X^\mu(\sigma)} :, \qquad k_\mu G^{\mu\nu} k_\nu = \alpha'^{-1},$$

$$V_V(\sigma) =: \zeta_\nu \partial_\sigma X^\nu e^{ik_\mu X^\nu(\sigma)} :, \quad k_\mu G^{\mu\nu} k_\nu = 0,$$

$$(G^{\mu\nu} = [(1+F)^{-1}\eta(1-F)^{-1}]^{\mu\nu} : \text{ open string metric}).$$

In computation of tachyon mass using Neumann coefficients, we enconunter

$$k_\mu G^{\mu\nu} k_\nu \left(\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{m=1}^{\infty} \frac{1}{m} \right)$$

at least naively. $\rightarrow$ We should take some *regularization*.

By truncating the level of string r as is proportional to $|\alpha_r|$, we obtain on-shell condition uniquely:

$$(-\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu} + (1+\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu} = 1 \text{ for } V_S$$

where $\beta = \alpha_1/\alpha_3$

$\rightarrow$ *open* string tachyon: $k_\mu G^{\mu\nu} k_\nu = \alpha'^{-1}$.

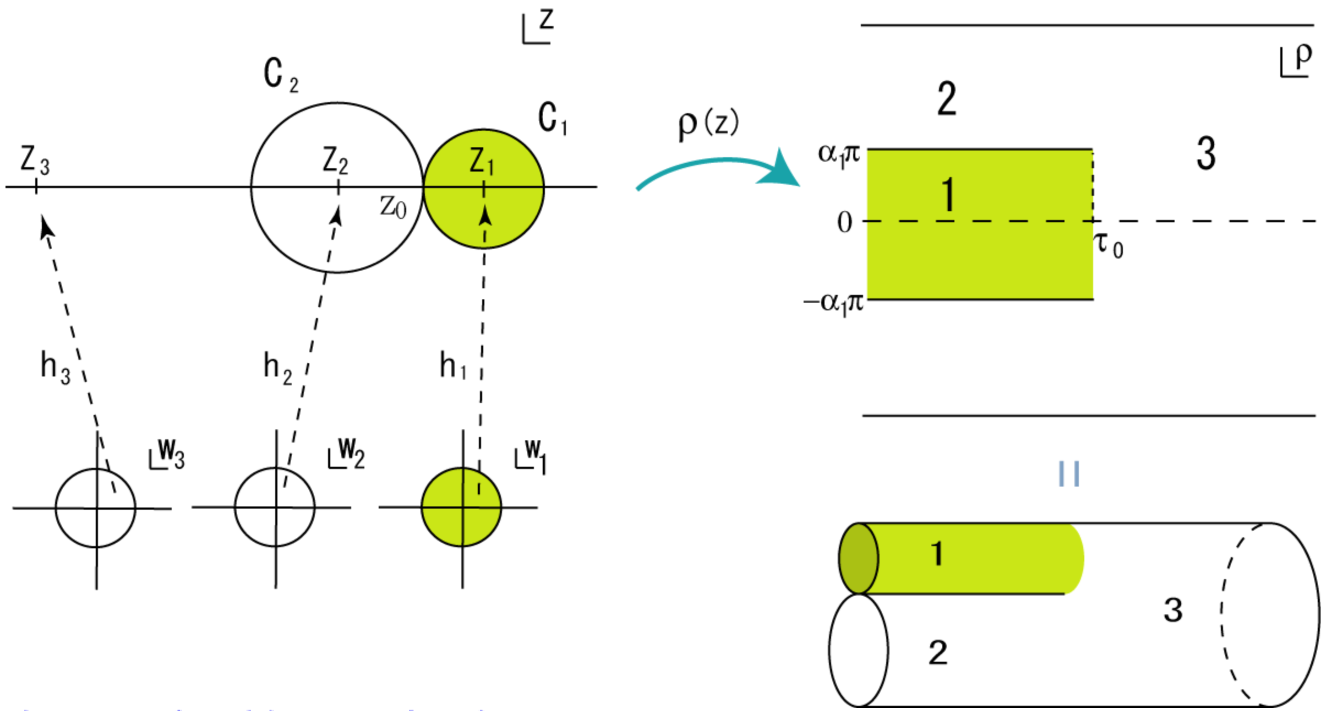
For vector type fluctuation $\delta_V \Phi_B$, we compute

$$
\begin{aligned}
&|\delta_V \Phi_B(\alpha_1)\rangle * |\Phi_B(\alpha_2)\rangle + |\Phi_B(\alpha_1)\rangle * |\delta_V \Phi_B(\alpha_2)\rangle \\
=\ & ((-\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu + 1} + (1+\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu + 1}) \mathcal{C} c_0^+ |\delta_V \Phi_B(\alpha_1+\alpha_2)\rangle \\
& + ((-\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu} - (1+\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu}) \\
& \quad \times \left[-i\zeta_\mu G^{\mu\nu} k_\nu \sum_{p=1}^{\infty} \frac{\sin^2 p\pi\beta}{\pi p} \mathcal{C} c_0^+ |\delta_S \Phi_B(\alpha_1+\alpha_2)\rangle + \cdots \right].
\end{aligned}
$$

We obtain massless condition $k_\mu G^{\mu\nu} k_\nu = 0$.

However, the transversality condition is subtle because $((-\beta)^0 - (1+\beta)^0) \sum_{p=1}^{\infty} \frac{\sin^2 \pi p \beta}{\pi p} \sim 0 \times \infty$.

Let us consider LPP formulation [LeClair-Peskin-Preitshopf(1989)] , which refers to CFT correlation function to define 3-string vertex:



$$
(\Phi_1(\alpha_1) * \Phi_2(\alpha_2)) \cdot \Phi_3(\alpha_3)
$$
$$
= 2\pi\delta(\alpha_1+\alpha_2+\alpha_3)(-1)^{|\Phi_2|}\left\langle h_1[b_0^- \wp \Phi_1]\, h_2[b_0^- \wp \Phi_2]\, h_3[b_0^- \wp \Phi_3] \right\rangle ,
$$

where

$$
\rho(z) = \alpha_1 \log(z-1) + \alpha_2 \log z ,
$$
$$
h_r(w_r) = \rho^{-1}(f_r(w_r)) , \quad f_r(w_r) = \alpha_r \log w_r + \tau_0 + i\beta_r .
$$

Using LPP formulation for the HIKKO closed SFT, the equation for the fluctuation is reduced to

$$\wp \left(\oint \frac{d\sigma_1}{2\pi} \Sigma_1[V(\sigma_1)] + \oint \frac{d\sigma_2}{2\pi} \Sigma_2[V(\sigma_2)] + \oint \frac{d\sigma_3}{2\pi} V(\sigma_3) \right) |B(x^\perp)\rangle = 0.$$

A *sufficient* condition for this solution : primary with weight 1

$$\Sigma_r[V(\sigma_r)]\, |B(x^\perp)\rangle = \frac{d}{d\sigma_r}\Sigma_r(\sigma_r)\, V(\Sigma_r(\sigma_r))\, |B(x^\perp)\rangle\,.$$

$\rightarrow$ *open* string spectrum!

However, Σ_r is a particular mapping.

Is this a *necessary* condition?

By modifying the vector type fluctuation [Murakami-Nakatsu(2002)] :

$$V_S(\sigma) = :e^{ik_\mu X^\mu(\sigma)}:, \quad V_V(\sigma) = :\zeta_\mu \partial_\sigma X^\mu(\sigma) e^{ik_\nu X^\nu(\sigma)}:,$$

$$\hat{V}_V(\sigma) \equiv V_V(\sigma) - (\zeta_\mu \theta^{\mu\nu} k_\nu / 4\pi) V_S(\sigma),$$

where $\theta \equiv \pi(\mathcal{O} - \mathcal{O}^T)/2 = -2\pi(1+F)^{-1}F(1-F)^{-1}$,

we obtain the finite transformation

$$(d\sigma)^\Delta V_S(\sigma)|B(x^\perp)\rangle = (d\lambda)^\Delta V_S(\lambda)|B(x^\perp)\rangle ,$$

$$(d\sigma)^{\Delta+1}\hat{V}_V(\sigma)|B(x^\perp)\rangle = (d\lambda)^{\Delta+1}\left[\hat{V}_V(\lambda)|B(x^\perp)\rangle - \Xi \frac{\partial_\lambda^2 \sigma}{\partial_\lambda \sigma} V_S(\lambda)|B(x^\perp)\rangle\right],$$

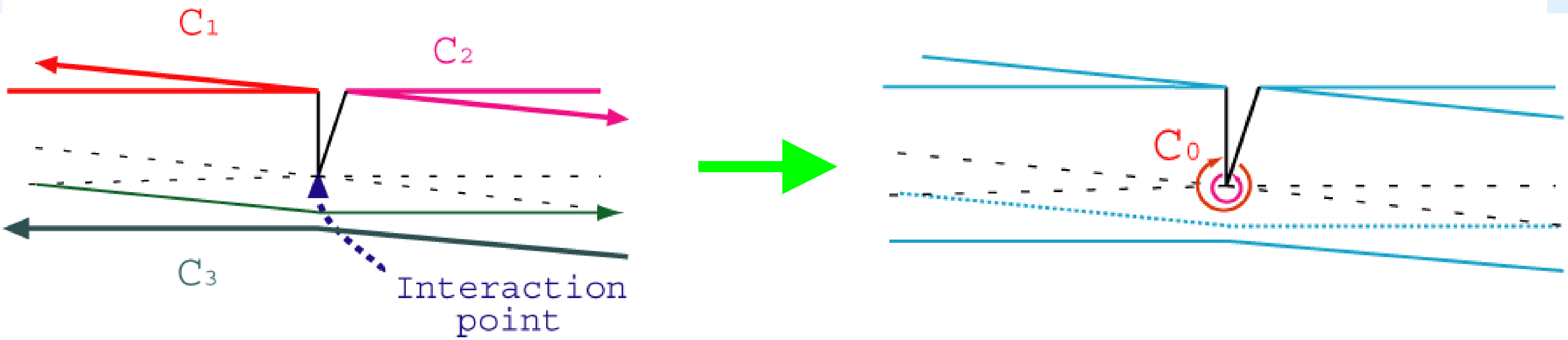
where

$$\Delta \equiv \alpha' k_\mu G^{\mu\nu} k_\nu, \qquad \Xi \equiv -i\zeta_\mu G^{\mu\nu} k_\nu / 2.$$

Σ_1, Σ_2 are linear mappings

$\rightarrow \quad \Delta = 1$ for V_S and $\Delta = 0$ for $\hat{V}_V$.

We should note the *singularity* at the interaction point for $\hat{V}_V$.



Around the interaction pt. for $\Delta = 0$, noting $\frac{d\rho}{dz} \sim \mathrm{const.}(z - z_0)$,

$$
\begin{aligned}
& d\sigma \hat{V}_V(\sigma)|B(x^\perp)\rangle \\
& = dz \left[\hat{V}_V(z)|B(x^\perp)\rangle - \Xi \left((z - z_0)^{-1} + \mathcal{O}((z - z_0)^0) \right) V_S(z)|B(x^\perp)\rangle \right]
\end{aligned}
$$

$\rightarrow$

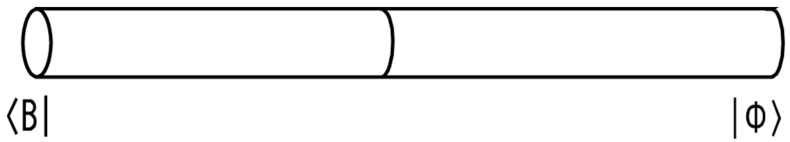
$$
\begin{aligned}
& \wp \left(\oint \frac{d\sigma_1}{2\pi} \Sigma_1[V(\sigma_1)] + \oint \frac{d\sigma_2}{2\pi} \Sigma_2[V(\sigma_2)] + \oint \frac{d\sigma_3}{2\pi} V(\sigma_3) \right) |B(x^\perp)\rangle \\
& = i\wp\, \Xi\, V_S(z_0)|B(x^\perp)\rangle = i\, \Xi \oint \frac{d\sigma}{2\pi} V_S(\sigma)|B(x^\perp)\rangle
\end{aligned}
$$

$\rightarrow$ the transversality condition $2i\Xi = \zeta_\mu G^{\mu\nu} k_\nu = 0$ is imposed.

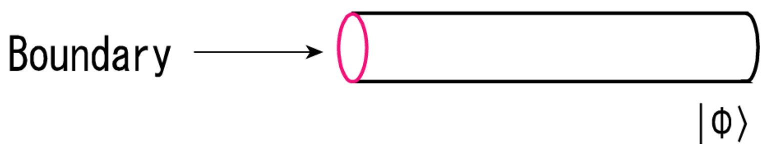
Correct open string spectrum!

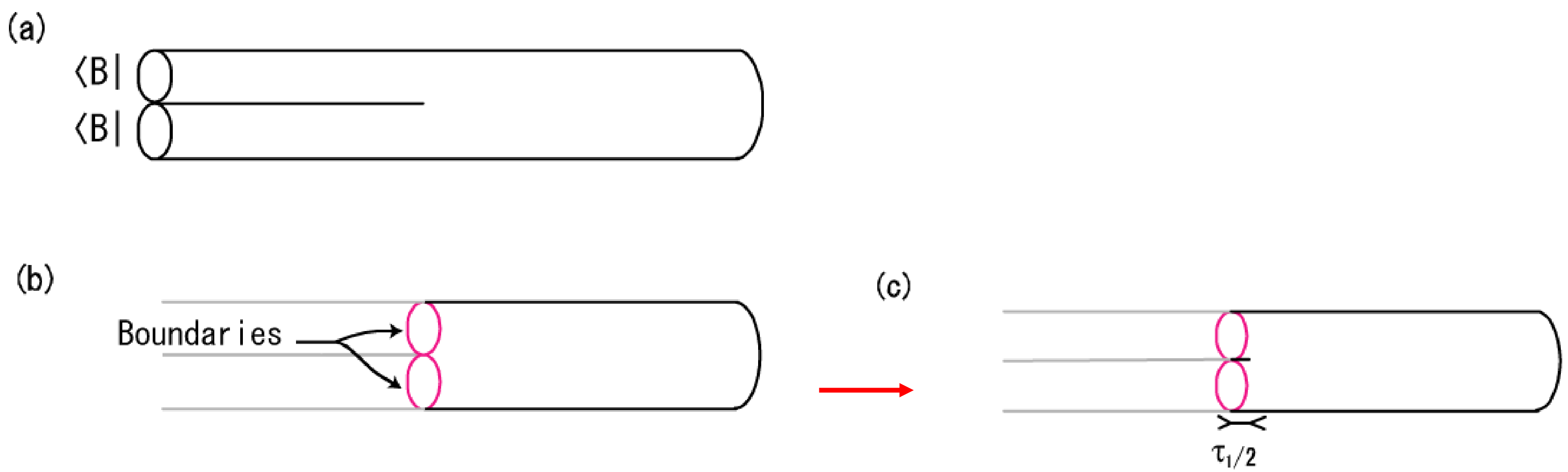
Overall factor

- Let us reconsider B ∗ B.



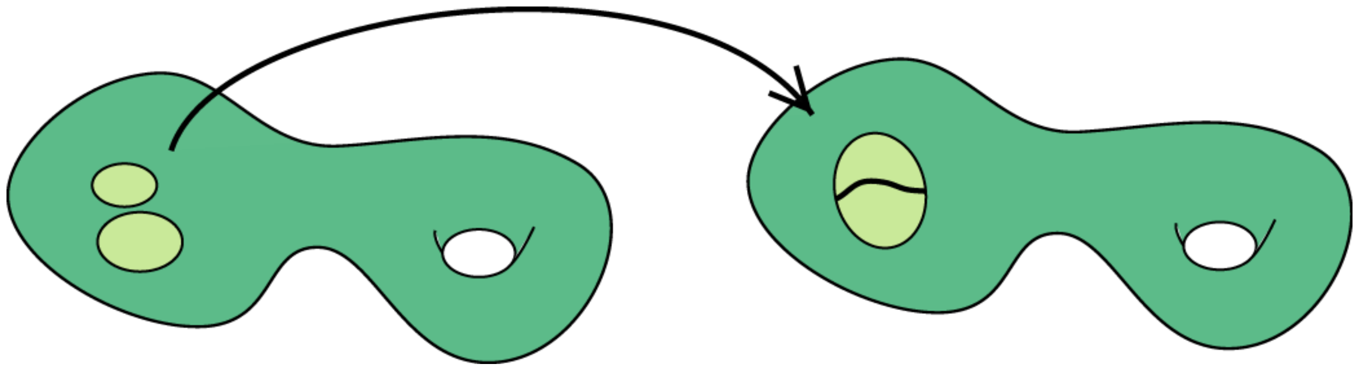
Note:





regularization

Conversely, it corresponds to a particular case of degeneration of a Riemann surface:



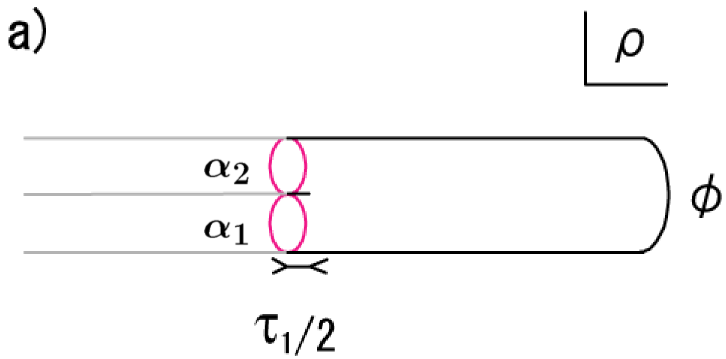
Generally, this process is described by factorization:

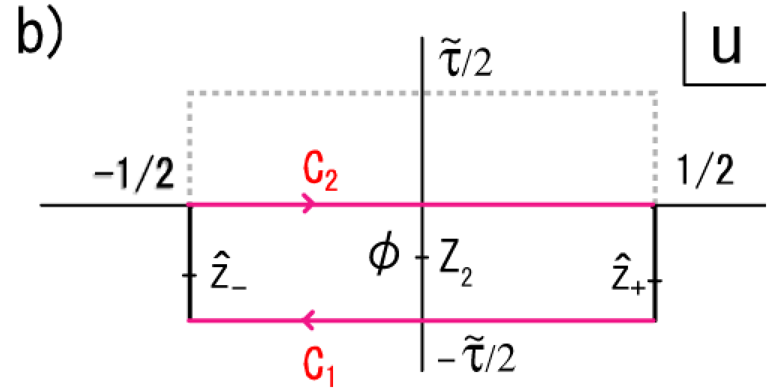
$$\langle \mathcal{O} \cdots \rangle \quad \longrightarrow \quad \sum_i \langle \mathcal{O} \cdots A_i(z_1) A_i(z_2) \rangle q^{\Delta_i}$$

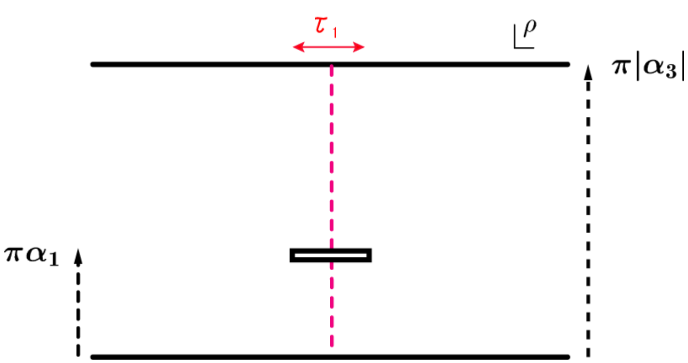
In the case of B $*$ B, it *roughly* implies

$$|B * B\rangle|_{\text{regularized}} \sim q^{-1} c(\sigma_1) c(\sigma_2) |B\rangle + (\text{less singular part}).$$

More precisely, we should consider modulus in terms of regulator and ghost structure in computation of $*$ product:







c.f. [Asakawa-Kugo-Takahashi(1999)]

Mandelstam mapping:

$$\rho(u) = (\alpha_1 + \alpha_2) \log \frac{\vartheta_1(u - \bar{Z}_2|\tilde{\tau})}{\vartheta_1(u - Z_2|\tilde{\tau})} - 2\pi i \alpha_1 u.$$

Modulus:

$$e^{-\frac{i\pi}{\tilde{\tau}}} = q^{1/2} \sim \frac{\tau_1}{8(\alpha_1+\alpha_2)\sin(\pi\alpha_1/(\alpha_1+\alpha_2))} \quad (\to 0)$$

• Evaluation of the coefficient

Using the idempotency equation, we get

$$\mathcal{C} = \left(\langle B_1 |_{\frac{\tau_1}{2\alpha_1}} b_0^+ c_0^- * \langle B_2 |_{\frac{\tau_1}{2\alpha_2}} b_0^+ c_0^- \right) |\phi\rangle / \langle B_2 | b_0^+ c_0^- c_0^+ |\phi\rangle \, .$$

In the following, we take $\phi = c\tilde{c}$ for simplicity.

From the previous figure b), we compute the numerator as

$$\begin{aligned} \mathcal{F}^{\mathrm{m}} &= \langle B_1^{\mathrm{m}} | \tilde{q}^{\frac{1}{2}\left(L_0 + \tilde{L}_0 - \frac{c}{12}\right)} | B_2^{\mathrm{m}} \rangle \\ &= q^{-\frac{c}{24}} \delta_{12} + (\text{higer order in } q) \end{aligned}$$

in the matter sector and

$$\begin{aligned} \mathcal{F}^{\mathrm{gh}}_{c\tilde{c}} &= 4\alpha_1\alpha_2(2\pi)^2 \int_{C_1} \frac{du_1}{2\pi i}\frac{du_1}{d\rho} \int_{C_2} \frac{du_2}{2\pi i}\frac{du_2}{d\rho} \left[\left.\frac{du}{dw_3}\right|_{w_3=0} \left.\frac{d\bar{u}}{d\bar{w}_3}\right|_{\bar{w}_3=0} \right]^{-1} \\ &\quad \times \langle B | \tilde{q}^{\frac{1}{2}\left(L_0 + \tilde{L}_0 + \frac{13}{6}\right)} b(2\pi i u_1) c(2\pi i Z_2) \tilde{c}(-2\pi i \bar{Z}_2) b(2\pi i u_2) | B \rangle \end{aligned}$$

in the ghost sector.

Combining matter and ghost contribution, the numerator is evaluated as:

$$\left(\langle B_1|_{\frac{\tau_1}{2\alpha_1}} b_0^+ c_0^- * \langle B_2|_{\frac{\tau_1}{2\alpha_2}} b_0^+ c_0^- \right) c_1 \tilde{c}_1 |0\rangle$$
$$= \mathcal{F}^{\mathrm{m}} \mathcal{F}^{\mathrm{gh}}_{c\tilde{c}} (\log q)^{-1}$$
$$\sim 32\, \delta_{12}\, \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \tau_1^{-3} q^{\frac{26-c}{24}}.$$

The facotor $(\log q)^{-1}$ comes from the identification $\alpha \sim p^+$.

The denominator is give by $\langle B_2| b_0^+ c_0^- c_0^+ c_1 \tilde{c}_1 |0\rangle = T_{B_2}$.

Namely, $\mathcal{C} \sim 32 \delta_{12}\, \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \tau_1^{-3} T_{B_2}^{-1}$

for c=26 and this is consistent with the correspondence of regularizations:

$$\tau_1 \sim T \sim |\alpha_3| / L\,.$$

Cardy states and idempotents

- On the flat (R^d) background, we have $*$ product formula for *Ishibashi states* :

$$|p_1^\perp\rangle\rangle_{\alpha_1} * |p_2^\perp\rangle\rangle_{\alpha_2} = \mathcal{C} c_0^+ |p_1^\perp + p_2^\perp\rangle\rangle_{\alpha_1+\alpha_2}.$$

$|p^\perp\rangle\rangle$ satisfies $(L_n - \tilde{L}_{-n})|p^\perp\rangle\rangle = 0$, but is *not* an idempotent. Its *Fourier transform* $|B(x^\perp)\rangle\rangle$ which is a Cardy state gives an idempotent.

<u>*Conjecture*</u>

Cardy states $\sim$ idempotents in closed SFT

even on nontrivial backgrounds.

Cardy states $|B\rangle$:

1. $(L_n - \tilde{L}_{-n})|B\rangle = 0.$
2. $\langle B|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|B'\rangle = \sum_i N^i_{BB'}\chi_i(q),$

   $N^i_{BB'}$:nonnegative integer.

Closed SFT:

1. $(L_n - \tilde{L}_{-n})|B\rangle = 0, \quad (L_n - \tilde{L}_{-n})|B'\rangle = 0,$

   $\rightarrow \quad (L_n - \tilde{L}_{-n})|B\rangle * |B'\rangle = 0.$
2. idempotency: $|B\rangle * |B'\rangle = \delta_{B,B'}\mathcal{C}\,|B\rangle.$

- Orbifold（M/Γ）

twisted sector: $X(\sigma+2\pi)=gX(\sigma) \quad (g\in\Gamma)$

$$(g\text{-twisted}) * (g'\text{-twisted}) \sim (gg'\text{-twisted})$$

$\rightarrow$ $*$ product of Ishibashi states should be

$$|g\rangle\rangle_{\alpha_1} * |g'\rangle\rangle_{\alpha_2} \sim |gg'\rangle\rangle_{\alpha_1+\alpha_2}$$

$\Updownarrow$

Group ring $C^{[\Gamma]}$: $\sum_{g\in\Gamma}\lambda_g e_g \in C^{[\Gamma]}, \ \lambda_g \in C$

$$e_g \star e_{g'} = e_{gg'}$$

Γ:nonabelian $\quad e_g \;\rightarrow\; e_i = \sum_{g\in\mathcal{C}_i} e_g \quad$ ($\mathcal{C}_i$: conjugacy class).

Formula: $\quad e_i \star e_j = \mathcal{N}_{ij}{}^k e_k$

$$\mathcal{N}_{ij}{}^k = \frac{1}{|\Gamma|}\sum_{\alpha:\text{irreps.}} \frac{|\mathcal{C}_i||\mathcal{C}_j|\zeta_i^{(\alpha)}\zeta_j^{(\alpha)}\zeta_k^{(\alpha)*}}{\zeta_1^{(\alpha)}}. \quad (\zeta_i^{(\alpha)}\text{ : character})$$

idempotents: $P^{(\alpha)} = \frac{\zeta_1^{(\alpha)}}{|\Gamma|}\sum_{i:\text{class}} \zeta_i^{(\alpha)} e_i, \quad P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha,\beta} P^{(\beta)}.$

$\Updownarrow$

Cardy states: $|\alpha\rangle = \frac{1}{\sqrt{|\Gamma|}}\sum_{i:\text{class}} \zeta_i^{(\alpha)}\sqrt{\sigma_i}|i\rangle\rangle, \quad |i\rangle\rangle := \sum_{g\in\mathcal{C}_i} |g\rangle\rangle$,

[cf. Billo et al.(2001)]

$$\sigma_i = \sigma(e,g), g\in\mathcal{C}_i, \;\; \chi_h^g(q) = \mathrm{Tr}_{\mathcal{H}_h}(g q^{L_0-\frac{c}{24}}) = \sigma(h,g)\chi_g^{h^{-1}}(\tilde{q})$$

$\rightarrow$ $|\alpha\rangle$: idempotents in closed SFT (?)

- <u>Fusion ring of RCFT</u>

$$e_i \star e_j = N_{ij}{}^k e_k, \quad N_{ij}{}^k = \sum_l \frac{S_{il} S_{jl} S^*_{kl}}{S_{1l}}$$

[Verlinde(1988)]

idempotents: $P^{(\alpha)} = S^*_{1\alpha} \sum_{i:\text{primary}} S_{i\alpha} e_i, \quad P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha,\beta} P^{(\beta)}.$

[T.Kawai (1989)]

↕

Cardy states: $|\alpha\rangle = \sum_{i:\text{primary}} \frac{S_{\alpha i}}{\sqrt{S_{1i}}} |i\rangle\rangle$

Suppose $|i\rangle\rangle_{\alpha_1} * |j\rangle\rangle_{\alpha_2} \sim N_{ij}{}^k |k\rangle\rangle_{\alpha_1+\alpha_2},$
then Cardy states $|\alpha\rangle \sim$ idempotents in closed SFT

- Conversely, suppose the idempotency relation

$$|a\rangle * |b\rangle = q^{-\frac{c}{24}} \delta_{ab} T_b^{-1} |b\rangle \,,$$

where $|a\rangle = \sum_j \frac{\psi_a^j}{\sqrt{S_{j1}}} |j\rangle\rangle$ is generalized Cardy state s.t.

[cf. Behrend et al.(1999)]

$$T_b = \frac{\psi_a^1}{\sqrt{S_{11}}} \,, \quad \sum_a \psi_a^i (\psi_a^j)^* = \delta_{ij} \,, \quad \sum_i \psi_a^i (\psi_b^i)^* = \delta_{ab} \,.$$

Then, the algebra of the Ishibashi states becomes

$$|i\rangle\rangle' * |j\rangle\rangle' = q^{-\frac{c}{24}} \sum_k \mathcal{N}_{ij}{}^k |k\rangle\rangle' ,$$

$$|i\rangle\rangle' \equiv (S_{i1} S_{11})^{-1/2} |i\rangle\rangle \,, \quad \mathcal{N}_{ij}{}^k = \sum_a \frac{(\psi_a^i)^* (\psi_a^j)^* \psi_a^k}{\psi_a^1} \,.$$

Can we check the above algebra in closed SFT on nontrivial background?

$T^D, T^D/Z_2$ compactification

Explicit formulation of closed SFT on $T^D, T^D/Z_2$ is known. [HIKKO(1987), Itoh-Kunitomo(1988)]

3-string vertex is modified:

$$(-1)^{p_2 w_2 - p_1 w_3} |V_0(1_u, 2_u, 3_u)\rangle,$$

$$(-1)^{p_1 n_3^f} \delta([n_3^f - n_2^f + w_1]) |V_0(1_u, 2_t, 3_t)\rangle$$

- cocycle factor $\leftarrow$ Jacobi identity,
- matter zero mode part.
- untwisted-twisted-twisted : different Neumann coefficients $\tilde{T}^{rs}_{n_r n_s}$,
- $\mathbf{Z}_2$ projection

We can compute $*$ product of Ishibashi states directly.

Ishibashi states:

$$|\iota(\mathcal{O},p,w)\rangle\rangle_u = e^{-\sum_{n=1}^{\infty}\frac{1}{n}\alpha^i_{-n}G_{ij}\mathcal{O}^j{}_k\tilde{\alpha}^k_{-n}}|p,w\rangle,$$

$$|\iota(\mathcal{O},n^f)\rangle\rangle_t = e^{-\sum_{r=1/2}^{\infty}\frac{1}{r}\alpha^i_{-r}G_{ij}\mathcal{O}^j{}_k\tilde{\alpha}^k_{-r}}|n^f\rangle,$$

$\mathcal{O}^T G\mathcal{O} = G$; p_i, w^j:integers such as $p_i = -F_{ij}w^j$,
$F = -(G+B-(G-B)\mathcal{O})(1+\mathcal{O})^{-1}$; $(n^f)^i = 0,1$: fixed point.

∗ products of these states are not diagonal.
→ We consider following linear combinations:

Dirichlet type ($\mathcal{O} = -1$)

$$|n^f\rangle_u := (\det(2G_{ij}))^{-\frac{1}{4}}\sum_{p_i}(-1)^{p\,n^f}|\iota(-1,p,0)\rangle\rangle_u,$$

$$|n^f\rangle_t := |\iota(-1,n^f)\rangle\rangle_t.$$

Neumann type ($\mathcal{O} \neq -1$)

$$|m^f,F\rangle_u := \left(\det(2G_O^{-1})\right)^{-\frac{1}{4}}\sum_{w}(-1)^{w\,m^f + wF_u w}|\iota(\mathcal{O},-Fw,w)\rangle\rangle_u,$$

$$|m^f,F\rangle_t := 2^{-\frac{D}{2}}\sum_{n^f\in\{0,1\}^D}(-1)^{m^f n^f + n^f F_u n^f}|\iota(\mathcal{O},n^f)\rangle\rangle_t,$$

where $(m^f)^i = 1,0$, $G_O^{-1} = (G+B+F)^{-1}G(G-B-F)^{-1}$.

∗ product (Dirichlet type)

$$|n_1^f, x^\perp, \alpha_1\rangle_u * |n_2^f, y^\perp, \alpha_2\rangle_u$$
$$= \left(\det(2G_{ij})\right)^{-\frac{1}{4}} (2\pi)^D \delta^D(0) \delta^D_{n_1,n_2} \delta^{d-p-1}(x^\perp - y^\perp)$$
$$\times \mu_u^2 \det{}^{-\frac{d+D-2}{2}}(1-(\tilde{N}^{33})^2) c_0^+ |n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_u,$$

$$|n_1^f, x^\perp, \alpha_1\rangle_u * |n_2^f, y^\perp, \alpha_2\rangle_t$$
$$= \left(\det(2G_{ij})\right)^{-\frac{1}{4}} (2\pi)^D \delta^D(0) \delta^D_{n_1,n_2} \delta^{d-p-1}(x^\perp - y^\perp)$$
$$\times \mu_t^2 \det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_t 3_t})^2) \det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2) c_0^+ |n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_t,$$

$$|n_1^f, x^\perp, \alpha_1\rangle_t * |n_2^f, y^\perp, \alpha_2\rangle_t$$
$$= \left(\det(2G_{ij})\right)^{\frac{1}{4}} \delta^D_{n_1,n_2} \delta^{d-p-1}(x^\perp - y^\perp)$$
$$\times \mu_t^2 \det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_u 3_u})^2) \det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2) c_0^+ |n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_u.$$

(Here we have included flat R^d sector and ghost sector.)

In the above formulae, twisted sector is not diagonalized.
We should take linear combinations of untwisted and twisted sector in order to get idempotents which include twisted sector.

- Neumann coefficients in the twisted sector

$$|V_0(1_u,2_t,3_t)\rangle = \mu_t^2\, e^{\frac{1}{2}a^{\dagger r}\tilde{T}^{rs}a^{\dagger s}+\frac{1}{2}\tilde{a}^{\dagger r}\tilde{T}^{rs}\tilde{a}^{\dagger s}}|p_1,w_1;n_2^f;n_3^f\rangle$$

$$\sum_{t,l_t}\tilde{T}^{rt}_{n_r l_t}\tilde{T}^{ts}_{l_t m_s}=\delta_{n_r,m_s}\delta_{r,s},\quad \sum_{t,l_t}\tilde{T}^{1t}_{0 l_t}\tilde{T}^{ts}_{l_t m_s}=-\tilde{T}^{1s}_{0m_s},\quad \sum_{t,l_t}\tilde{T}^{1t}_{0 l_t}\tilde{T}^{t1}_{l_t 0}=-2T^{11}_{00},$$

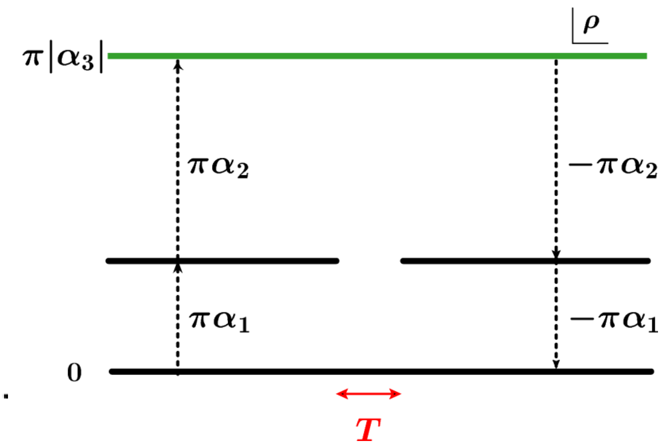
$$\tilde{T}^{rs}_{n_r m_s}=\frac{\alpha_1 n_r m_s}{\alpha_r m_s+\alpha_s n_r}\tilde{T}^{r1}_{n_r 0}\tilde{T}^{s1}_{m_s 0}$$

$$T^{11}_{00}-\sum_{r,s=2,3}\tilde{T}^{1r}_{0\cdot}[(1+\tilde{T})^{-1}]^{rs}\tilde{T}^{s1}_{\cdot 0}=-2\sum_{n=1}^{\infty}\frac{\cos^2\left(\frac{\alpha_1}{\alpha_3}n\pi\right)}{n}=-\infty$$

We have used the above relations to compute $*$ product.

Note:

$$\begin{aligned}\mathcal{C} &:= \mu_u^2\,\det{}^{-\frac{d+D-2}{2}}(1-(\tilde{N}^{33})^2)\\ &= \mu_t^2\,\det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_t3_t})^2)\det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2),\\ &\sim |\alpha_1\alpha_2\alpha_3|T^{-3}\end{aligned}$$



follows from *Cremmer-Gervais identity* for $D+d=26$.

$\mathcal{C}' := \mu_t^2\,\det^{-\frac{D}{2}}(1-(\tilde{T}^{3_u3_u})^2)\,\det^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2)$ cannot be evaluated similarly.

We can conclude that

$$|n^f, x^\perp, \alpha\rangle_\pm = \frac{1}{2}(2\pi\delta(0))^{-D}\left((\det(2G_{ij}))^{\frac{1}{4}}|n^f, x^\perp, \alpha\rangle_u \pm c_t(2\pi\delta(0))^{\frac{D}{2}}|n^f, x^\perp, \alpha\rangle_t\right)$$

are idempotents:

$$|n_1^f, x^\perp, \alpha_1\rangle_\pm * |n_2^f, y^\perp, \alpha_2\rangle_\pm = \delta^D_{n_1^f, n_2^f}\delta^{d-p-1}(x^\perp - y^\perp)\,\mathcal{C}\, c_0^+|n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_\pm,$$

$$|n_1^f, x^\perp, \alpha_1\rangle_\pm * |n_2^f, y^\perp, \alpha_2\rangle_\mp = 0.$$

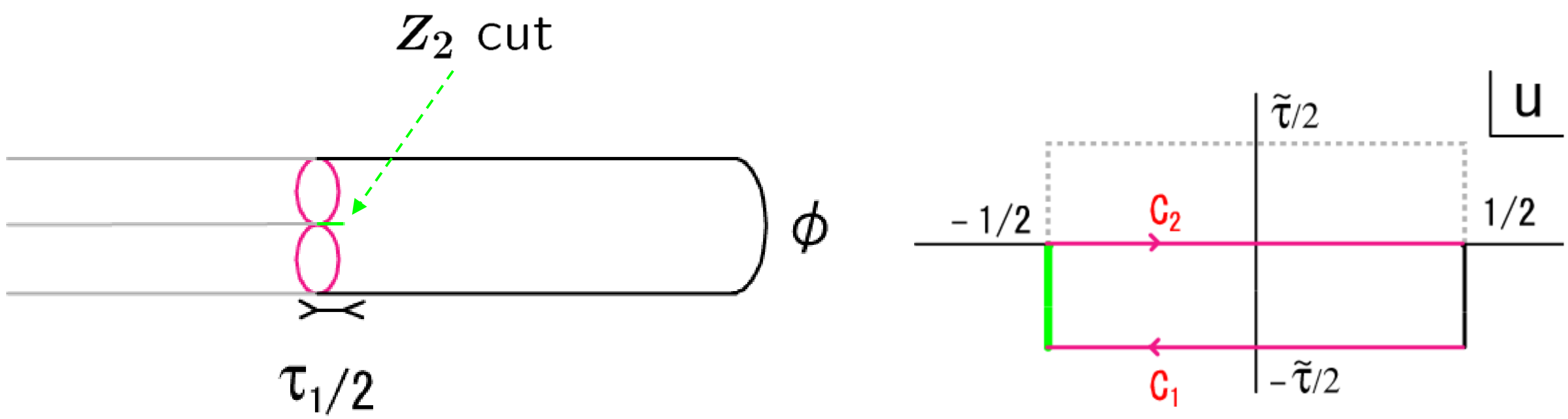
c_t is given by

$$c_t = \sqrt{\frac{\mathcal{C}}{\mathcal{C}'}} = \left(e^{-\frac{\tau_0}{4}(\alpha_1^{-1}+\alpha_2^{-1})}\frac{\det(1-(\tilde{T}^{1_u 1_u}(\alpha_3,\alpha_1,\alpha_2))^2)}{\det(1-(\tilde{N}^{33}(\alpha_1,\alpha_2,\alpha_3))^2)}\right)^{\frac{D}{4}},$$

which is evaluated by *1-loop amplitude* as

$$c_t(2\pi\delta(0))^{\frac{D}{2}} = 2^{\frac{D}{4}}(\det(2G))^{\frac{1}{4}} = \sqrt{\sigma(e,g)}\,(2\pi)^{-\frac{D}{2}}.$$

$\rightarrow$ $|n^f, x^\perp, \alpha\rangle_\pm$: Cardy state for fractional D-brane.



Ratio of 1-loop amplitude :

$$\langle B_t|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0)}|B_t\rangle / \langle B_u|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0)}|B_u\rangle$$

$$\sim \tilde{q}^{\frac{D}{48}} \prod_{n\geq 1}(1-\tilde{q}^{n-\frac{1}{2}})^{-D} \left((2\pi\delta(0))^{-D} \tilde{q}^{-\frac{D}{24}} \prod_{n\geq 1}(1-\tilde{q}^{n})^{-D} \sum_{p\in Z^D} \tilde{q}^{\frac{1}{4}pG^{-1}p} \right)^{-1}$$

$$= \left(\frac{\eta(\tilde{\tau})}{\vartheta_0(\tilde{\tau})}\right)^{\frac{D}{2}} \left((2\pi\delta(0))^{-D} \eta(\tilde{\tau})^{-D} \sum_{p\in Z^D} \tilde{q}^{\frac{1}{4}pG^{-1}p} \right)^{-1}$$

$$= \left(\frac{\eta(\tau)}{\vartheta_2(\tau)}\right)^{\frac{D}{2}} \left((2\pi\delta(0))^{-D} \det{}^{\frac{1}{2}}(2G)\, \eta(\tau)^{-D} \sum_{m\in Z^D} q^{mGm} \right)^{-1}$$

$$\to 2^{-\frac{D}{2}} (2\pi\delta(0))^{D} \det{}^{-\frac{1}{2}}(2G) = \frac{\mathcal{C}'}{\mathcal{C}}$$

$\tilde{\tau} \to +i0$:degenerating limit

Similarly, we obtain Neumann type idempotents:

$$|m^f, F, x^\perp, \alpha\rangle_\pm = \frac{1}{2}\frac{\det^{\frac{1}{4}}(2G_O^{-1})}{(2\pi\delta(0))^D}\left[|m^f, F, x^\perp, \alpha\rangle_u \pm 2^{\frac{D}{4}}|m^f, F, x^\perp, \alpha\rangle_t\right],$$

$$|m_1^f, F, x^\perp, \alpha_1\rangle_\pm * |m_2^f, F, y^\perp \alpha_2\rangle_\pm = \delta^D_{m_1^f, m_2^f}\,\delta(x^\perp - y^\perp)\mathcal{C}\, c_0^+|m_2^f, F, x^\perp, \alpha_1+\alpha_2\rangle_\pm,$$

$$|m_1^f, F, x^\perp, \alpha_1\rangle_\pm * |m_2^f, F, y^\perp, \alpha_2\rangle_\mp = 0.$$

（※）Neumann type idempotents are obtained from Dirichlet type by T-duality :

$$\mathcal{U}_g^\dagger|n^f, \alpha\rangle_{\pm,E} = |m^f = n^f, F, \alpha\rangle_{\pm, g(E)}.$$

In fact, we can prove

$$\mathcal{U}_g^\dagger|A * B\rangle_E = |(\mathcal{U}_g^\dagger A) * (\mathcal{U}_g^\dagger B)\rangle_{g(E)}, \quad g = \begin{pmatrix} -F & 1 \\ 1 & 0 \end{pmatrix} \in O(D, D; \mathbb{Z})$$

for both *uuu* and *utt* 3-string vertices. $(E = G + B)$

$\mathcal{U}_g$ is given by *Kugo-Zwiebach's transformation* for the untwisted sector and

$$\mathcal{U}_g^\dagger \alpha_r(E)\mathcal{U}_g = -E^{T-1}\alpha_r(g(E)), \quad \mathcal{U}_g^\dagger \tilde\alpha_r(E)\mathcal{U}_g = E^{-1}\tilde\alpha_r(g(E)),$$

$$\mathcal{U}_g^\dagger|n^f\rangle_E = 2^{-\frac{D}{2}}\sum_{m^f\in\{0,1\}^D}(-1)^{n^f m^f + m^f F_u m^f}|n^f\rangle_{g(E)},$$

for the twisted sector. $(F_u)_{ij} := F_{ij}\ (i<j),\ 0\ (\text{otherwise}).$

Comment on the Seiberg-Witten limit

KT operator which was introduced to represent noncommutativety in SFT on constant B-field background : [Kawano-Takahashi]

$$V_{\theta,\sigma_c} = \exp\left(-\frac{i}{4}\int_{\sigma_c}^{2\pi+\sigma_c} d\sigma \int_{\sigma_c}^{2\pi+\sigma_c} d\sigma' P_i(\sigma)\theta^{ij}\epsilon(\sigma,\sigma')P_j(\sigma')\right).$$

In fact, noting $V_\theta \partial_\sigma X^i(\sigma) V_\theta^{-1} = \partial_\sigma X^i(\sigma) - \theta^{ij} P_j(\sigma)$, KT operator induces a map from Dirichlet boundary state to Neumann one with constant flux at least naively.

More precisely, we find the identity:

$$V_{\theta,\sigma_c}|p\rangle\rangle_D =: e^{ipX(\sigma_c)} : |B(F_{ij} = -(\theta^{-1})_{ij})\rangle .$$

Dirichlet type Ishibashi state

Neumann type boundary state

In the Seiberg-Witten limit: $\alpha' \sim \epsilon^{1/2}$, $g_{ij} \sim \epsilon$, $\epsilon \to 0$, deformed Ishibashi states form a closed algebra:

$$V_{\theta,\sigma_c}|p_1\rangle\rangle_{D,\alpha_1} * V_{\theta,\sigma_c}|p_2\rangle\rangle_{D,\alpha_2} \sim a_\beta(p_1,p_2) V_{\theta,\sigma_c}|p_1+p_2\rangle\rangle_{D,\alpha_1+\alpha_2},$$

$$a_\beta(p_1,p_2) = \det{}^{-\frac{d}{2}}(1-(\tilde{N}^{33})^2)\frac{\sin(\beta p_1\theta p_2)}{\beta p_1\theta p_2}\frac{\sin((1+\beta)p_1\theta p_2)}{(1+\beta)p_1\theta p_2}, \qquad \beta = \frac{-\alpha_1}{\alpha_1+\alpha_2}.$$

In terms of coefficients function:

$$_{\alpha_1+\alpha_2}\langle x| \left[\int dy f_{\alpha_1}(y)\hat{V}_{\theta,\sigma_c}|B(y)\rangle_{\alpha_1} * \int dy' g_{\alpha_2}(y')\hat{V}_{\theta,\sigma_c}|B(y')\rangle_{\alpha_2} \right]$$

$$\sim [\det{}^{-\frac{d}{2}}(1-(\tilde{N}^{33})^2)\, 2\pi\delta(0)]\, f_{\alpha_1}(x)\frac{\sin(-\beta\lambda)\sin((1+\beta)\lambda)}{(-\beta)(1+\beta)\lambda^2} g_{\alpha_2}(x)$$ where $\lambda = \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x^i}\theta^{ij}\frac{\overrightarrow{\partial}}{\partial x^j}$

By taking the Laplace transformation with an ansatz:
$f_\alpha(x) = \alpha^{\delta-1} f(x)$ the idempotency equation is reduced to

$$f(x)\frac{\sin\lambda}{\lambda}f(x) = f(x)$$

i.e., projector eq. with respect to the ***Strachan product*** (or one of the generalized star product: $*_2$) which is commutative and non-associative.

↑

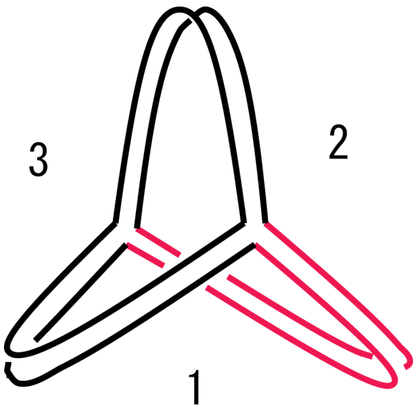
feature of the HIKKO ***closed SFT*** $*$ product

Summary and discussion

- Cardy states satisfy idempotency equation
  in closed SFT (on $R^D, T^D, T^D/Z_2$).
- Variation around idempotents gives open string spectrum.
- Idempotents ～ Cardy states
  more detailed and general correspondence?
  (Proof of necessary and sufficient conditions)
- Closed version of VSFT? (Veneziano amplitude,…)
- Supersymmetric extension? (HIKKO's NSR vertex, Green-Schwarz LCSFT, Witten type, Berkovits type…)

- 3-string vertex in Nonpolynomial closed SFT

[Saadi-Zwiebach,Kugo-Kunitomo-Suehiro,Kugo-Suehiro,Kaku,…]



← closed string version of Witten's $*$ product

We can also prove idempotency straightforwardly:

$$|\Phi_B(x^\perp)\rangle * |\Phi_B(y^\perp)\rangle = \delta(x^\perp - y^\perp) \mathcal{C}_W c_0^+ b_0^- |\Phi_B(x^\perp)\rangle$$

(Computation is simplified by closed sting version of MSFT. [Bars-Kishimoto-Matsuo])

- n-string vertices (n≧4) in nonpolynomial closedSFT?

$$(|i\rangle\!\rangle, |j\rangle\!\rangle, |k\rangle\!\rangle) := \langle\!\langle i|\langle\!\langle j|\langle\!\langle k|V_4\rangle = ?, \quad \cdots .$$

- Green-Schwarz-Brink's closed light-cone super SFT

We will be able to compute straightforwardly as

$$\langle B|\langle B|V_3\rangle = G|B\rangle.$$

The boundary state |B⟩ is given in terms of Green-Schwarz formulation, which is constructed in [Green-Gutperle(1996)]:

$$|B\rangle = e^{\sum_{n\geq 1}\left(\frac{1}{n}M_{IJ}\alpha^{I}_{-n}\tilde{\alpha}^{J}_{-n} - iM_{ab}S^{a}_{-n}\tilde{S}^{b}_{-n}\right)}|B_0\rangle,$$

where G comes from the prefactor of interaction vertex and the determinant of Neumann matrices.

- We have checked the above relation for "D-instanton" (up to zero mode dependence) by direct computation.

What is the explicit form and meaning of G?

- super SFT in terms of NSR formulation

There are explicit representations of <u>3-string vertex</u> for both HIKKO and Witten type in *open* super SFT

[HIKKO, Gross-Jevicki, Suehiro, Samuel…]

$$|V(1_{\rm NS},2_{\rm NS},3_{\rm NS})\rangle, \quad |V(1_{\rm NS},2_{\rm R},3_{\rm R})\rangle.$$

Boundary states are known :

$$|B\rangle = \mathcal{P}_{\rm GSO}|B,\eta\rangle_{\rm NSNS} + \mathcal{P}_{\rm GSO}|B,\eta\rangle_{\rm RR}.$$

•Define 3-string vertex for *closed* super SFT.

•Compute $*$ product of boundary states:

$$|B\rangle_{\rm NSNS} * |B\rangle_{\rm NSNS} \sim |B\rangle_{\rm NSNS}, \quad |B\rangle_{\rm NSNS} * |B\rangle_{\rm RR} \sim |B\rangle_{\rm RR},$$
$$|B\rangle_{\rm RR} * |B\rangle_{\rm RR} \sim |B\rangle_{\rm NSNS}.$$

•Can we find a *universal* relation for boundary states ?

•Divergence of the coefficients would vanish
 and the idempotency equation might become regular.