Boundary states and idempotency in closed string field theory

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References

I.K., Y.Matsuo, PLB590(2004)303, NPB707(2005)3 [KM1,2]

I.K., Y.Matsuo, E.Watanabe,PRD68(2003)126006, PTP111(2004) 433 [KMW1,2]

Introduction and motivation

- String field theory (SFT) is one possible approach to the construction of nonperturbative formulation of string theory.

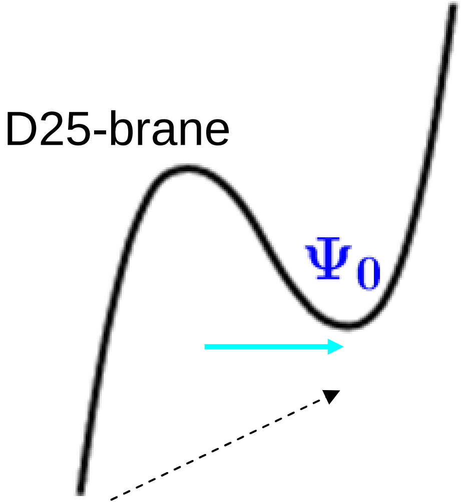
Well-known (?) string field theories at the critical dimension :

Light-cone gauge SFT (Kaku-Kikkawa)
Witten's open string SFT: bosonic, cubic
HIKKO open SFT: bosonic, quartic
Witten's open superstring SFT: NSR, cubic
Modified cubic open superstring SFT: NSR, cubic
Berkovits' open superstring field theory : NS sector, WZW-type
…
HIKKO closed SFT: bosonic, cubic ←
Nonpolynomial closed SFT: bosonic, nonpolynomial
Green-Schwarz SFT
Heterotic SFT (Berkovits-Okawa-Zwiebach) : NS sector, WZW-like
…

- Witten's open SFT

$$S=\frac{1}{2}\Psi\cdot Q_B\Psi+\frac{1}{3}\Psi\cdot\Psi\star\Psi$$

∃ tachyon vacuum Ψ_0



- Vacuum String Field Theory (VSFT)

[(Gaiotto)-Rastelli-Sen-Zwiebach(2000/2001)]

$$S=\frac{1}{2}\Psi\cdot Q\Psi+\frac{1}{3}\Psi\cdot\Psi\star\Psi \qquad Q=c(\pi/2)$$:pure ghost BRST operator

In VSFT, D-brane ~ classical solution of $Q|\Psi_0\rangle+|\Psi_0\rangle\star|\Psi_0\rangle=0$
~ Projector with respect to Witten's ★ product in the matter sector:
Sliver, Butterfly,… are constructed explicitly. $|\Xi\rangle\star|\Xi\rangle=|\Xi\rangle$

Essentially, they are the same as noncommutative solitons
because Witten's ★ can be expressed as the Moyal product. [Bars(2001),…]

On the other hand,

D-brane ∼ Boundary state ← closed string

Closed SFT description is more natural (!?)

↓

$$S = \frac{1}{2}\Phi \cdot Q\Phi + \frac{1}{3}\Phi \cdot \Phi * \Phi \ (+\cdots)$$

HIKKO cubic closed SFT (or Nonpolynomial closed SFT)

In this framework, we will characterize the boundary states by a universal nonlinear relation :

$$|B\rangle * |B\rangle \sim |B\rangle$$

Contents

- Introduction and motivation
- $*$ product of boundary state [KMW1]
- Idempotency equation [KMW1,KMW2,KM2]
- Cardy states and idempotents [KM1,KM2]
- T^D,T^D/Z_2 compactification [KM2]
- Comment on the Seiberg-Witten limit [KM2]
- Toward super version (NSR and GS)
- Summary and discussion

Star product in closed SFT

$*$ product is defined by 3-string vertex:

$$|\Phi_1 * \Phi_2\rangle_3 = {}_1\langle\Phi_1|{}_2\langle\Phi_2|V(1,2,3)\rangle$$

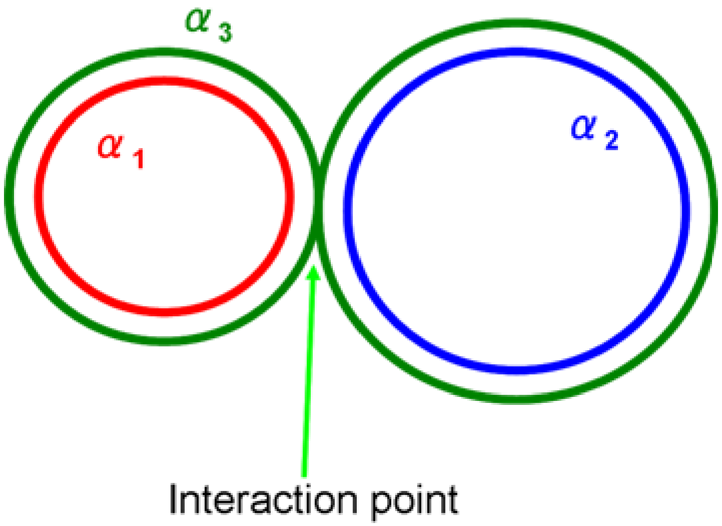
- HIKKO (Hata-Itoh-Kugo-Kunitomo-Ogawa) type

$$(X^{(3)} - \Theta_1 X^{(1)} - \Theta_2 X^{(2)})|V_0(1,2,3)\rangle = 0$$

and ghost sector (to be compatible with BRST invariance) with projection:

$$|V(1,2,3)\rangle = \wp_1\wp_2\wp_3|V_0(1,2,3)\rangle,$$

$$\wp_r := \oint \frac{d\theta}{2\pi} e^{i\theta(L_0^{(r)} - \tilde{L}_0^{(r)})}$$



- Explicit representation of the 3-string vertex:

  solution to overlapping condition [HIKKO]

$$|V(1,2,3)\rangle = \int \delta(1,2,3)[\mu(1,2,3)]^2 \wp_1\wp_2\wp_3 \frac{\alpha_1\alpha_2}{\alpha_3} \Pi_c\, \delta\left(\sum_{r=1}^3 \alpha_r^{-1}\pi_c^{0(r)}\right)$$
$$\times \prod_{r=1}^3 \left[1+2^{-\frac{1}{2}} w_I^{(r)} \bar{c}_0^{(r)}\right] e^{F(1,2,3)} |p_1,\alpha_1\rangle_1 |p_2,\alpha_2\rangle_2 |p_3,\alpha_3\rangle_3$$

$$\begin{aligned} F(1,2,3) \; = \; & \sum_{r,s=1}^{3} \sum_{m,n\geq 1} \tilde{N}^{rs}_{mn} \left[\frac{1}{2} a_m^{(r)\dagger} a_n^{(s)\dagger} + \sqrt{m}\alpha_r c_{-m}^{(r)} (\sqrt{n}\alpha_s)^{-1} b_{-n}^{(s)} \right. \\ & \left. + \frac{1}{2} \tilde{a}_m^{(r)\dagger} \tilde{a}_n^{(s)\dagger} + \sqrt{m}\alpha_r \tilde{c}_{-m}^{(r)} (\sqrt{n}\alpha_s)^{-1} \tilde{b}_{-n}^{(s)} \right] \\ & + \frac{1}{2} \sum_{r=1}^{3} \sum_{n\geq 1} \tilde{N}^r_n (a_n^{(r)\dagger} + \tilde{a}_n^{(r)\dagger}) \mathrm{P} - \frac{\tau_0}{4\alpha_1\alpha_2\alpha_3} \mathrm{P}^2 \end{aligned}$$

Various relations among Neumann coefficients: [Mandelstam,Green-Schwarz,…]
In particular, Yoneya formulae are essential to computation of B ∗ B.

$$\sum_{t=1}^{3} \sum_{p=1}^{\infty} \tilde{N}^{rt}_{mp} \tilde{N}^{ts}_{pn} = \delta_{r,s}\delta_{m,n}, \quad \sum_{t=1}^{3} \sum_{p=1}^{\infty} \tilde{N}^{rt}_{mp} \tilde{N}^{t}_{p} = -\tilde{N}^r_m, \quad \sum_{t=1}^{3} \sum_{p=1}^{\infty} \tilde{N}^{t}_{p} \tilde{N}^{t}_{p} = \frac{2\tau_0}{\alpha_1\alpha_2\alpha_3}$$

∗ product of boundary state

The boundary state for Dp-brane with constant flux:

$$
\begin{aligned}
|B(x^{\perp})\rangle &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\mathcal{O}\tilde{\alpha}_{-n}+\sum_{n=1}^{\infty}(c_{-n}\tilde{b}_{-n}+\tilde{c}_{-n}b_{-n})\right)\\
&\quad\times c_0^{+}c_1\tilde{c}_1|p^{\|}=0,x^{\perp}\rangle\otimes|0\rangle_{gh},\\
\mathcal{O}^{\mu}{}_{\nu} &= \left[(1+F)^{-1}(1-F)\right]^{\mu}{}_{\nu}\,, \qquad \mu,\nu=0,1,\cdots,p\,, \\
\mathcal{O}^{i}{}_{j} &= -\delta^{i}_{j}\,, \qquad\qquad i,j=p+1,\cdots,d-1\,.
\end{aligned}
$$

(Neumann)

(Dirichlet)

We *define* the string field: $|\Phi_B(x^{\perp},\alpha)\rangle = c_0^{-}b_0^{+}|B(x^{\perp})\rangle\otimes|\alpha\rangle$

Note1: $|B(x^{\perp})\rangle * |B(y^{\perp})\rangle = 0,$ which follows from $b_0^{-}|B(x^{\perp})\rangle = 0.$

Note 2: $|\Phi\rangle = c_0^{-}|\phi\rangle + c_0^{-}c_0^{+}|\psi\rangle + |\chi\rangle + c_0^{+}|\eta\rangle$

ϕ : “physical sector” i.e., $\frac{1}{2}\Phi\cdot Q_B\Phi = \frac{1}{2}\langle I[\phi](L_0+\tilde{L}_0-2)\phi\rangle+\cdots\,.$

$|\Phi_B(x^\perp,\alpha)\rangle$ and $|V(1,2,3)\rangle$ are "Gaussian." $\mathcal{O}$ is orthogonal.
Using *Yoneya formula* for Neumann matrices, we have obtained

$$|\Phi_B(x^\perp,\alpha_1)\rangle * |\Phi_B(y^\perp,\alpha_2)\rangle = \delta(x^\perp - y^\perp)\mathcal{C}c_0^+|\Phi_B(x^\perp,\alpha_1+\alpha_2)\rangle$$

"idempotency equation"

$$\mathcal{C} = [\mu(1,2,3)]^2[\det(1-(\tilde{N}^{33})^2)]^{-\frac{d-2}{2}} \ ; \quad \mu(1,2,3) = e^{-\tau_0\sum_{r=1}^{3}\alpha_r^{-1}}.$$

$\mathcal{C}$ is divergent because $\tilde{N}^{33}_{mn}$ is $\infty \times \infty$ matrix.

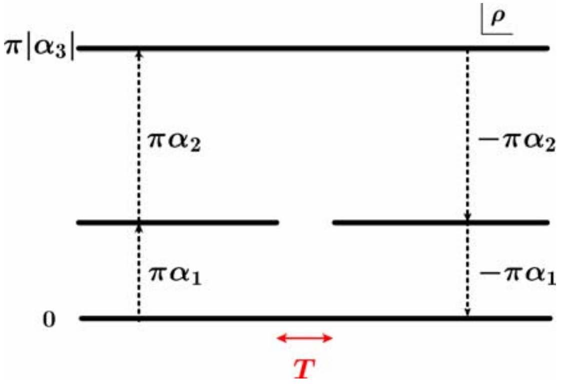
Regularization: $\tilde{N}^{33}_{mn} \to \tilde{N}^{33}_{mn}e^{-(m+n)\frac{T}{|\alpha_3|}}$

Using *Cremmer-Gervais identity*, we can evaluate as

$\mathcal{C} = 2^5\,T^{-3}|\alpha_1\alpha_2\alpha_3|$ $(T \to +0)$ at $d = 26$.

By level truncation, we numerically observed

$\mathcal{C} \sim L^3|(\alpha_1/\alpha_3)(\alpha_2/\alpha_3)|$, therefore, $T \sim |\alpha_3|/L$.



Idempotency equation (universal version):

$$|\Phi(\alpha_1)\rangle * |\Phi(\alpha_2)\rangle = K^3 \hat{\alpha}^2 c_0^+ |\Phi(\alpha_1+\alpha_2)\rangle$$

where $c_0^+ = \frac{1}{2}(c_0 + \tilde{c}_0)$, $K(\sim T^{-1} \to \infty)$: constant and $\alpha_1\alpha_2 > 0$

$\hat{\alpha}^2 c_0^+$ is a "pure ghost" BRST operator which is nilpotent, partial integrable and derivation with respect to $*$ product.

Boundary state for Dp-brane is a solution to the above equation:

$$|\Phi_f(\alpha)\rangle = \int d^{d-p-1}x^\perp f(x^\perp) |\Phi_B(x^\perp, \alpha)\rangle/\alpha$$

$f(x^\perp)$ is a solution to $f(x^\perp)^2 = f(x^\perp)$.

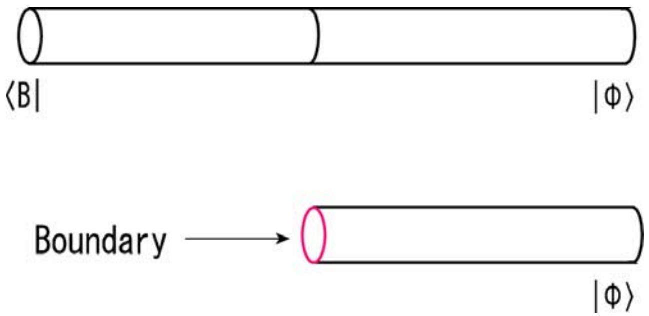
Namely, "commutative soliton" $f(x^\perp) = \begin{cases} 1 & (x^\perp \in \Sigma) \\ 0 & (\text{otherwise}) \end{cases}$

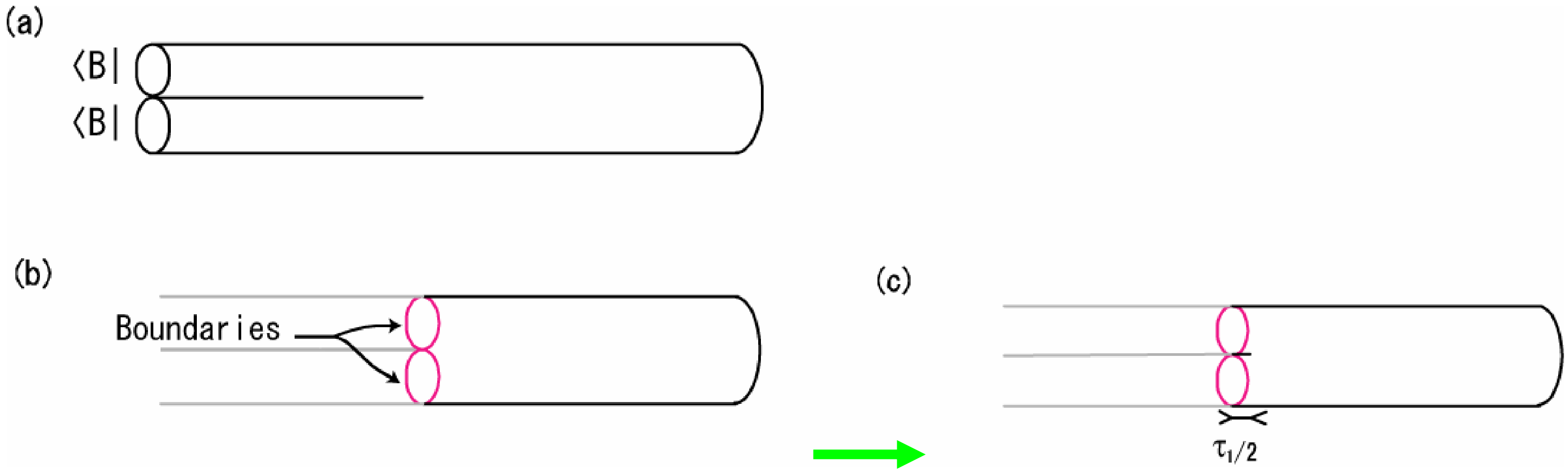
for some subset Σ of R^{d-p-1}.

Overall factor

- Let us reconsider B $*$ B.

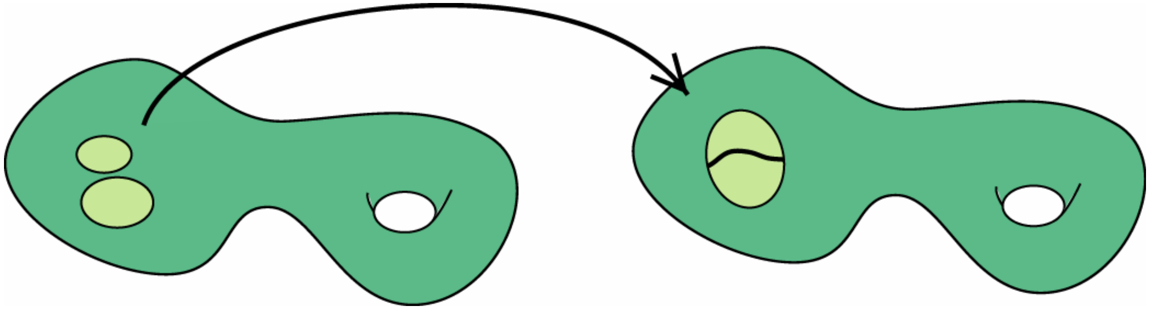
Note:





regularization

Conversely, it corresponds to a particular case of degeneration of a Riemann surface:



Generally, this process is described by factorization:

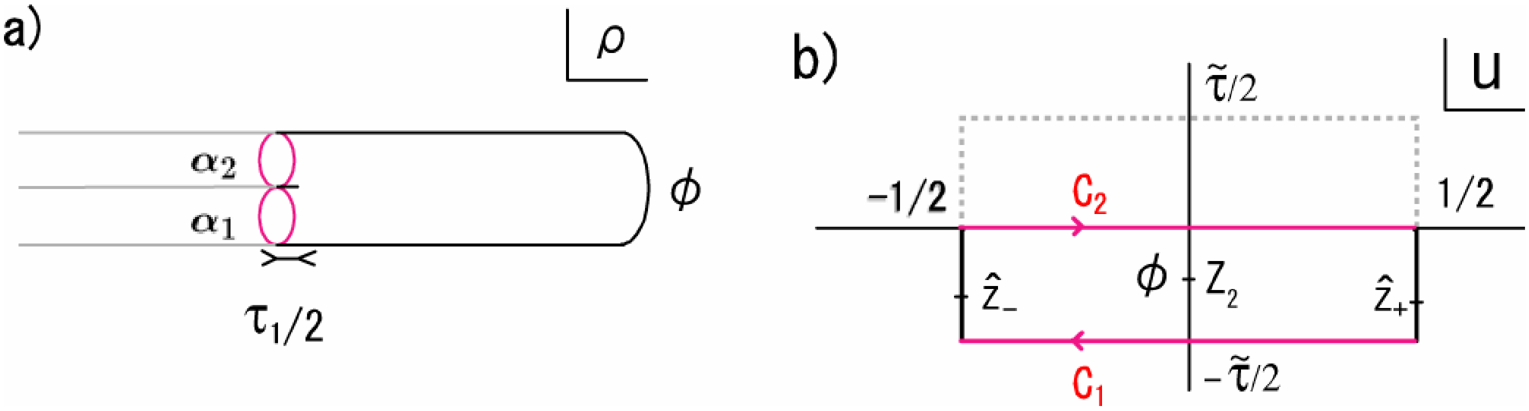
$$\langle \mathcal{O} \cdots \rangle \quad \longrightarrow \quad \sum_i \langle \mathcal{O} \cdots A_i(z_1) A_i(z_2) \rangle q^{\Delta_i}$$

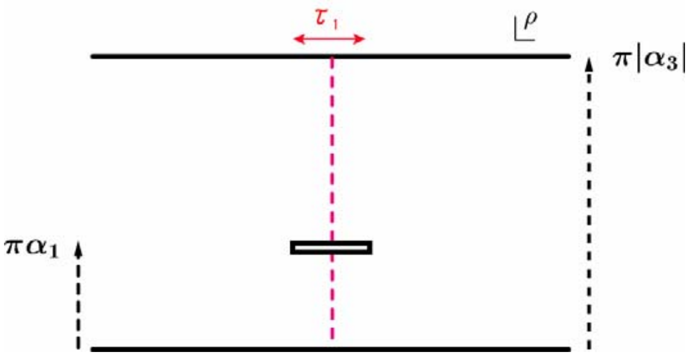
In the case of B ∗ B, it *roughly* implies

$$|B * B\rangle|_{\text{regularized}} \sim q^{-1} c(\sigma_1) c(\sigma_2) |B\rangle + (\text{less singular part}).$$

open string tachyon

More precisely, we should consider modulus in terms of regulator and ghost structure in computation of the $*$ product:





c.f. [Asakawa-Kugo-Takahashi(1999)]

Mandelstam mapping:

$$\rho(u) = (\alpha_1 + \alpha_2) \log \frac{\vartheta_1(u - \bar{Z}_2|\tilde{\tau})}{\vartheta_1(u - Z_2|\tilde{\tau})} - 2\pi i \alpha_1 u.$$

Modulus:

$$e^{-\frac{i\pi}{\tilde{\tau}}} = q^{1/2}$$
$$\sim \frac{\tau_1}{8(\alpha_1+\alpha_2)\sin(\pi\alpha_1/(\alpha_1+\alpha_2))} \ (\to 0)$$

- Evaluation of the coefficient

From idempotency equation, $\mathcal{C} = \left(\langle B_1|_{\frac{\tau_1}{2\alpha_1}} b_0^+ c_0^- * \langle B_2|_{\frac{\tau_1}{2\alpha_2}} b_0^+ c_0^- \right) |\phi\rangle / \langle B_2| b_0^+ c_0^- c_0^+ |\phi\rangle \, .$

In the following, we take $\phi = c\tilde{c}$ for simplicity.

From figure b), the numerator in matter sector:

$$\mathcal{F}^{\mathrm{m}} = \langle B_1^{\mathrm{m}} | \tilde{q}^{\frac{1}{2}\left(L_0 + \tilde{L}_0 - \frac{c}{12}\right)} | B_2^{\mathrm{m}} \rangle = q^{-\frac{c}{24}} \delta_{12} + (\text{higer order in } q),$$

and in ghost sector: $$\mathcal{F}^{\mathrm{gh}}_{c\tilde{c}} = 4\alpha_1\alpha_2(2\pi)^2 \int_{C_1} \frac{du_1}{2\pi i} \frac{du_1}{d\rho} \int_{C_2} \frac{du_2}{2\pi i} \frac{du_2}{d\rho} \left[\left.\frac{du}{dw_3}\right|_{w_3=0} \left.\frac{d\bar{u}}{d\bar{w}_3}\right|_{\bar{w}_3=0} \right]^{-1}$$
$$\times \langle B | \tilde{q}^{\frac{1}{2}\left(L_0 + \tilde{L}_0 + \frac{13}{6}\right)} b(2\pi i u_1) c(2\pi i Z_2) \tilde{c}(-2\pi i \bar{Z}_2) b(2\pi i u_2) | B \rangle .$$

Combining matter and ghost contribution, noting $\alpha \sim p^+$, the numerator is

$$\mathcal{F}^{\mathrm{m}} \mathcal{F}^{\mathrm{gh}}_{c\tilde{c}} (\log q)^{-1} \sim 32\, \delta_{12}\, \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \tau_1^{-3} q^{\frac{26-c}{24}} .$$

The denominator is given by $\langle B_2 | b_0^+ c_0^- c_0^+ c_1 \tilde{c}_1 | 0 \rangle = T_{B_2}.$

Namely, we conclude $$\mathcal{C} \sim 32 \delta_{12}\, \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \tau_1^{-3} T_{B_2}^{-1}$$

for c=26 and this is consistent with the correspondence of regularizations:

$$\tau_1 \sim T \sim |\alpha_3| / L \, .$$

Cardy states and idempotents

- On the flat (R^d) background, we have $*$ product formula for *Ishibashi states* :

$$|p_1^\perp\rangle\rangle_{\alpha_1} * |p_2^\perp\rangle\rangle_{\alpha_2} = \mathcal{C} c_0^+ |p_1^\perp + p_2^\perp\rangle\rangle_{\alpha_1+\alpha_2}.$$

$|p^\perp\rangle\rangle$ satisfies $(L_n - \tilde{L}_{-n})|p^\perp\rangle\rangle = 0$, but is *not* an idempotent. Its *Fourier transform* $|B(x^\perp)\rangle\rangle$ which is a Cardy state gives an idempotent.

Conjecture

Cardy states $\sim$ idempotents in closed SFT

even on nontrivial backgrounds.

Cardy states $|B\rangle$:

1. $(L_n - \tilde{L}_{-n})|B\rangle = 0.$
2. $\langle B|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|B'\rangle = \sum_i N^i_{BB'}\chi_i(q),$

   $N^i_{BB'}$:nonnegative integer.

↕

Closed SFT:

1. $(L_n - \tilde{L}_{-n})|B\rangle = 0, \quad (L_n - \tilde{L}_{-n})|B'\rangle = 0,$

   $\rightarrow \quad (L_n - \tilde{L}_{-n})|B\rangle * |B'\rangle = 0.$
2. idempotency: $|B\rangle * |B'\rangle = \delta_{B,B'}\,\mathcal{C}\,|B\rangle.$

- Orbifold（M/Γ）

twisted sector: $X(\sigma+2\pi)=gX(\sigma)\quad(g\in\Gamma)$

$$(g\text{-twisted}) * (g'\text{-twisted}) \sim (gg'\text{-twisted})$$

$\rightarrow$ $*$ product of Ishibashi states should be

$$|g\rangle\!\rangle_{\alpha_1} * |g'\rangle\!\rangle_{\alpha_2} \sim |gg'\rangle\!\rangle_{\alpha_1+\alpha_2}$$

$$\updownarrow$$

Group ring $\mathrm{C}^{[\Gamma]}$: $\sum_{g\in\Gamma}\lambda_g e_g \in \mathrm{C}^{[\Gamma]},\ \ \lambda_g \in \mathrm{C}$

$$e_g \star e_{g'} = e_{gg'}$$

Γ:nonabelian $\quad e_g \;\to\; e_i = \sum_{g\in\mathcal{C}_i} e_g \quad$ ($\mathcal{C}_i$: conjugacy class).

Formula: $\quad e_i \star e_j = \mathcal{N}_{ij}^{\ \ k} e_k$

$$\mathcal{N}_{ij}^{\ \ k} = \frac{1}{|\Gamma|}\sum\nolimits_{\alpha:\text{irreps.}} \frac{|\mathcal{C}_i||\mathcal{C}_j|\zeta_i^{(\alpha)}\zeta_j^{(\alpha)}\zeta_k^{(\alpha)*}}{\zeta_1^{(\alpha)}} . \quad (\zeta_i^{(\alpha)} \text{ : character})$$

idempotents: $P^{(\alpha)} = \dfrac{\zeta_1^{(\alpha)}}{|\Gamma|}\sum_{i:\text{class}} \zeta_i^{(\alpha)} e_i, \quad P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha,\beta} P^{(\beta)}$.

↕

Cardy states: $|\alpha\rangle = \dfrac{1}{\sqrt{|\Gamma|}}\sum_{i:\text{class}} \zeta_i^{(\alpha)} \sqrt{\sigma_i}|i\rangle\!\rangle, \quad |i\rangle\!\rangle := \sum_{g\in\mathcal{C}_i} |g\rangle\!\rangle$,

[cf. Billo et al.(2001)]

$$\sigma_i = \sigma(e,g), g \in \mathcal{C}_i, \;\; \chi_h^g(q) = \mathrm{Tr}_{\mathcal{H}_h}(g q^{L_0-\frac{c}{24}}) = \sigma(h,g)\chi_g^{h^{-1}}(\tilde{q})$$

→ $|\alpha\rangle$: idempotents in closed SFT (?)

Fusion ring of RCFT

$$e_i \star e_j = N_{ij}{}^{k} e_k, \quad N_{ij}{}^{k} = \sum_l \frac{S_{il} S_{jl} S^*_{kl}}{S_{1l}}$$

[Verlinde(1988)]

idempotents: $P^{(\alpha)} = S^*_{1\alpha} \sum_{i:\text{primary}} S_{i\alpha} e_i,$ $P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha,\beta} P^{(\beta)}.$

[T.Kawai (1989)]

↕

Cardy states: $|\alpha\rangle = \sum_{i:\text{primary}} \frac{S_{\alpha i}}{\sqrt{S_{1i}}} |i\rangle\rangle$

Suppose $|i\rangle\rangle_{\alpha_1} * |j\rangle\rangle_{\alpha_2} \sim N_{ij}{}^{k} |k\rangle\rangle_{\alpha_1+\alpha_2},$
then Cardy states $|\alpha\rangle \sim$ idempotents in closed SFT

$T^D, T^D/Z_2$ compactification

Explicit formulation of closed SFT on $T^D, T^D/Z_2$ is known. [HIKKO(1987), Itoh-Kunitomo(1988)]

3-string vertex is modified:

$$(-1)^{p_2 w_2 - p_1 w_3} |V_0(1_u, 2_u, 3_u)\rangle,$$

$$(-1)^{p_1 n_3^f} \delta([n_3^f - n_2^f + w_1]) |V_0(1_u, 2_t, 3_t)\rangle$$

- cocycle factor ← Jacobi identity,
- matter zero mode part.
- untwisted-twisted-twisted : different Neumann coefficients $\tilde{T}^{rs}_{n_r n_s}$,
- Z_2 projection

We can compute $*$ product of Ishibashi states directly.

Ishibashi states:

$$|\iota(\mathcal{O},p,w)\rangle\!\rangle_u = e^{-\sum_{n=1}^{\infty}\frac{1}{n}\alpha^i{}_{-n}G_{ij}\mathcal{O}^j{}_k\tilde{\alpha}^k{}_{-n}}|p,w\rangle,$$

$$|\iota(\mathcal{O},n^f)\rangle\!\rangle_t = e^{-\sum_{r=1/2}^{\infty}\frac{1}{r}\alpha^i{}_{-r}G_{ij}\mathcal{O}^j{}_k\tilde{\alpha}^k{}_{-r}}|n^f\rangle,$$

$\mathcal{O}^T G\mathcal{O} = G$; p_i, w^j:integers such as $p_i = -F_{ij}w^j$,

$F = -(G+B-(G-B)\mathcal{O})(1+\mathcal{O})^{-1}$; $(n^f)^i = 0,1$: fixed point.

$\ast$ products of these states are not diagonal.

→ We consider following linear combinations:

Dirichlet type ($\mathcal{O} = -1$)

$$|n^f\rangle_u := (\det(2G_{ij}))^{-\frac{1}{4}}\sum_{p_i}(-1)^{p\,n^f}|\iota(-1,p,0)\rangle\!\rangle_u,$$

$$|n^f\rangle_t := |\iota(-1,n^f)\rangle\!\rangle_t.$$

Neumann type ($\mathcal{O} \neq -1$)

$$|m^f,F\rangle_u := \left(\det(2G_O^{-1})\right)^{-\frac{1}{4}}\sum_{w}(-1)^{w\,m^f+wF_uw}|\iota(\mathcal{O},-Fw,w)\rangle\!\rangle_u,$$

$$|m^f,F\rangle_t := 2^{-\frac{D}{2}}\sum_{n^f\in\{0,1\}^D}(-1)^{m^fn^f+n^fF_un^f}|\iota(\mathcal{O},n^f)\rangle\!\rangle_t,$$

where $(m^f)^i = 1,0$, $G_O^{-1} = (G+B+F)^{-1}G(G-B-F)^{-1}$.

- Neumann coefficients in the twisted sector

$$|V_0(1_u,2_t,3_t)\rangle = \mu_t^2\, e^{\frac{1}{2}a^{\dagger r}\tilde{T}^{rs}a^{\dagger s}+\frac{1}{2}\tilde{a}^{\dagger r}\tilde{T}^{rs}\tilde{a}^{\dagger s}}|p_1,w_1;n_2^f;n_3^f\rangle$$

$$\sum_{t,l_t}\tilde{T}^{rt}_{n_r l_t}\tilde{T}^{ts}_{l_t m_s} = \delta_{n_r,m_s}\delta_{r,s}, \quad \sum_{t,l_t}\tilde{T}^{1t}_{0l_t}\tilde{T}^{ts}_{l_t m_s} = -\tilde{T}^{1s}_{0m_s}, \quad \sum_{t,l_t}\tilde{T}^{1t}_{0l_t}\tilde{T}^{t1}_{l_t 0} = -2T^{11}_{00},$$

$$\tilde{T}^{rs}_{n_r m_s} = \frac{\alpha_1 n_r m_s}{\alpha_r m_s + \alpha_s n_r}\tilde{T}^{r1}_{n_r 0}\tilde{T}^{s1}_{m_s 0}$$

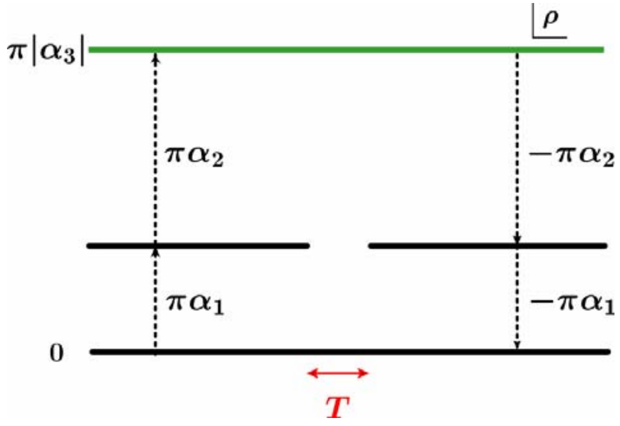
$$T^{11}_{00} - \sum_{r,s=2,3}\tilde{T}^{1r}_{0\cdot}[(1+\tilde{T})^{-1}]^{rs}\tilde{T}^{s1}_{\cdot 0} = -2\sum_{n=1}^{\infty}\frac{\cos^2\left(\frac{\alpha_1}{\alpha_3}n\pi\right)}{n} = -\infty$$

We have used the above relations to compute $*$ product.

Note:

$$\begin{aligned}\mathcal{C} \;&:=\; \mu_u^2\,\det{}^{-\frac{d+D-2}{2}}(1-(\tilde{N}^{33})^2)\\ &=\; \mu_t^2\,\det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_t 3_t})^2)\det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2),\\ &\sim\; |\alpha_1\alpha_2\alpha_3|T^{-3}\end{aligned}$$

follows from *Cremmer-Gervais identity* for $D+d=26$.



$\mathcal{C}' := \mu_t^2\,\det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_u 3_u})^2)\,\det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2)$ cannot be evaluated similarly.

We conclude that

$$|n^f, x^\perp, \alpha\rangle_\pm = \frac{1}{2}(2\pi\delta(0))^{-D}\left((\det(2G_{ij}))^{\frac{1}{4}}|n^f, x^\perp, \alpha\rangle_u \pm c_t(2\pi\delta(0))^{\frac{D}{2}}|n^f, x^\perp, \alpha\rangle_t\right)$$

are idempotents:

$$|n_1^f, x^\perp, \alpha_1\rangle_\pm * |n_2^f, y^\perp, \alpha_2\rangle_\pm = \delta^D_{n_1^f, n_2^f}\delta^{d-p-1}(x^\perp - y^\perp)\,\mathcal{C}\,c_0^+|n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_\pm,$$

$$|n_1^f, x^\perp, \alpha_1\rangle_\pm * |n_2^f, y^\perp, \alpha_2\rangle_\mp = 0.$$

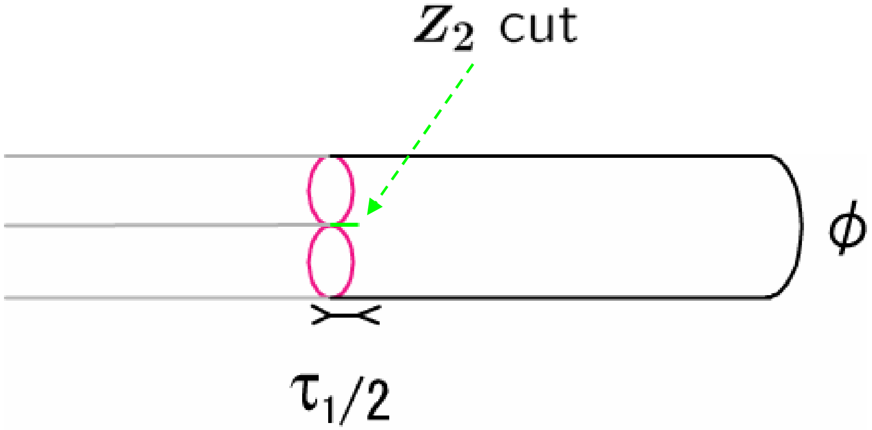
c_t is given by

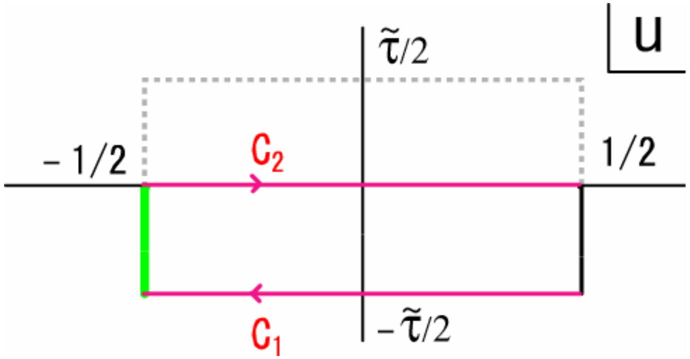
$$c_t = \sqrt{\frac{\mathcal{C}}{\mathcal{C}'}} = \left(e^{-\frac{\tau_0}{4}(\alpha_1^{-1}+\alpha_2^{-1})}\frac{\det(1-(\tilde{T}^{1_u1_u}(\alpha_3,\alpha_1,\alpha_2))^2)}{\det(1-(\tilde{N}^{33}(\alpha_1,\alpha_2,\alpha_3))^2)}\right)^{\frac{D}{4}},$$

which is evaluated by *1-loop amplitude* as

$$c_t(2\pi\delta(0))^{\frac{D}{2}} = 2^{\frac{D}{4}}(\det(2G))^{\frac{1}{4}} = \sqrt{\sigma(e,g)}\,(2\pi)^{-\frac{D}{2}}.$$

→ $|n^f, x^\perp, \alpha\rangle_\pm$: Cardy state for fractional D-brane.





Ratio of 1-loop amplitude :

$$\langle B_t|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0)}|B_t\rangle/\langle B_u|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0)}|B_u\rangle$$

$$\sim \tilde{q}^{\frac{D}{48}}\prod_{n\geq 1}(1-\tilde{q}^{n-\frac{1}{2}})^{-D}\left((2\pi\delta(0))^{-D}\tilde{q}^{-\frac{D}{24}}\prod_{n\geq 1}(1-\tilde{q}^{n})^{-D}\sum_{p\in Z^D}\tilde{q}^{\frac{1}{4}pG^{-1}p}\right)^{-1}$$

$$= \left(\frac{\eta(\tilde{\tau})}{\vartheta_0(\tilde{\tau})}\right)^{\frac{D}{2}}\left((2\pi\delta(0))^{-D}\eta(\tilde{\tau})^{-D}\sum_{p\in Z^D}\tilde{q}^{\frac{1}{4}pG^{-1}p}\right)^{-1}$$

$$= \left(\frac{\eta(\tau)}{\vartheta_2(\tau)}\right)^{\frac{D}{2}}\left((2\pi\delta(0))^{-D}\det{}^{\frac{1}{2}}(2G)\eta(\tau)^{-D}\sum_{m\in Z^D}q^{mGm}\right)^{-1}$$

$$\rightarrow 2^{-\frac{D}{2}}(2\pi\delta(0))^{D}\det{}^{-\frac{1}{2}}(2G)=\frac{\mathcal{C}'}{\mathcal{C}}$$

$\tilde{\tau}\rightarrow +i0$:degenerating limit

Similarly, we obtain Neumann type idempotents:

$$|m^f, F, x^\perp, \alpha\rangle_\pm = \frac{1}{2}\frac{\det^{\frac{1}{4}}(2G_O^{-1})}{(2\pi\delta(0))^D}\left[|m^f, F, x^\perp, \alpha\rangle_u \pm 2^{\frac{D}{4}}|m^f, F, x^\perp, \alpha\rangle_t\right],$$

$$|m_1^f, F, x^\perp, \alpha_1\rangle_\pm * |m_2^f, F, y^\perp\alpha_2\rangle_\pm = \delta^D_{m_1^f, m_2^f}\,\delta(x^\perp - y^\perp)\mathcal{C}\, c_0^+|m_2^f, F, x^\perp, \alpha_1 + \alpha_2\rangle_\pm,$$

$$|m_1^f, F, x^\perp, \alpha_1\rangle_\pm * |m_2^f, F, y^\perp, \alpha_2\rangle_\mp = 0.$$

※ Neumann type idempotents are obtained from Dirichlet type by T-dual :

$$\mathcal{U}_g^\dagger|n^f, \alpha\rangle_{\pm,E} \;=\; |m^f = n^f, F, \alpha\rangle_{\pm,g(E)}.$$

In fact, we can prove

$$\mathcal{U}_g^\dagger|A * B\rangle_E = |(\mathcal{U}_g^\dagger A) * (\mathcal{U}_g^\dagger B)\rangle_{g(E)}, \quad g = \begin{pmatrix} -F & 1 \\ 1 & 0 \end{pmatrix} \in O(D, D; \mathbb{Z})$$

for both *uuu* and *utt* 3-string vertices. $(E = G + B)$

$\mathcal{U}_g$ is given by *Kugo-Zwiebach's transformation* for the untwisted sector and

$$\mathcal{U}_g^\dagger\alpha_r(E)\mathcal{U}_g = -E^{T-1}\alpha_r(g(E)), \quad \mathcal{U}_g^\dagger\tilde{\alpha}_r(E)\mathcal{U}_g = E^{-1}\tilde{\alpha}_r(g(E)),$$

$$\mathcal{U}_g^\dagger|n^f\rangle_E = 2^{-\frac{D}{2}}\sum_{m^f \in \{0,1\}^D}(-1)^{n^f m^f + m^f F_u m^f}|n^f\rangle_{g(E)},$$

for the twisted sector. $(F_u)_{ij} := F_{ij}\ (i < j),\ 0\ (\text{otherwise}).$

Comment on the Seiberg-Witten limit

KT operator which was introduced to represent noncommutativety in SFT on constant B-field background : [Kawano-Takahashi(1999/2000)]

$$V_{\theta,\sigma_c} = \exp\left(-\frac{i}{4}\int_{\sigma_c}^{2\pi+\sigma_c} d\sigma \int_{\sigma_c}^{2\pi+\sigma_c} d\sigma' P_i(\sigma)\theta^{ij}\epsilon(\sigma,\sigma')P_j(\sigma')\right).$$

In fact, noting $V_\theta \partial_\sigma X^i(\sigma) V_\theta^{-1} = \partial_\sigma X^i(\sigma) - \theta^{ij}P_j(\sigma)$, KT operator induces a map from Dirichlet boundary state to Neumann one with constant flux at least naively. [cf. Okuyama(2000)]

More precisely, we find the following identity by explicit computation:

$$V_{\theta,\sigma_c}\underline{|p\rangle\rangle}_D =: e^{ipX(\sigma_c)} : \underline{|B(F_{ij} = -(\theta^{-1})_{ij})\rangle}.$$

Dirichlet type Ishibashi state

Neumann type boundary state

We can directly compute the star product:

$$
\begin{aligned}
&V_{\theta,\sigma_c}|p_1\rangle\!\rangle_{D,\alpha_1} * V_{\theta,\sigma_c}|p_2\rangle\!\rangle_{D,\alpha_2} \\
&= (-\beta)^{\alpha' p_1 G_O^{-1} p_1}(1+\beta)^{\alpha' p_2 G_O^{-1} p_2} \det{}^{-\frac{d}{2}}(1-(\tilde{N}^{33})^2) \\
&\quad \times \oint \frac{d\sigma_1}{2\pi} \oint \frac{d\sigma_2}{2\pi} \wp \, \underbrace{:e^{ip_1 X(\sigma^{(1)}(\sigma_1))}::e^{ip_2 X(\sigma^{(2)}(\sigma_2))}:|B(F=-\theta^{-1})\rangle_{\alpha_1+\alpha_2}}.
\end{aligned}
$$

operator product of tachyon vertices on the Neumann type boundary state

[cf. Murakami-Nakatsu(2002)]

In the Seiberg-Witten limit: $\alpha' \sim \epsilon^{1/2}$, $g_{ij} \sim \epsilon$, $\epsilon \to 0$, deformed Ishibashi states form a closed algebra:

$$V_{\theta,\sigma_c}|p_1\rangle\!\rangle_{D,\alpha_1} * V_{\theta,\sigma_c}|p_2\rangle\!\rangle_{D,\alpha_2} \sim a_\beta(p_1,p_2) V_{\theta,\sigma_c}|p_1+p_2\rangle\!\rangle_{D,\alpha_1+\alpha_2},$$

$$a_\beta(p_1,p_2) = \det{}^{-\frac{d}{2}}(1-(\tilde{N}^{33})^2) \frac{\sin(\beta p_1 \theta p_2)}{\beta p_1 \theta p_2} \frac{\sin((1+\beta) p_1 \theta p_2)}{(1+\beta) p_1 \theta p_2}, \quad \beta = \frac{-\alpha_1}{\alpha_1+\alpha_2}.$$

In terms of coefficients function:

$${}_{\alpha_1+\alpha_2}\langle x| \left[\int dy\, f_{\alpha_1}(y) V_{\theta,\sigma_c} |B(y)\rangle_{\alpha_1} * \int dy'\, g_{\alpha_2}(y') V_{\theta,\sigma_c} |B(y')\rangle_{\alpha_2} \right]$$

$$\sim\ [\det{}^{-\frac{d}{2}}(1-(\tilde{N}^{33})^2)\, 2\pi\delta(0)]\ f_{\alpha_1}(x) \frac{\sin(-\beta\lambda)\sin((1+\beta)\lambda)}{(-\beta)(1+\beta)\lambda^2} g_{\alpha_2}(x) \quad \text{where} \quad \lambda = \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x^i}\theta^{ij}\frac{\overrightarrow{\partial}}{\partial x^j}$$

By taking the Laplace transformation with an ansatz:
$f_\alpha(x) = \alpha^{\delta-1} f(x)$ the idempotency equation is reduced to

$$f(x)\frac{\sin\lambda}{\lambda}f(x) = f(x)$$

i.e., projector eq. with respect to the ***Strachan product*** , or one of the generalized star product: $*_2$, which is commutative and non-associative.

↑

feature of the HIKKO ***closed SFT*** $*$ product

Roughly, in the Seiberg-Witten limit,

$|S\rangle$:lump sol. of VSFT ⟷ $V_{\theta,\sigma_c}|B(x)\rangle$:deformed D(-1)-brane B.S.

Moyal product
(noncommutative associative) ⟷ Strachan product
(commutative nonassociative)

Witten's <u>open</u> SFT ⟷ HIKKO's <u>closed</u> SFT

Supersymmetric case (NSR)

- Constructing 3-stirng vertex as the LPP formulation

Here, we bosonize $\psi^{\mu}\ \beta\ \gamma$ due to a technical reason (unbosonize version [HIKKO(1987)]):

$$\psi^{\pm a} = e^{\pm\phi^a} c_{\pm e^a},\ \tilde{\psi}^{\pm a} = e^{\pm\tilde{\phi}^a} \tilde{c}_{\pm e^a},\quad (a = 1, \cdots, 5)$$:matter fermion

$$\beta = e^{-\phi}\partial e^{\chi},\ \gamma = e^{\phi-\chi},\quad \tilde{\beta} = e^{-\tilde{\phi}}\bar{\partial} e^{\tilde{\chi}},\ \tilde{\gamma} = e^{\tilde{\phi}-\tilde{\chi}}$$:superghost

For each sectors, LPP vertex is given by

$$\phi(y)\phi(z) \sim \varepsilon \log(y-z), \qquad T(z) = \frac{1}{2}\varepsilon(\partial\phi)^2 - \frac{1}{2}Q\partial^2\phi,$$

$$\langle V_3^{\mathrm{LPP}}|A_1\rangle|A_2\rangle|A_3\rangle = \langle h_1[\mathcal{O}_{A_1}] h_2[\mathcal{O}_{A_2}] h_3[\mathcal{O}_{A_3}]\rangle$$

with a particular conformal map. [LeClair-Peskin-Preitschopf (1989)]

- Oscillator representation

$$\langle V_3^{\mathrm{LPP}}| = \sum_{q_1,q_2,q_3} \delta_{q_1+q_2+q_3+Q,0}\langle -q_i - Q| e^{\frac{1}{2}\varepsilon \sum_{m,n\geq 0}\sum_{r,s=1}^{3} j_m^r \mathcal{N}_{mn}^{rs} j_n^s}$$

Neumann coefficients: $\mathcal{N}_{mn}^{rs} = \bar{N}_{mn}^{rs} = \frac{1}{\sqrt{mn}}\tilde{N}_{mn}^{rs},$

$$\mathcal{N}_{0m}^{rs} = \bar{N}_{0m}^{\prime rs} - \frac{1}{2}K_m^{\prime s}, \cdots$$

$\bar{N}_{mn}^{rs}, \bar{N}_{0m}^{\prime rs}$ are the same as those of bosonic SFT one.

$K_m^{\prime s}$ come from the background charge Q : which are computed as

$$K_m^{\prime 1} = \frac{\alpha_3}{\alpha_1}\frac{e^{m\frac{\tau_0}{\alpha_1}}}{m}\sum_{k=0}^{m-1}\frac{\Gamma(-m\alpha_2/\alpha_1+1)}{k!\,\Gamma(-m\alpha_2/\alpha_1-k+1)}\left(\frac{\alpha_3}{\alpha_1}\right)^{m-1-k}, \cdots$$

and are contribution of the pole at **the interaction point**.

$$\langle V_3^{\mathrm{LPP}}| \sim \langle v_3| e^{-\frac{Q}{2}\phi(z_{\mathrm{int}})}$$

↑

determined by naïve connection condition

- Boundary states for Dp brane [Callan et.al.(1987),…,Yost(1989)]

$$(\alpha_n^\mu + S^\mu{}_\nu \tilde{\alpha}^\nu{}_{-n})|B;\eta\rangle = 0,\ \ (\psi_r^\mu - i\eta S^\mu{}_\nu \tilde{\psi}^\nu{}_{-r})|B;\eta\rangle = 0,$$

$$S^\mu{}_\nu = \delta^\mu_\nu \text{ (Neumann)}; -\delta^\mu_\nu \text{ (Dirichlet)},$$

$$(c_n + \tilde{c}_{-n})|B;\eta\rangle = 0,\ \ (b_n - \tilde{b}_{-n})|B;\eta\rangle = 0,$$

$$(\gamma_t + i\eta\tilde{\gamma}_{-t})|B;\eta\rangle = 0,\ \ (\beta_t + i\eta\tilde{\beta}_{-t})|B;\eta\rangle = 0, \qquad (\eta = \pm 1)$$

(bosonized) solution: $|B;\eta\rangle_P = |B_{\text{bosonic}}\rangle \otimes |B;\eta\rangle^\psi \otimes |B;\eta\rangle_P^{\beta\gamma}$

$$|B;\eta\rangle^\psi = \sum_{s^1,\cdots,s^5} \prod_{b=1}^{5} (\eta\eta^b)^{s^b+c} e^{-\sum_{n\geq 1}\frac{1}{n} j^a_{-n}\tilde{j}^a_{-n}} |s^a, -s^a\rangle, \qquad (c = 0\,(\mathrm{NS}^2);\ -\frac{1}{2}\,(\mathrm{R}^2))$$

$$\eta^b = 1 \text{ (Neumann)}; -1 \text{ (Dirichlet)},$$

$$|B;\eta\rangle_P^{\beta\gamma} = \sum_s (i\eta)^{s-P} e^{\sum_{n\geq 1}\frac{1}{n} j_{-n}\tilde{j}_{-n}} |s, -s-2\rangle_\phi \otimes |B_{P-s}\rangle_\chi, \quad ((P, -P-2)\text{-picture})$$

Where χ sector is

$$\begin{aligned} |B_m\rangle_\chi &:= \eta_0 e^{-\sum_{n\geq 1}\frac{1}{n}\chi_{-n}\tilde{\chi}_{-n}} |m+1, -m\rangle_\chi = \tilde{\eta}_0 e^{-\sum_{n\geq 1}\frac{1}{n}\chi_{-n}\tilde{\chi}_{-n}} |m, -m+1\rangle_\chi \\ &= \oint \frac{d\theta}{2\pi} e^{-im\theta} e^{-\sum_{n\geq 1}\frac{1}{n}(\chi_{-n}\tilde{\chi}_{-n} + \chi_{-n}e^{in\theta} + \tilde{\chi}_{-n}e^{-in\theta})} |m, -m\rangle_\chi \end{aligned}$$

• $\ast$-product of the boundary states

As in the case of bosonic SFT, we define $|\Phi_B\rangle = c_0^- b_0^+ |B\rangle$

and the $\ast$-product as

$$\langle \Phi_B \ast \Phi_B| \sim \langle V_3^{\mathrm{LPP}}|(X(z_{\mathrm{int}})\tilde{X}(\bar{z}_{\mathrm{int}}))b_0^- |\Phi_B\rangle_1 \, b_0^- |\Phi_B\rangle_2$$

We insert the picture changing operator:

$$X(z) = e^{\phi}(i\psi^{\mu}\partial X_{\mu}) + c\partial e^{\chi} - e^{2\phi}(\partial e^{-\chi})b - \partial(e^{2\phi-\chi}b)$$

from the analogy of ***open*** superstring field theory (Witten version, i.e., NSNS sector is in the (-1,-1) picture)

- We use various relations among Neumann coefficients for computation:

$\bar{N}^{rs}_{mn}$ for Green-Schwarz/Yoneya formula, and

$$K'^{r}_{m} = \sum_{k=1}^{\infty} (\cos k\sigma^{(3)}_{\mathrm{int}})(k^{-1}\delta^{3,r}\delta_{k,m} - \bar{N}^{3r}_{km})$$

- matter sector

$$\langle V_3^{\rm LPP}|B(x_\perp);\eta_1\rangle_1|B(y_\perp);\eta_2\rangle_2 \\ = \delta^{9-p}(x_\perp - y_\perp)(2\pi\delta(0))^5\,\delta_{\eta_1,\eta_2}\,\det{}^{-\frac{15}{2}}(1-(\tilde{N}^{33})^2)C_{12}\langle B(y_\perp);\eta_2|$$

C_{12} is +1 for NSNS∗NSNS～NSNS, and
$\eta_2(-1)^{(9-p)/2}$ for NSNS∗RR～RR, RR∗NSNS～RR, RR∗RR～NSNS

⇨ Although the determinant factors do not cancel because of bosonized version, the boundary states are idempotent as in the case of bosonic closed SFT.
※There is another factor such as $\psi^\mu\partial X_\mu\tilde{\psi}^\nu\bar{\partial}X_\nu$ if the picture changing operator is inserted.

- ghost sector

bc sector is the same as bosonic closed SFT:

$$\langle V_3^{\rm LPP}|b_0^+|B\rangle_1\, b_0^+|B\rangle_2 = \mu^2\det(1-(\tilde{N}^{33})^2)\langle B|c_0^-$$

※There is another factor if the picture changing operator is inserted.

- superghost sector

$$\begin{aligned}&\langle V_3^{\mathrm{LPP}}|e^{\phi(z_{\mathrm{int}})}e^{\tilde{\phi}(\bar{z}_{\mathrm{int}})}|B;\eta_1\rangle_{P_1}|B;\eta_2\rangle_{P_2}\\&=2\pi\delta(0)\delta_{\eta_1,-\eta_2}\mu^{-\frac{3}{4}}\det{}^{-1}(1-(\tilde{N}^{33})^2)C_{\phi\chi}\\&\quad\times{}_{(-P_1-P_2-3)}\langle B^{\sigma^{(3)}_{\mathrm{int}}};\eta_2|\int_{C_1}\frac{dz}{2\pi i}e^{-\chi(z)}\int_{C_2}\frac{d\bar{z}}{2\pi i}e^{-\tilde{\chi}(\bar{z})}\end{aligned}$$

where $\quad {}_P\langle B^{\sigma^{(3)}_{\mathrm{int}}};\eta|\oint\frac{d\theta}{2\pi}e^{i\theta(L_0-\tilde{L}_0)}\sim{}_P\langle B;\eta|$

$$C_{\phi\chi}=(-\beta)(1+\beta)e^{-\frac{7\tau_0}{4\alpha_3}-\sum_{n\geq1}\frac{\cos^2 n\sigma^{(3)}_{\mathrm{int}}}{4n}-\frac{1}{4}\sum_{m,n\geq1}\bar{N}^{33}_{mn}\cos m\sigma^{(3)}_{\mathrm{int}}\cos n\sigma^{(3)}_{\mathrm{int}}}$$

⇨ We should determine the cocycle factor for the 3-string vertex correctly in order to impose GSO projection including the matter part, which should be consistent with gauge invariance of the SFT action.
In addition, appropriate regularization is necessary to evaluate the overall factor becasuse it does not cancel trivially.

※There are other terms by the picture changing operator.

Supersymmetric case (GS)

- Green-Schwarz formalism

light-cone qunatization $(i, a = 1, 2, \cdots, 8)$

$$X^i(\sigma) = x^i + i \sum_{n\neq 0} \frac{1}{n}(\alpha^i_n e^{in\sigma} + \tilde{\alpha}^i_n e^{-in\sigma}),$$

$$\vartheta^a(\sigma) = e^{-\frac{i\pi}{4}} \sum_n S^a_n e^{in\sigma} + e^{\frac{i\pi}{4}} \sum_n \tilde{S}^a_n e^{-in\sigma}.$$

- connection condition for 3-string interaction :

$$\delta^8(\vartheta^{(3)}(\sigma^{(3)}) - \Theta_1 \vartheta^{(1)}(\sigma^{(1)}) - \Theta_2 \vartheta^{(2)}(\sigma^{(2)}))$$

The same form as X^i sector (bosonic closed SFT, HIKKO type)

- The 3-string vertex is constructed by respecting space-time SUSY algebra.

- 3-string vertex [Green-Schwarz-Brink(1983)]

$$|V_3\rangle = X^i \tilde{X}^j v_{ij}(Y)|v_3\rangle,$$

$$X^i = \mathrm{P}^i - \sum_{r=1}^{3} \sum_{n\geq 1} \frac{\alpha_{123}}{\alpha_r}(\bar{N}^r C)_n \alpha_{-n}^{i(r)}, \quad \tilde{X}^i = \mathrm{P}^i - \sum_{r=1}^{3} \sum_{n\geq 1} \frac{\alpha_{123}}{\alpha_r}(\bar{N}^r C)_n \tilde{\alpha}_{-n}^{i(r)},$$

$$v_{ij}(Y) = \delta^{ij} - \frac{i}{\alpha_{123}}\gamma_{ab}^{ij} Y^a Y^b + \frac{1}{6(\alpha_{123})^2}\gamma_{[ab}^{ik}\gamma_{cd]}^{jk} Y^a Y^b Y^c Y^d$$

$$- \frac{4i}{6!(\alpha_{123})^3}\gamma_{ab}^{ij}\varepsilon^{abcdefgh} Y^c Y^d Y^e Y^f Y^g Y^h + \frac{16}{8!(\alpha_{123})^4}\delta^{ij}\varepsilon^{abcdefgh} Y^a Y^b Y^c Y^d Y^e Y^f Y^g Y^h,$$

$$Y^a = \Lambda^a - \alpha_{123} \sum_{r=1}^{3} \sum_{n\geq 1} \hat{N}_n^r (e^{i\pi/4} S_{-n}^{(r)} + e^{-i\pi/4} \tilde{S}_{-n}^{(r)}),$$

$|v_3\rangle$ is the naïve overlap part whose bosonic part is the same as bosonic SFT. Fermionic part is

$$|v_3\rangle = \int d^8\lambda_1^a d^8\lambda_2^a d^8\lambda_3^a \delta^8(\lambda_1^a + \lambda_2^a + \lambda_3^a) e^{E_Q} |\lambda_r^a\rangle$$

$$E_Q = \frac{1}{2} S_{-m}^{(r)} \hat{X}_{mn}^{rs} S_{-n}^{(s)} + \frac{1}{2} \tilde{S}_{-m}^{(r)} \hat{X}_{mn}^{rs} \tilde{S}_{-n}^{(s)}$$

$$+ \frac{i}{2}\alpha_{123} S_{-m}^{(r)} \hat{N}_m^r \hat{N}_n^s \tilde{S}_{-n}^{(s)} - \Lambda \hat{N}_n^r (e^{-i\pi/4} S_{-n}^{(r)} + e^{i\pi/4} \tilde{S}_{-n}^{(r)})$$

- Boundary state [Green-Gutperle(1996)]

$$|B;\eta_\pm\rangle = e^{M_{ij}\sum_{n=1}^{\infty}\frac{1}{n}\alpha^i_{-n}\tilde{\alpha}^j_{-n} - i\eta_\pm M_{ab}\sum_{n=1}^{\infty}S^a_{-n}\tilde{S}^b_{-n}}|B_0\rangle$$

where $\eta_\pm = \pm 1$, $|B_0\rangle$ is the zero mode part:

$$|B_0\rangle = |B_0\rangle_{\mathrm{bosonic}} \otimes \left(M_{ij}|i\rangle|j\rangle - i\eta_\pm M_{\dot{a}\dot{b}}|\dot{a}\rangle|\dot{b}\rangle\right)$$

constant matrix M which preserves 1/2 SUSY:

$$M_{ij} = \left(e^{\Omega_{kl}\Sigma^{kl}}\right)_{ij}, \quad (\Sigma^{kl}_{ij} = \delta^k_i\delta^l_j - \delta^l_i\delta^k_j), \quad M_{ab} = \left(e^{\frac{1}{2}\Omega_{kl}\gamma^{kl}}\right)_{ab}, \quad M_{\dot{a}\dot{b}} = \left(e^{\frac{1}{2}\Omega_{kl}\gamma^{kl}}\right)_{\dot{a}\dot{b}}$$

Dp brane is characterized by Ω_{kl}.

Fermion zero mode dependence:

$$\langle B_0|\lambda\rangle = \frac{1}{4}(\mathrm{Tr}M_{ij} + \eta_\pm \mathrm{Tr}M_{\dot{a}\dot{b}})\, e^{\frac{i}{\alpha}\lambda^a\Theta_{ab}\lambda^b}, \qquad \Theta = \begin{cases} \tanh\left(\frac{1}{4}\Omega_{kl}\gamma^{kl}\right) & (\eta_\pm = +1) \\ \coth\left(\frac{1}{4}\Omega_{kl}\gamma^{kl}\right) & (\eta_\pm = -1) \end{cases}$$

- Non-linear relation among boundary states

We can compute "the star product" of boundary states with $|B_0\rangle_{\mathrm{bosonic}} = |p^i_\perp, \alpha\rangle$ directly:

$$\langle B(-p^i_{\perp 1},-\alpha_1)|\langle B(-p^i_{\perp 2},-\alpha_2)|V_3\rangle = (4\epsilon)^{-4} V_{kl} E^{kl} e^{\Delta E} |B(p^i_{\perp 1}+p^i_{\perp 2},\alpha_1+\alpha_2)\rangle$$

The same relation as bosonic SFT holds except for the prefactor.
The prefactor depends on M (or Ω_{kl}) :

$$E^{kl} = \epsilon^{-2}\left[\frac{1}{2}\alpha_1\alpha_2\delta_{k,p} + (\mathrm{P}^k + \alpha_1\alpha_2(BC^{\frac{1}{2}})_m\alpha^k_{-m})(\mathrm{P}^p + \alpha_1\alpha_2(BC^{\frac{1}{2}})_n\tilde{\alpha}^p_{-n})\right] M_{pl} \,,$$

$$\Delta E = \begin{cases} 0 & (M_{ab} = M_{ba}) \\ -\frac{\alpha_1\alpha_2}{4}(S^\dagger_m - i\eta_\pm \tilde{S}^\dagger_m M^T)^a (BC^{\frac{1}{2}})_m (\Theta^{-1})_{ab} (BC^{\frac{1}{2}})_n (S^\dagger_n - i\eta_\pm M\tilde{S}^\dagger_n)^b & (M_{ab} \neq M_{ba}) \end{cases} ,$$

$$V_{kl} = \frac{1}{6}\det_{a,b}(1-(1-\epsilon)\eta_\pm M)[\mathrm{Tr}(M_{ij}) + \eta_\pm \mathrm{Tr}(M_{\dot{a}\dot{b}})]\alpha_3^{\frac{4}{3}} \int d^8\Lambda\, v_{kl}(\Lambda) e^{-\frac{i}{\alpha_{123}}\Lambda^a\Theta_{ab}\Lambda^b}$$

Example of V_{kl} :

D(-1) / D7 brane $\quad V_{kl} = \pm\epsilon^8\,\frac{8}{3}(\alpha_1\alpha_2)^{-4}\,\delta_{k,l}$

anti- D(-1) / anti-D7 brane $\quad V_{kl} = \pm\frac{2^{16}}{3^2}(\alpha_1\alpha_2)^{-4}\,\delta_{k,l}$

※The orders of $\epsilon\ (\to 0)$ in the prefactor are different.

• Some comments

- regularization : $\epsilon := 1 + \alpha_{123} \sum_{r,s=1,2} \sum_{m,n\geq 1} (\tilde{N}^r C \alpha_r^{-1})_m [(1+\tilde{N})^{-1}]^{rs}_{mn} \tilde{N}^s_n$

which vanishes if one use the Green-Schwarz’s formula naïvely.

（By level truncation, it behaves as L～ 1/L or 1/log(L)（？））

- In computing the fermionic non zero mode, we have used

$$\sum_{t=1}^{3}\sum_{k=1}^{\infty} \hat{X}^{rt}_{mk}\hat{X}^{ts}_{kn} = \delta^{r,s}\delta_{m,n}, \quad \sum_{t=1}^{3}\sum_{k=1}^{\infty} \hat{N}^t_k \hat{N}^t_k = 0, \quad \sum_{t=1}^{3}\sum_{k=1}^{\infty} \hat{X}^{rt}_{mk}\hat{N}^t_k = -\hat{N}^r_m.$$

In particular, the determinant factor which comes from fermionic nonzero mode is formally evaluated as

$$\det{}^4_{r,s=1,2}(\delta^{r,s}\delta_{m,n} - \hat{X}^{rt}_{mk}\hat{X}^{ts}_{kn}) = \det{}^4(1-(\tilde{N}^{33})^2)(4\epsilon)^{-4}$$

- nonzero mode dependence in the prefactor is along only one “direction” :

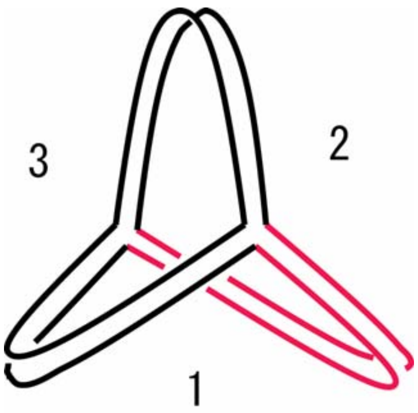
$$(BC^{\frac{1}{2}})_n = -\frac{2\alpha_3}{\pi\alpha_1\alpha_2}\frac{\sin n\sigma^{(3)}_{\mathrm{int}}}{n}$$

Summary and discussion

- Cardy states satisfy idempotency equation in closed SFT (explicitly checked on $R^D, T^D, T^D/Z_2$).
- Variation around idempotents gives open string spectrum on the D-brane. [KMW1,KMW2]
- We have directly checked $B * B = (\ldots)B$ for NSR and BGS type 3-string vertex although the prefactor is complicated and does not seem to be universal.

- Idempotency equation $\sim$ Cardy condition
  more detailed and general correspondence?
  (Proof of necessary and sufficient conditions)
- Closed version of VSFT? (Veneziano amplitude,…)
- Precise construction of 3-string vertex (SFT) in NSR formalism

- 3-string vertex in Nonpolynomial closed SFT



[Saadi-Zwiebach,Kugo-Kunitomo-Suehiro,Kugo-Suehiro,Kaku,…]

← closed string version of Witten's $*$ product

We can also prove idempotency straightforwardly [KMW2]:

$$|\Phi_B(x^\perp)\rangle\!\rangle * |\Phi_B(y^\perp)\rangle\!\rangle = \delta(x^\perp - y^\perp)\mathcal{C}_W c_0^+ b_0^- |\Phi_B(x^\perp)\rangle\!\rangle$$

Computation is simplified by closed sting version of MSFT. [Bars-Kishimoto-Matsuo(2004)]

- n-string vertices (n≧4) in nonpolynomial closedSFT?

$$|[|i\rangle\!\rangle, |j\rangle\!\rangle, |k\rangle\!\rangle]\rangle := \langle\!\langle i|\langle\!\langle j|\langle\!\langle k|V_4\rangle = ?, \quad \cdots .$$

Consistent regularization is indispensable.
Direct computation seems to be difficult. [c.f. Moeller(2004)]

- Berkovits’ pure spinor formalism?

Boundary states for D-branes are proposed recently.
[Schiappa-Wyllard, Mukhopadhyay]

We should construct 3-string vertex to investigate their idempotency.

String field is functional of $X^\mu, \theta^\alpha, \tilde{\theta}^\alpha, \lambda^\alpha, \tilde{\lambda}^\alpha \ (\lambda\gamma^\mu\lambda = \tilde{\lambda}\gamma^\mu\tilde{\lambda} = 0).$

Naively, 3-string vertex using LPP prescription will be defined as

$$\langle V_3|A_1\rangle|A_2\rangle|A_3\rangle \sim \langle \underline{Y^{11}\tilde{Y}^{11}} h_1[\mathcal{O}_{A_1}] h_2[\mathcal{O}_{A_2}] h_3[\mathcal{O}_{A_3}]\rangle .$$

“picture changing operator” $Y^{11} = \prod_{I=1}^{11} C_{I\alpha}\theta^\alpha(z_I)\delta(C_{I\alpha}\lambda^\alpha(z_I))$

It seems to be complicated to perform explicit computation….
We should extensively use Fierz transformation to compute zero modes in addition to Neumann coefficients.