Comments on Schnabl's marginal and scalar solutions in open string field theory

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Introduction

- Witten's bosonic open string field theory (d=26):

$$S[\Psi] = -\frac{1}{g^2}\left(\frac{1}{2}\langle\Psi, Q_{\mathrm{B}}\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right)$$

- There were various attempts to prove Sen's conjecture since around 1999.
- Numerically, it has been checked with "level truncation approximation." (c.f. … Gaiotto-Ratelli "Experimental string field theory"(2002))
- Analytically, some solutions have been constructed.
- Here, we generalize "Schnabl's analytical solutions" (2005, 2007) which include "tachyon vacuum solution" in Sen's conjecture and "marginal solutions."

Main result

Suppose $\hat{\phi}$ is BRST invariant and nilpotent:

$$Q_{\mathrm{B}}\hat{\phi} = 0, \quad \hat{\phi} * \hat{\phi} = 0.$$

Then,

$$\Psi^{(r,s)} = |r\rangle * \frac{1}{1 + \hat{\phi} * A^{(r+s-1)}} * \hat{\phi} * |s\rangle, \quad A^{(r+s-1)} \equiv \frac{\pi}{2}\int_1^{r+s-1} dr' B_1^L |r'\rangle$$

gives a solution.

Ex.) $\hat{\phi} = U_1^\dagger U_1 \lambda J(0)|0\rangle, \quad r = s = 3/2$

Schnabl / Kiermaier-Okawa-Rastelli-Zwiebach's marginal solution is reproduced.

$\hat{\phi} = \hat{\lambda} Q_{\mathrm{B}} U_1^\dagger U_1 B_1^L c_1 |0\rangle, \quad r = s = 3/2, \ \hat{\lambda} = \infty$

Schnabl's tachyon vacuum solution is reproduced.

Witten's bosonic open string field theory

Action: $$S[\Psi]=-\frac{1}{g^2}\left(\frac{1}{2}\langle\Psi,Q_{\rm B}\Psi\rangle+\frac{1}{3}\langle\Psi,\Psi*\Psi\rangle\right)$$

String field: (infinitely many fields are included.)

$$|\Psi\rangle=\phi(x)c_1|0\rangle+A_\mu(x)\alpha^\mu_{-1}c_1|0\rangle+iB(x)c_0|0\rangle+\cdots$$

BRST operator:

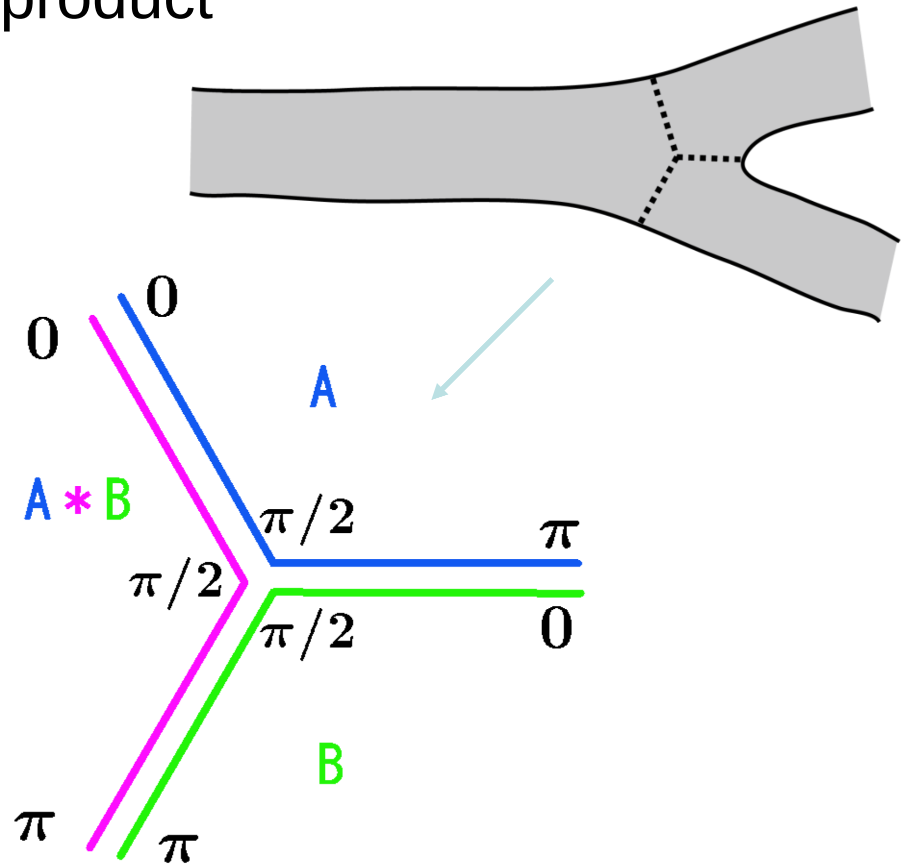
$$Q_{\rm B}=\oint\frac{dz}{2\pi i}\left(cT^{\rm m}+bc\partial c+\frac{3}{2}\partial^2 c\right)$$

(nilpotent for $c^m=26$.)

Kinetic term:

$$\langle\Psi,Q_{\rm B}\Psi\rangle$$
$$=\int d^{26}x\left(\phi(-\alpha'\partial^2-1)\phi-\alpha' A_\mu\partial^2 A^\mu+2\sqrt{2\alpha'}B\partial_\mu A^\mu+2B^2+\cdots\right)$$

Interaction term: the Witten star product



$$\int dx\,(\phi(x))^3$$
$$= \int dx_1 dx_2 dx_3 \delta(x_1 - x_2)\delta(x_2 - x_3) \phi(x_1)\phi(x_2)\phi(x_3)$$

$$\langle \Psi, \Psi * \Psi \rangle$$
$$= \langle V_3(1,2,3)|\Psi\rangle_1|\Psi\rangle_2|\Psi\rangle_3$$
$$\sim \int \prod_{0\leq\sigma\leq\pi/2} \big(\delta(X^{(1)}(\pi-\sigma) - X^{(2)}(\sigma))\delta(X^{(2)}(\pi-\sigma) - X^{(3)}(\sigma))$$
$$\times\, \delta(X^{(3)}(\pi-\sigma) - X^{(1)}(\sigma))(bc \text{ ghost} \cdots)\big)$$
$$\times\, \Psi[X^{(1)}(\sigma),\cdots]\Psi[X^{(2)}(\sigma),\cdots]\Psi[X^{(3)}(\sigma),\cdots]$$

equation of motion: $$Q_{\mathrm{B}}\Psi + \Psi * \Psi = 0$$

gauge transformation: $$\delta_{\Lambda}\Psi = Q_{\mathrm{B}}\Lambda + \Psi * \Lambda - \Lambda * \Psi$$

$$\longrightarrow \quad \delta_{\Lambda}S = 0$$

(※) $$Q_{\mathrm{B}}^2 = 0, \quad \langle A, Q_{\mathrm{B}}B\rangle = -(-1)^{|A|}\langle Q_{\mathrm{B}}A, B\rangle,$$

$$Q_{\mathrm{B}}(A * B) = (Q_{\mathrm{B}}A) * B + (-1)^{|A|}A * (Q_{\mathrm{B}}B),$$

$$\langle A, B\rangle = \langle B, A\rangle, \qquad \langle A, B * C\rangle = \langle B, C * A\rangle,$$

$$(A * B) * C = A * (B * C)$$: associative

Note : $A * B \neq B * A$ in general.

Schnabl's tachyon vacuum solution

- "sliver frame": $\tilde{z} = \arctan z$ (z :UHP)

  For a primary field ϕ with dim=h,

$$\tilde{\phi}(\tilde{z}) = \left(\frac{dz}{d\tilde{z}}\right)^h \phi(z) = (\cos\tilde{z})^{-2h}\phi(\tan\tilde{z}),$$

$$\tilde{\phi}(\tilde{z}) = \sum_n \tilde{\phi}_n \tilde{z}^{-n-h}, \qquad \phi(z) = \sum_n \phi_n z^{-n-h},$$

$$\tilde{\phi}_n = \oint_0 \frac{d\tilde{z}}{2\pi i}\tilde{z}^{n+h-1}\tilde{\phi}(\tilde{z}) = \oint_0 \frac{dz}{2\pi i}(\arctan z)^{n+h-1}(1+z^2)^{h-1}\phi(z)$$

$$= \sum_{m=n}^{\infty}\phi_m \oint_0 \frac{d\tilde{z}}{2\pi i}\tilde{z}^{n+h-1}(\cos\tilde{z})^{-2h}(\tan\tilde{z})^{-m-h} = \sum_{m=n}^{\infty}\phi_m \oint_0 \frac{dz}{2\pi i}(\arctan z)^{n+h-1}(1+z^2)^{h-1}z^{-m-h},$$

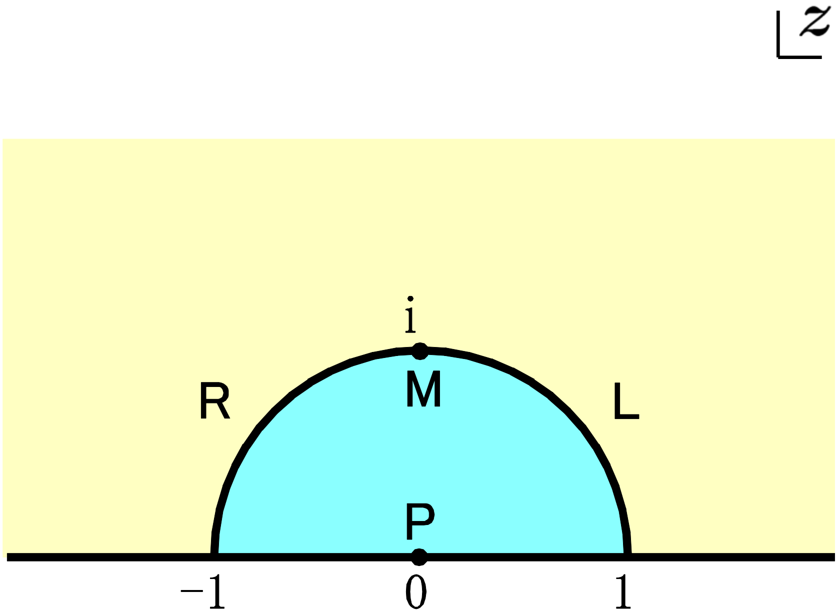
In particular, we often use

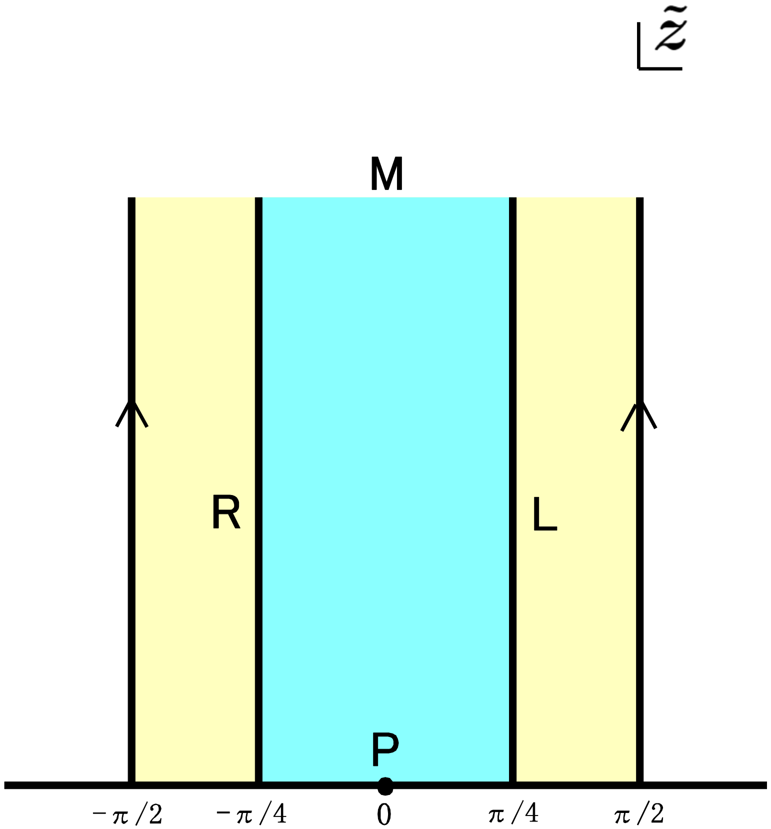
$$\mathcal{L}_0 \equiv \tilde{L}_0 = L_0 + \sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{4k^2-1}L_{2k}, \qquad K_1 \equiv \tilde{L}_{-1} = L_1 + L_{-1},$$

$$\mathcal{B}_0 \equiv \tilde{b}_0 = b_0 + \sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{4k^2-1}b_{2k}, \qquad B_1 \equiv \tilde{b}_{-1} = b_1 + b_{-1},$$

and

$$\hat{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_0^\dagger, \quad K_1^{L/R} = \frac{1}{2}K_1 \pm \frac{1}{\pi}\hat{\mathcal{L}}, \quad \hat{\mathcal{B}} = \mathcal{B}_0 + \mathcal{B}_0^\dagger, \quad B_1^{L/R} = \frac{1}{2}B_1 \pm \frac{1}{\pi}\hat{\mathcal{B}}.$$





$$\mathbf{arctan}\, z = \tilde{z}$$

Using $U_r = \left(\frac{2}{r}\right)^{\mathcal{L}_0} = \left(\frac{2}{r}\right)^{L_0} e^{-\frac{r^2-4}{3r^2}L_2 + \frac{r^4-16}{30r^4}L_4 + \cdots}$ we have a formula for the star product:

$$U_r^\dagger U_r \tilde{\phi}_1(\tilde{x}_1)\cdots\tilde{\phi}_n(\tilde{x}_n)|0\rangle * U_s^\dagger U_s \tilde{\psi}_1(\tilde{y}_1)\cdots\tilde{\psi}_m(\tilde{y}_m)|0\rangle$$
$$= U_{r+s-1}^\dagger U_{r+s-1}\tilde{\phi}_1(\tilde{x}_1 + \frac{\pi}{4}(s-1))\cdots\tilde{\phi}_n(\tilde{x}_n + \frac{\pi}{4}(s-1))\tilde{\psi}_1(\tilde{y}_1 - \frac{\pi}{4}(r-1))\cdots\tilde{\psi}_m(\tilde{y}_m - \frac{\pi}{4}(r-1))|0\rangle$$

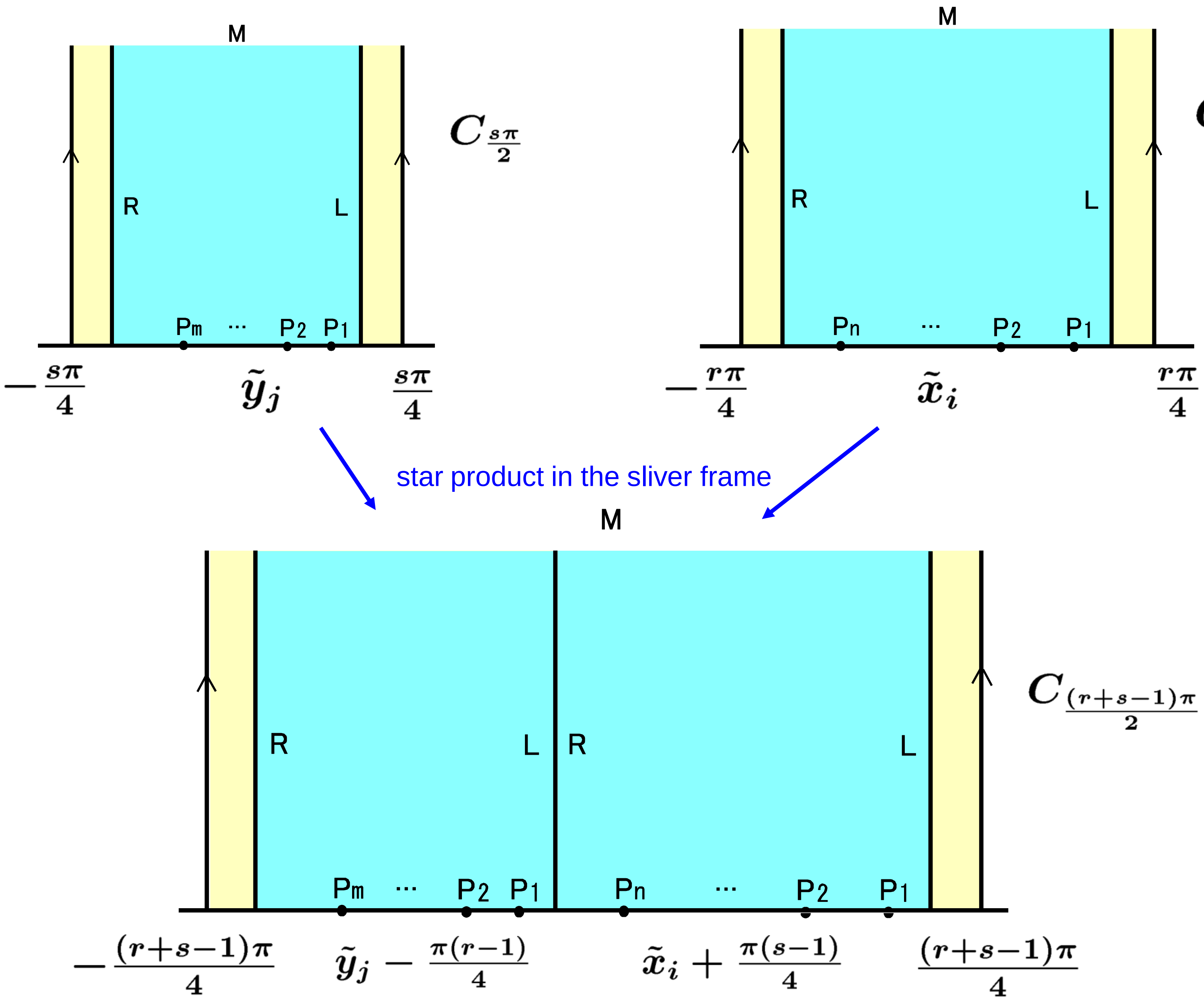
In particular, for the wedge state: $|r\rangle = U_r^\dagger|0\rangle = U_r^\dagger U_r|0\rangle$

$$|r\rangle * |s\rangle = |r+s-1\rangle$$

$|r=1\rangle = I$: identity state

$|r=2\rangle = |0\rangle$: conformal vacuum

$|r=\infty\rangle$: sliver state


M
R
L
Pm ··· P2 P1
$C_{\frac{s\pi}{2}}$
$-\frac{s\pi}{4}$
$\tilde{y}_j$
$\frac{s\pi}{4}$
M
R
L
Pn ··· P2 P1
$C_{\frac{r\pi}{2}}$
$-\frac{r\pi}{4}$
$\tilde{x}_i$
$\frac{r\pi}{4}$
star product in the sliver frame
M
R
L
R
L
Pm ··· P2 P1
Pn ··· P2 P1
$C_{\frac{(r+s-1)\pi}{2}}$
$-\frac{(r+s-1)\pi}{4}$
$\tilde{y}_j - \frac{\pi(r-1)}{4}$
$\tilde{x}_i + \frac{\pi(s-1)}{4}$
$\frac{(r+s-1)\pi}{4}$

Note: the wedge state can be rewritten as

$$|r\rangle \;=\; e^{-\frac{r-2}{2}\hat{\mathcal{L}}}|0\rangle = e^{-(r-1)\frac{\pi}{2}K_1^L}|I\rangle$$

As a surface state, $r \geq 1$ for the wedge state.

However, *if one uses the last expression formally,* the wedge state with "negative angle" $r < 1$, which satisfies $|r\rangle * |s\rangle = |r+s-1\rangle$, might be considered.
In fact, this algebra can be formally obtained using following properties:

$$A * I = I * A = A, \qquad \forall A,$$
$$K_1^L(A * B) = (K_1^L A) * B, \quad \forall A, B.$$

Schnabl's solution in [hep-th/0511286] is given by

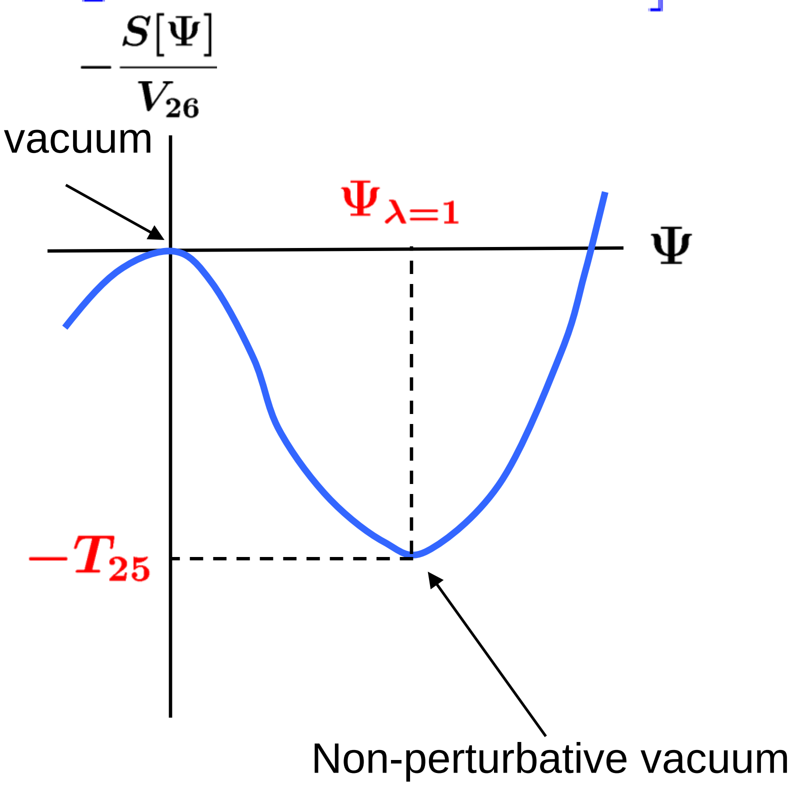
[Okawa]

$$\Psi_\lambda = -\sum_{n=0}^{\infty} \lambda^{n+1} \partial_t \psi_{t+n}|_{t=0} = \lambda Q_{\mathrm{B}} \Lambda_0 * (1-\lambda\Lambda_0)^{-1}, \qquad \Lambda_0 = B_1^L c_1|0\rangle\,,$$

$$\psi_n = \frac{2}{\pi} U_{n+2}^{\dagger} U_{n+2} \left[-\frac{1}{\pi} \hat{\mathcal{B}} \tilde{c}(\pi n/4) \tilde{c}(-\pi n/4) + \frac{1}{2}(\tilde{c}(\pi n/4) + \tilde{c}(-\pi n/4)) \right] |0\rangle\,.$$

It turned out to be

$$S[\Psi_\lambda]/V_{26} = \begin{cases} \frac{1}{2\pi^2 g^2} & (\lambda = 1) \\ 0 & (|\lambda| < 1) \end{cases} .$$



Ellwood and Schnabl [hep-th/0606142] have shown the triviality of the new BRST operator around $\Psi_{\lambda=1}$ using a relation: $Q_{\Psi_{\lambda=1}} A = I$.

Schnabl / KORZ's marginal solution

[Schnabl,Nov.1 (2006) Hawaii, hep-th/0701248],
[Kiermaier-Okawa-Rastelli-Zwiebach,hep-th/0701249]

For some matter primary operators with weight 1: J^a, one can construct a solution:

$$\Psi_\lambda^J = \lambda_a c J^a(0)|0\rangle + \sum_{k=1}^{\infty}\left(\frac{-\pi}{2}\right)^k \int_0^1 dr_1 \cdots \int_0^1 dr_k \psi_k^J(r_1,\cdots,r_k)$$

$$= |3/2\rangle * \frac{1}{1+\hat{\phi}^J * A} * \hat{\phi}^J * |3/2\rangle,$$

$$\psi_k^J(r_1,\cdots,r_k) = U^\dagger_{2+\sum_{l=1}^k r_l} U_{2+\sum_{l=1}^k r_l} \prod_{\overrightarrow{m=0}}^{k} \lambda_{a_m} \tilde{J}^{a_m}\left(\frac{\pi}{4}(-\sum_{l=1}^m r_l + \sum_{l=m+1}^k r_l)\right)$$

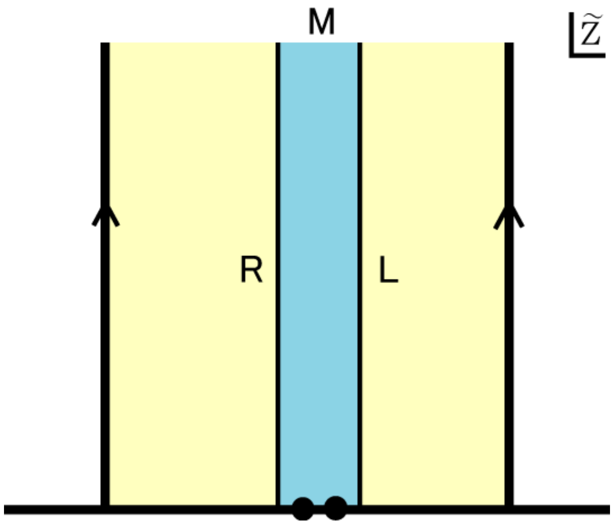
$$\times \left[-\frac{1}{\pi}\hat{\mathcal{B}}\tilde{c}(\frac{\pi}{4}\sum_{l=1}^k r_l)\tilde{c}(-\frac{\pi}{4}\sum_{l=1}^k r_l) + \frac{1}{2}\left(\tilde{c}(\frac{\pi}{4}\sum_{l=1}^k r_l) + \tilde{c}(-\frac{\pi}{4}\sum_{l=1}^k r_l)\right)\right]|0\rangle.$$

$$A = \frac{\pi}{2}\int_1^2 dr B_1^L |r\rangle, \qquad \hat{\phi}^J = U_1^\dagger U_1 \lambda_a c J^a(0)|0\rangle\,.$$

Here, we have supposed "non-singularity" of the current:

$$\lambda_a\lambda_b g^{ab} = 0, \qquad J^a(y)J^b(z) \sim \frac{-g^{ab}}{(y-z)^2} + \frac{1}{y-z} f^{ab}{}_c J^c(z) + \cdots$$

Due to the non-singularity condition for the current, we find nilpotency with respect to the star product: $\hat{\phi}^J * \hat{\phi}^J = 0$.



$$c\lambda_a J^a(\epsilon)\, c\lambda_b J^b(0) \sim 0$$

Solution generation

Note: $Q_{\rm B}A = I - |r=2\rangle = I - |0\rangle, \quad Q_{\rm B}\hat{\phi}^J = 0, \quad \hat{\phi}^J * \hat{\phi}^J = 0\,.$

$\hat{\phi}_{\rm s} \equiv Q_{\rm B}U_1^\dagger U_1 \Lambda_0 = Q_{\rm B}U_1^\dagger U_1 B_1^L c_1|0\rangle\,, \quad Q_{\rm B}\hat{\phi}_{\rm s} = 0, \quad \hat{\phi}_{\rm s} * \hat{\phi}_{\rm s} = 0\,.$

Suppose $\hat{\phi}$ **is BRST invariant and nilpotent:**

$Q_{\rm B}\hat{\phi} = 0, \quad \hat{\phi} * \hat{\phi} = 0.$ **Then,**

$$\Psi^{(r,s)} = |r\rangle * \frac{1}{1 + \hat{\phi} * A^{(r+s-1)}} * \hat{\phi} * |s\rangle, \quad A^{(r+s-1)} \equiv \frac{\pi}{2}\int_1^{r+s-1} dr' B_1^L |r'\rangle$$

gives a solution.

∵

$$
\begin{aligned}
Q_{\mathrm{B}}\Psi^{(r,s)} &= |r\rangle * Q_{\mathrm{B}}\left(\frac{1}{1+\hat{\phi}*A^{(r+s-1)}}\right) * \hat{\phi} * |s\rangle \\
&= -|r\rangle * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * (Q_{\mathrm{B}}(I+\hat{\phi}*A^{(r+s-1)})) * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * \hat{\phi} * |s\rangle \\
&= |r\rangle * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * \hat{\phi} * (Q_{\mathrm{B}}A^{(r+s-1)}) * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * \hat{\phi} * |s\rangle \\
&= |r\rangle * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * \hat{\phi} * (I - |r+s-1\rangle) * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * \hat{\phi} * |s\rangle \\
&= |r\rangle * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * \underbrace{\hat{\phi} * \hat{\phi}}_{=\,0} * \frac{1}{1+A^{(r+s-1)}*\hat{\phi}} * |s\rangle \\
&\quad -|r\rangle * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * \hat{\phi} * |s\rangle * |r\rangle * \frac{1}{1+\hat{\phi}*A^{(r+s-1)}} * \hat{\phi} * |s\rangle \\
&= -\Psi^{(r,s)} * \Psi^{(r,s)}\,.
\end{aligned}
$$

Generalization of Schnabl's marginal solution

From a BRST invariant, nilpotent $\hat{\phi}^J = U_1^\dagger U_1 \lambda_a c J^a(0)|0\rangle$ which satisfies

$(\mathcal{B}_0 - \mathcal{B}_0^\dagger)\hat{\phi}^J = 0$, we can generate a solution

$$
\begin{aligned}
\Psi_\lambda^{J(r,s)} &= |r\rangle * \hat{\phi}^J * |s\rangle + \sum_{k=1}^{\infty} (-1)^k |r\rangle * (\hat{\phi}^J * A^{(r+s-1)})^k * \hat{\phi} * |s\rangle = \sum_{n=1}^{\infty} \phi_n^J , \\
\phi_{k+1}^J &= \left(-\frac{\pi}{2}\right)^k \int_0^{r+s-2} dr_1 \cdots \int_0^{r+s-2} dr_k U^\dagger_{r+s-1+\sum_{l=1}^k r_l} U_{r+s-1+\sum_{l=1}^k r_l} \left[\prod_{\overrightarrow{m=0}}^{k} \lambda_{a_m} \tilde{J}^{a_m} \left(\frac{\pi}{4}(s - r - \sum_{l=1}^{m} r_l + \sum_{l=m+1}^{k} r_l) \right) \right] \\
&\quad \times \left[-\frac{1}{\pi} \hat{\mathcal{B}} \tilde{c}(\frac{\pi}{4}(s - r + \sum_{l=1}^{k} r_l)) \tilde{c}(\frac{\pi}{4}(s - r - \sum_{l=1}^{k} r_l)) + \frac{1}{2} \left(\tilde{c}(\frac{\pi}{4}(s - r + \sum_{l=1}^{k} r_l)) + \tilde{c}(\frac{\pi}{4}(s - r - \sum_{l=1}^{k} r_l)) \right) \right] |0\rangle .
\end{aligned}
$$

Actually, we can show the following relations by explicit computation:

$$
Q_{\mathrm{B}} \phi_1^J = 0, \quad \mathcal{B}^{(r,s)} \phi_1^J = 0, \quad \phi_{k+1}^J = -\frac{\mathcal{B}^{(r,s)}}{\mathcal{L}^{(r,s)}} \sum_{l=1}^{k} \phi_l^J * \phi_{k-l+1}^J,
$$

$$
\mathcal{B}^{(r,s)} = \frac{1}{2}(r + s - 3)\hat{\mathcal{B}} + \mathcal{B}_0 + \frac{\pi}{4}(r - s) B_1 , \quad \mathcal{L}^{(r,s)} \equiv \{Q_{\mathrm{B}}, \mathcal{B}^{(r,s)}\} .
$$

In particular, this solution satisfy a "generalized Schnabl gauge":

$$
\mathcal{B}^{(r,s)} \Psi_\lambda^{J(r,s)} = 0
$$

ex.1) Rolling tachyon $\lambda_a J^a = \lambda :e^{X^0}:$

$$\Psi_\lambda^{J(r,r)} = \Big[\lambda e^{X^0} - \frac{64\cot^3\frac{\pi(2r-1)}{2(4r-3)}}{3(4r-3)^3}\lambda^2 e^{2X^0} + \cdots$$
$$+(\sim r^{-k^2-2k} \text{ for } r \gg 1)\lambda^{k+1}e^{(k+1)X^0}\Big]c_1|0\rangle + \cdots$$

ex. 2) light-cone-like deformation $\lambda_a J^a = \lambda i\partial X^+$

$$\Psi_\lambda^{J(r,r)} = \left[\lambda\alpha_{-1}^+ - \frac{4\cot\frac{\pi(2r-1)}{2(4r-3)}}{4r-3}\lambda^2\alpha_{-1}^+\alpha_{-1}^+ + \cdots\right]c_1|0\rangle + \cdots$$

Generalization of Schnabl's tachyon vacuum solution

- From a BRST invariant, nilpotent $\hat{\phi}_{\rm s} = Q_{\rm B} U_1^\dagger U_1 B_1^L c_1 |0\rangle$ which satisfies $(\mathcal{B}_0 - \mathcal{B}_0^\dagger)\hat{\phi}_{\rm s} = 0$, we can generate a solution:

$$\begin{aligned}\hat{\Psi}_{\hat{\lambda}}^{(r,s)} &= |r\rangle * \frac{\hat{\lambda}}{1+\hat{\lambda}\hat{\phi}_{\rm s} * A^{(r+s-1)}} * \hat{\phi}_{\rm s} * |s\rangle = \sum_{k=0}^{\infty}(-1)^k \hat{\lambda}^{k+1} |r\rangle * \hat{\phi}_{\rm s} * (A^{(r+s-1)} * \hat{\phi}_{\rm s})^k * |s\rangle \\ &= \sum_{k=0}^{\infty} \hat{\lambda}^{k+1} Q_{\rm B}\Lambda_0^{(r,s)} * (\Lambda_0^{(r,s)} - I)^k = \frac{\hat{\lambda}}{1+\hat{\lambda}} Q_{\rm B}\Lambda_0^{(r,s)} * \frac{1}{1-\frac{\hat{\lambda}}{1+\hat{\lambda}}\Lambda_0^{(r,s)}} = \Psi^{(r,s)}_{\lambda=\frac{\hat{\lambda}}{1+\hat{\lambda}}}\end{aligned}$$

where we have defined

$$\Psi_\lambda^{(r,s)} = -\sum_{n=0}^{\infty}\lambda^{n+1}\partial_t \psi_{t,n}^{(r,s)}|_{t=0} = -\sum_{n=0}^{\infty}\lambda^{n+1}|r-1/2\rangle * \partial_t\psi_{t+n(r+s-2)}|_{t=0} * |s-1/2\rangle\,.$$

In the case of $\lambda = 1\ (\leftrightarrow \hat{\lambda} = \infty)$, we regularize the above solution as

$$\Psi_{\lambda=1}^{(r,s)} = \frac{1}{r+s-2}\sum_{n=0}^{\infty}\frac{B_n}{n!}(r+s-2)^n \partial_t^n \psi_{t,n=0}^{(r,s)}|_{t=0} = \lim_{N\to\infty}\left(\frac{1}{r+s-2}\psi_{t=0,N}^{(r,s)} - \sum_{n=0}^{N}\partial_t\psi_{t,n}^{(r,s)}|_{t=0}\right).$$

Using the identity: $\mathcal{B}^{(r,s)} e^{\frac{\pi}{4}(s-r)K_1}(r+s-2)^{\frac{D}{2}} = e^{\frac{\pi}{4}(s-r)K_1}(r+s-2)^{\frac{D}{2}}\mathcal{B}_0$

($K_1 = L_1 + L_{-1},\ D = \mathcal{L}_0 - \mathcal{L}_0^\dagger$: derivations, BPZ odd)

we find a formula $\Psi_\lambda^{(r,s)} = e^{\frac{\pi}{4}(s-r)K_1}(r+s-2)^{\frac{D}{2}}\Psi_\lambda^{(\frac{3}{2},\frac{3}{2})}.$

⇨ We can show following relations:

$$Q_{\mathrm{B}}\partial_t\psi_{t,0}^{(r,s)}|_{t=0} = 0, \quad \mathcal{B}^{(r,s)}\partial_t\psi_{t,0}^{(r,s)}|_{t=0} = 0,$$

$$\partial_t\psi_{t,n}^{(r,s)}|_{t=0} - \partial_t\psi_{t,0}^{(r,s)}|_{t=0} = \frac{\mathcal{B}^{(r,s)}}{\mathcal{L}^{(r,s)}}\sum_{m=0}^{n-1}\partial_t\psi_{t,m}^{(r,s)}|_{t=0} * \partial_t\psi_{t,n-1-m}^{(r,s)}|_{t=0}.$$

$$\mathcal{B}^{(r,s)}\Psi_\lambda^{(r,s)} = 0,$$ (generalized Schnabl gauge)

$$S[\Psi_\lambda^{(r,s)}]/V_{26} = \begin{cases} \frac{1}{2\pi^2 g^2} & (\lambda = 1) \\ 0 & (|\lambda| < 1) \end{cases},$$

$$Q_{\Psi_{\lambda=1}^{(r,s)}} A^{(r+s-1)} = I.$$

(Sen's conjecture)

Comments

- In the case of $r = s = 3/2$, Schnabl / KORZ's marginal solutions and the Schnabl's tachyon vacuum solution are reproduced.

- In the case of $r = s = 1$, the original $\hat{\phi}$ is reproduced and a relation $\Psi_\lambda^{(r,s)} = e^{\frac{\pi}{4}(s-r)K_1}(r+s-2)^{\frac{D}{2}}\Psi_\lambda^{(\frac{3}{2},\frac{3}{2})}$ becomes singular.

⇨ Direct evaluation of the action at the identity based, BRST invariant and nilpotent solution $\hat{\phi}_s = Q_B U_1^\dagger U_1 B_1^L c_1|0\rangle$ is difficult.
(This situation is similar to the Takahashi-Tanimoto's "universal solution" which is identity based solution.)

- If we use "wedge state *with negative angle*," the above solution is gauge equivalent to the original identity based solution formally:

$$\Psi^{(r,s)} = V^{-1} * \hat{\phi} * V + V^{-1} * Q_B V,$$

$$V = (I + \hat{\phi} * A^{(r+s-1)}) * |2-r\rangle, \quad V^{-1} = |r\rangle * \frac{1}{1 + \hat{\phi} * A^{(r+s-1)}}.$$

Future problems

- Other solutions?
  How about the solutions which is generated by $\hat{\phi} = \hat{\phi}_{\mathrm{s}} + \hat{\phi}^{J}$.

- Physical interpretation of the generated solutions?
  BRST cohomology around them?

- Marginal solution for singular currents? ($\lambda_a \lambda_b g^{ab} \neq 0$)
  [KORZ], [Fuchs-Kroyter-Potting, arXiv:07042222]

- How to define the space of string field?
  What are *regular* string fields?

- Generalization of our method to Berkovits' WZW type superstring field theory?

Generalization to super SFT

Berkovits' open super SFT.

The action for NS(+) sector is given by

WZW type

$$S[\Phi]$$
$$= \frac{1}{2g^2}\langle\langle (e^{-\Phi}Q_{\mathrm{B}}e^{\Phi})(e^{-\Phi}\eta_0 e^{\Phi}) - \int_0^1 dt (e^{-t\Phi}\partial_t e^{t\Phi})\{(e^{-t\Phi}Q_{\mathrm{B}}e^{t\Phi}), (e^{-t\Phi}\eta_0 e^{t\Phi})\}\rangle\rangle$$
$$= -\frac{1}{g^2}\int_0^1 dt \langle\langle (\eta_0\Phi)(e^{-t\Phi}Q_{\mathrm{B}}e^{t\Phi})\rangle\rangle$$

[Berkovits-Okawa-Zwiebach(2004)]

$$= -\frac{1}{g^2}\sum_{M,N=0}^{\infty}\frac{(-1)^M}{(M+N+2)(M+N+1)M!N!}\langle\langle (\eta_0\Phi)\Phi^M(Q_{\mathrm{B}}\Phi)\Phi^N\rangle\rangle\,.$$

String field Φ : ghost number 0, picture number 0, Grassmann even,
represented by matter and ghosts b, c, ϕ, ξ, η $(\beta = e^{-\phi}\partial\xi, \gamma = \eta e^{\phi})$:

$$Q_{\mathrm{B}} = \oint \frac{dz}{2\pi i}(c(T^{\mathrm{m}} - \tfrac{1}{2}(\partial\phi)^2 - \partial^2\phi + \partial\xi\eta) + bc\partial c + \eta e^{\phi}G^{\mathrm{m}} - \eta\partial\eta e^{2\phi}b)(z)$$

$$\eta_0 = \oint \frac{dz}{2\pi i}\eta(z)$$

Equation of motion: $\eta_0(e^{-\Phi}Q_{\mathrm{B}}e^{\Phi}) = 0$

Expand with respect to a formal parameter λ: $\Phi = \sum_{n\geq 1} \lambda^{n+1}\Phi_n$

The above EOM can be rewritten as:

$$
\begin{aligned}
&\eta_0 Q_{\mathrm{B}}\Phi_0 = 0\,,\\
&\eta_0\left(Q_{\mathrm{B}}\Phi_1 - \frac{1}{2}[\Phi_0, Q_{\mathrm{B}}\Phi_0]\right) = 0,\\
&\eta_0\left(Q_{\mathrm{B}}\Phi_2 - \frac{1}{2}([\Phi_1, Q_{\mathrm{B}}\Phi_0] + [\Phi_0, Q_{\mathrm{B}}\Phi_1]) + \frac{1}{6}[\Phi_0, [\Phi_0, Q_{\mathrm{B}}\Phi_0]]\right) = 0,\\
&\vdots\\
&\eta_0\left(Q_{\mathrm{B}}\Phi_n + \sum_{k=1}^{n}\frac{(-1)^k}{(k+1)!}\sum_{\substack{l_1+\cdots+l_{k+1}=n-k,\\ l_1,\cdots,l_{k+1}\geq 0}} \mathrm{ad}_{\Phi_{l_1}}\mathrm{ad}_{\Phi_{l_2}}\cdots\mathrm{ad}_{\Phi_{l_k}}(Q_{\mathrm{B}}\Phi_{l_{k+1}})\right) = 0
\end{aligned}
$$

The higher terms can be determined by the lowest one *formally*:

$$
\Phi_n = \frac{\tilde{\mathcal{G}}_0^-}{\mathcal{L}_0}\frac{\mathcal{B}_0}{\mathcal{L}_0}\eta_0 \sum_{k=1}^{n}\frac{(-1)^{k+1}}{(k+1)!}\sum_{\substack{l_1+\cdots+l_{k+1}=n-k,\\ l_1,\cdots,l_{k+1}\geq 0}} \mathrm{ad}_{\Phi_{l_1}}\mathrm{ad}_{\Phi_{l_2}}\cdots\mathrm{ad}_{\Phi_{l_k}}(Q_{\mathrm{B}}\Phi_{l_{k+1}})
$$

Notation

$$\mathrm{ad}_A B = [A, B] = A * B - B * A$$

$$\begin{aligned} \tilde{G}^-(z) &= [Q, \xi b(z)] \\ &= -\xi T(z) + e^{\phi} G^{\mathrm{m}} b(z) + c\partial\xi b(z) + b\partial b\eta e^{2\phi}(z) - \partial^2\xi(z) \end{aligned}$$

$$\tilde{\mathcal{G}}_0^- = \oint \frac{dw}{2\pi i}(1+w^2)(\arctan w)\tilde{G}^-(w) = \tilde{G}_0^- + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1}\tilde{G}_{2k}^-$$

Note

$$\{\eta_0, \tilde{\mathcal{G}}_0^-\} = -\mathcal{L}_0, \quad \{Q_{\mathrm{B}}, \tilde{\mathcal{G}}_0^-\} = 0, \quad \{\mathcal{B}_0, \tilde{\mathcal{G}}_0^-\} = 0, \quad [\mathcal{L}_0, \tilde{\mathcal{G}}_0^-] = 0$$

The above formal solution satisfies the gauge condition:

$$\mathcal{B}_0 \Phi_k = 0, \quad \tilde{\mathcal{G}}_0^- \Phi_k = 0$$

In bosonic SFT, roughly $\frac{\mathcal{B}_0}{\mathcal{L}_0} \rightarrow *A*$

$$(Q_B A = I - |r=2\rangle,\ Q_B\hat{\phi} = 0,\ \hat{\phi} * \hat{\phi} = 0)$$

We guess: $\frac{\tilde{\mathcal{G}}_0^-}{\mathcal{L}_0}\frac{\mathcal{B}_0}{\mathcal{L}_0} \rightarrow *\hat{A}*$

$$\hat{A} \equiv -\frac{\pi}{2}\int_0^1 du \int_0^1 dv J_1^{--L}|uv+1\rangle - \left(\frac{\pi}{2}\right)^2 \int_0^1 du \int_0^1 dv\, uv \tilde{G}_1^{-L} B_1^L |uv+1\rangle$$

$$J^{--} = \xi b, \quad \tilde{G}^- = [Q_B, J^{--}],$$

$$\eta_0 Q_B \hat{A} = I - |r=2\rangle, \quad \eta_0 \hat{A} = -\frac{\pi}{2}\int_1^2 dr B_1^L |r\rangle, \quad Q_B \hat{A} = -\frac{\pi}{2}\int_1^2 dr \tilde{G}_1^{-L}|r\rangle.$$

$$\hat{\Phi}_0 \equiv U_1^\dagger U_1 c\xi e^{-\phi}\lambda_a \psi^a(0)|0\rangle$$

$$\eta_0 Q_B \hat{\Phi}_0 = 0,\ \hat{\Phi}_0 * Q_B\hat{\Phi}_0 = 0,\ (Q_B\hat{\Phi}_0) * \hat{\Phi}_0 = 0,\ \hat{\Phi}_0 * \hat{\Phi}_0 = 0,\ \hat{\Phi}_0 * \eta_0\hat{\Phi}_0 = 0,\ (\eta_0\hat{\Phi}_0) * \hat{\Phi}_0 = 0.$$

These equations follow from non-singularity, which we suppose, of the super current.

$$\lambda_a \lambda_b \Omega^{ab} = 0$$

$$\begin{aligned}
\psi^a(y)\psi^b(z) &\sim (y-z)^{-1}\frac{1}{2}\Omega^{ab}, \\
J^a(y)\psi^b(z) &\sim (y-z)^{-1} f^{ab}{}_c \psi^c(z), \\
\psi^a(y)J^b(z) &\sim (y-z)^{-1} f^{ab}{}_c \psi^c(z), \\
J^a(y)J^b(z) &\sim (y-z)^{-2}\frac{1}{2}\Omega^{ab} + (y-z)^{-1} f^{ab}{}_c J^c(z), \\
G^{\mathrm{m}}(y)\psi^a(z) &\sim (y-z)^{-1} J^a(z), \\
T^{\mathrm{m}}(y)\psi^a(z) &\sim (y-z)^{-2}\frac{1}{2}\psi^a(z) + (y-z)^{-1}\partial\psi^a(z), \\
G^{\mathrm{m}}(y)J^a(z) &\sim (y-z)^{-2}\psi^a(z) + (y-z)^{-1}\partial\psi^a(z), \\
T^{\mathrm{m}}(y)J^a(z) &\sim (y-z)^{-2}J^a(z) + (y-z)^{-1}\partial J^a(z)
\end{aligned}$$

For $\Phi_0 = |3/2\rangle * \hat{\Phi}_0 * |3/2\rangle$
we have explicitly computed as

$$
\begin{aligned}
\Phi_1 &= \frac{1}{2}\frac{\tilde{\mathcal{G}}_0^-}{\mathcal{L}_0}\frac{\mathcal{B}_0}{\mathcal{L}_0}(\eta_0\Phi_0 * Q_{\rm B}\Phi_0 + Q_{\rm B}\Phi_0 * \eta_0\Phi_0) \\
&= \frac{1}{2}\tilde{\mathcal{G}}_0^- \int_0^\infty dT_1 e^{-T_1\mathcal{L}_0}\mathcal{B}_0 \int_0^\infty dT_2 e^{-T_2\mathcal{L}_0}(\eta_0\Phi_0 * Q_{\rm B}\Phi_0 + Q_{\rm B}\Phi_0 * \eta_0\Phi_0) \\
&= \frac{1}{2}\int_0^1 du \int_0^1 dv U^\dagger_{uv+2}U_{uv+2}\Big[\frac{1}{2}\hat{\mathcal{J}}^{--}\tilde{\psi}_\eta(\tilde{x})\tilde{\psi}_Q(-\tilde{x}) \\
&\quad -\frac{\pi}{4}(J_1^{--}\tilde{\psi}_\eta(\tilde{x})\tilde{\psi}_Q(-\tilde{x}) - \tilde{\psi}_\eta(\tilde{x})J_1^{--}\tilde{\psi}_Q(-\tilde{x}))\Big]|0\rangle \\
&\quad +\frac{1}{2}\int_0^1 du \int_0^1 dv \frac{\pi}{4}uvU^\dagger_{uv+2}U_{uv+2}\Big[\frac{1}{\pi}\hat{\tilde{\mathcal{G}}}^-\hat{\mathcal{B}}\tilde{\psi}_\eta(\tilde{x})\tilde{\psi}_Q(-\tilde{x}) \\
&\qquad + \frac{1}{2}\hat{\mathcal{B}}(\tilde{G}_1^-\tilde{\psi}_\eta(\tilde{x})\tilde{\psi}_Q(-\tilde{x})) + \tilde{\psi}_\eta(\tilde{x})\tilde{G}_1^-\tilde{\psi}_Q(-\tilde{x})) \\
&\qquad - \frac{1}{2}\hat{\tilde{\mathcal{G}}}^-(B_1\tilde{\psi}_\eta(\tilde{x})\tilde{\psi}_Q(-\tilde{x})) + \tilde{\psi}_\eta(\tilde{x})B_1\tilde{\psi}_Q(-\tilde{x})) \\
&\qquad + \frac{\pi}{4}\Big(\tilde{G}_1^-\tilde{\psi}_\eta(\tilde{x})B_1\tilde{\psi}_Q(-\tilde{x})) - B_1\tilde{\psi}_\eta(\tilde{x})\tilde{G}_1^-\tilde{\psi}_Q(-\tilde{x})) \\
&\qquad\quad - B_1\tilde{G}_1^-\tilde{\psi}_\eta(\tilde{x})\tilde{\psi}_Q(-\tilde{x})) - \tilde{\psi}_\eta(\tilde{x})B_1\tilde{G}_1^-\tilde{\psi}_Q(-\tilde{x}))\Big)\Big]|0\rangle \\
&\quad +(\tilde{\psi}_\eta \leftrightarrow \tilde{\psi}_Q) \\
&= -\frac{1}{2}|3/2\rangle * (\eta_0\hat{\Phi}_0 * \hat{A} * Q_{\rm B}\hat{\Phi}_0 + Q_{\rm B}\hat{\Phi}_0 * \hat{A} * \eta_0\hat{\Phi}_0) * |3/2\rangle.
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{G}_1^- \equiv \oint \frac{dz}{2\pi i}(1+z^2)\tilde{G}^-(z) = \tilde{G}_1^- + \tilde{G}_{-1}^-, \\
&\hat{\tilde{\mathcal{G}}}^- \equiv \tilde{\mathcal{G}}_0^- + \tilde{\mathcal{G}}_0^{-\dagger}, \\
&J_1^{--} \equiv \oint \frac{dz}{2\pi i}(1+z^2)J^{--}(z) = J_1^{--} + J_{-1}^{--}, \\
&\hat{\mathcal{J}}^{--} \equiv \mathcal{J}_0^{--} - \mathcal{J}_0^{--\dagger}, \\
&\psi_Q(z) \equiv [Q_{\rm B}, \varphi_0(z)] = \lambda_a(cJ^a + \eta e^{\phi}\psi^a)(z), \\
&\psi_\eta(z) \equiv [\eta_0, \varphi_0(z)] = -\lambda_a c e^{-\phi}\psi^a(z), \\
&\vdots
\end{aligned}
$$

For the next order, using the above guess, we have obtained

$$\begin{aligned}\Phi_2 \;=\; & -\frac{1}{16}|3/2\rangle * \Big(\{\{\eta_0\hat{\Phi}_0, Q_B\hat{\Phi}_0\}_{\hat{A}}, Q_B\hat{\Phi}_0\}_{\eta_0\hat{A}} + \{\{\eta_0\hat{\Phi}_0, Q_B\hat{\Phi}_0\}_{\eta_0\hat{A}}, Q_B\hat{\Phi}_0\}_{\hat{A}} \\ & \qquad + \{\{Q_B\hat{\Phi}_0, \eta_0\hat{\Phi}_0\}_{\hat{A}}, \eta_0\hat{\Phi}_0\}_{Q_B\hat{A}} + \{\{Q_B\hat{\Phi}_0, \eta_0\hat{\Phi}_0\}_{Q_B\hat{A}}, \eta_0\hat{\Phi}_0\}_{\hat{A}} \Big) * |3/2\rangle \\ & + \frac{1}{24}|3/2\rangle * \Big(\{\eta_0\hat{\Phi}_0, \{Q_B\hat{\Phi}_0, \hat{\Phi}_0\}_{Q_B\hat{A}}\}_{\eta_0\hat{A}} - \{Q_B\hat{\Phi}_0, \{\eta_0\hat{\Phi}_0, \hat{\Phi}_0\}_{Q_B\hat{A}}\}_{\eta_0\hat{A}} \\ & \qquad + \{Q_B\hat{\Phi}_0, \{\eta_0\hat{\Phi}_0, \hat{\Phi}_0\}_{\eta_0\hat{A}}\}_{Q_B\hat{A}} - \{\eta_0\hat{\Phi}_0, \{Q_B\hat{\Phi}_0, \hat{\Phi}_0\}_{\eta_0\hat{A}}\}_{Q_B\hat{A}} \Big) * |3/2\rangle,\end{aligned}$$

where

$$\begin{aligned}\{\Psi_1, \Psi_2\}_{\hat{A}} &= \Psi_1 * \hat{A} * \Psi_2 + \Psi_2 * \hat{A} * \Psi_1, \\ \{\Psi_1, \Psi_2\}_{\eta_0\hat{A}} &= \Psi_1 * \eta_0\hat{A} * \Psi_2 + \Psi_2 * \eta_0\hat{A} * \Psi_1, \\ \{\Psi_1, \Psi_2\}_{Q_B\hat{A}} &= \Psi_1 * Q_B\hat{A} * \Psi_2 + \Psi_2 * Q_B\hat{A} * \Psi_1 \,.\end{aligned}$$

In fact, it satisfies

$$\begin{aligned}\eta_0 Q_B\Phi_2 &= \frac{1}{2}(\{\eta_0\Phi_1, Q_B\Phi_0\} + \{\eta_0\Phi_0, Q_B\Phi_1\} + [\Phi_0, \eta_0 Q_B\Phi_1]) \\ &\quad -\frac{1}{6}(\{\eta_0\Phi_0, [\Phi_0, Q_B\Phi_0]\} + [\Phi_0, \{\eta_0\Phi_0, Q_B\Phi_0\}]) \\ &= \frac{1}{2}(\{\eta_0\Phi_1, Q_B\Phi_0\} + \{\eta_0\Phi_0, Q_B\Phi_1\}) \\ &\quad -\frac{1}{12}(\{[\eta_0\Phi_0, \Phi_0], Q_B\Phi_0\} + \{\eta_0\Phi_0, [\Phi_0, Q_B\Phi_0]\}) \\ &= \frac{1}{4}|3/2\rangle * (\{\{\eta_0\hat{\Phi}_0, Q_B\hat{\Phi}_0\}_{Q_B\hat{A}}, \eta_0\hat{\Phi}_0\} + \{\{Q_B\hat{\Phi}_0, \eta_0\hat{\Phi}_0\}_{\eta_0\hat{A}}, Q_B\hat{\Phi}_0\}) * |3/2\rangle \\ &\quad -\frac{1}{12}(\{[\eta_0\Phi_0, \Phi_0], Q_B\Phi_0\} + \{\eta_0\Phi_0, [\Phi_0, Q_B\Phi_0]\}) \,.\end{aligned}$$

• Closed form for all order terms?

Recently, Erler / Okawa constructed full order form in arXiv.0704.0930 / 0704.0936 [hep-th]

Here, we have generalized Okawa's solutions as in the bosonic case and found four types of solutions.

$$\Phi^{(r,s)}_{(1)} = \log(1+|r\rangle * f_{(1)} * |s\rangle), \qquad f_{(1)} = \frac{1}{1-\eta_0\hat{\Phi}_0 * Q_{\rm B}\hat{A}^{(r+s-1)}} * \hat{\Phi}_0,$$

$$\Phi^{(r,s)}_{(2)} = \log(1+|r\rangle * f_{(2)} * |s\rangle), \qquad f_{(2)} = \hat{\Phi}_0 * \frac{1}{1-\eta_0\hat{A}^{(r+s-1)} * Q_{\rm B}\hat{\Phi}_0},$$

$$\Phi^{(r,s)}_{(3)} = -\log(1-|r\rangle * f_{(3)} * |s\rangle), \qquad f_{(3)} = \frac{1}{1-Q_{\rm B}\hat{\Phi}_0 * \eta_0\hat{A}^{(r+s-1)}} * \hat{\Phi}_0,$$

$$\Phi^{(r,s)}_{(4)} = -\log(1-|r\rangle * f_{(4)} * |s\rangle), \qquad f_{(4)} = \hat{\Phi}_0 * \frac{1}{1-Q_{\rm B}\hat{A}^{(r+s-1)} * \eta_0\hat{\Phi}_0},$$

where

$$\hat{A}^{(r+s-1)} = -\frac{\pi}{2}\int_0^{\sqrt{r+s-2}} du \int_0^{\sqrt{r+s-2}} dv J_1^{--L}|uv+1\rangle - \left(\frac{\pi}{2}\right)^2 \int_0^{\sqrt{r+s-2}} du \int_0^{\sqrt{r+s-2}} dv\, uv\, \tilde{G}_1^{-L} B_1^L |uv+1\rangle\,,$$

$$\eta_0 Q_{\rm B}\hat{A}^{(r+s-1)} = I - |r+s-1\rangle\,,$$

$$\eta_0\hat{A}^{(r+s-1)} = -\frac{\pi}{2}\int_1^{r+s-1} dr B_1^L|r\rangle\,, \qquad Q_{\rm B}\hat{A}^{(r+s-1)} = -\frac{\pi}{2}\int_1^{r+s-1} dr \tilde{G}_1^{-L}|r\rangle\,.$$

$$
\begin{aligned}
\eta_0(e^{-\Phi^{(r,s)}_{(i)}} Q_{\rm B} e^{\Phi^{(r,s)}_{(i)}}) &= |r\rangle * \frac{1}{1+f_{(i)}*|r+s-1\rangle} \\
&\quad * \left(\eta_0 Q_{\rm B} f_{(i)} - \eta_0 f_{(i)} * |r+s-1\rangle * \frac{1}{1+f_{(i)}*|r+s-1\rangle} * Q_{\rm B} f_{(i)}\right) * |s\rangle \\
&= 0, \quad (i=1,2)
\end{aligned}
$$

$$
\begin{aligned}
\eta_0(e^{-\Phi^{(r,s)}_{(i)}} Q_{\rm B} e^{\Phi^{(r,s)}_{(i)}}) &= |r\rangle * \left(\eta_0 Q_{\rm B} f_{(i)} - Q_{\rm B} f_{(i)} * \frac{1}{1-|r+s-1\rangle * f_{(i)}} * |r+s-1\rangle * \eta_0 f_{(i)}\right) \\
&\quad * \frac{1}{1-|r+s-1\rangle * f_{(i)}} * |s\rangle \\
&= 0, \quad (i=3,4)
\end{aligned}
$$

In the case of $\Phi^{(r,s)}_{(3)}$, $\Phi^{(r,s)}_{(4)}$ and r=s=3/2, they reproduce Okawa's solutions.

As in the bosonic Schnabl type solution, we have found that the above solutions are obtained by gauge transformation from identity based solution $\hat{\Phi}_0$ in the following sense if we use "wedge state with negative angle."

$$
\begin{aligned}
e^{\Phi^{(r,s)}_{(1)}} &= \frac{1}{1+Q_{\rm B}(|r\rangle * \eta_0\hat{\Phi}_0 * \hat{A}^{(r+s-1)} * |2-r\rangle)} * |r\rangle * e^{\hat{\Phi}_0} * |2-r\rangle * (1-\eta_0(|r\rangle * \hat{\Phi}_0 * Q_{\rm B}\hat{A}^{(r+s-1)} * |2-r\rangle)) \\
&= e^{Q_{\rm B}\Lambda_{(1)}} * e^{|r\rangle * \hat{\Phi}_0 * |2-r\rangle} * e^{\eta_0\Lambda'_{(1)}}\,, \\
e^{\Phi^{(r,s)}_{(2)}} &= (1+Q_{\rm B}(|2-s\rangle * \eta_0\hat{A}^{(r+s-1)} * \hat{\Phi}_0 * |s\rangle)) * |2-s\rangle * e^{\hat{\Phi}_0} * |s\rangle * \frac{1}{1-\eta_0(|2-s\rangle * \hat{A}^{(r+s-1)} * Q_{\rm B}\hat{\Phi}_0 * |s\rangle)} \\
&= e^{Q_{\rm B}\Lambda_{(2)}} * e^{|2-s\rangle * \hat{\Phi}_0 * |s\rangle} * e^{\eta_0\Lambda'_{(2)}}\,, \\
e^{\Phi^{(r,s)}_{(3)}} &= \frac{1}{1-Q_{\rm B}(|r\rangle * \hat{\Phi}_0 * \eta_0\hat{A}^{(r+s-1)} * |2-r\rangle)} * |r\rangle * e^{\hat{\Phi}_0} * |2-r\rangle * (1+\eta_0(|r\rangle * Q_{\rm B}\hat{\Phi}_0 * \hat{A}^{(r+s-1)} * |2-r\rangle)) \\
&= e^{Q_{\rm B}\Lambda_{(3)}} * e^{|r\rangle * \hat{\Phi}_0 * |2-r\rangle} * e^{\eta_0\Lambda'_{(3)}}\,, \\
e^{\Phi^{(r,s)}_{(4)}} &= (1-Q_{\rm B}(|2-s\rangle * \hat{A}^{(r+s-1)} * \eta_0\hat{\Phi}_0 * |s\rangle)) * |2-s\rangle * e^{\hat{\Phi}_0} * |s\rangle * \frac{1}{1+\eta_0(|2-s\rangle * Q_{\rm B}\hat{A}^{(r+s-1)} * \hat{\Phi}_0 * |s\rangle)} \\
&= e^{Q_{\rm B}\Lambda_{(4)}} * e^{|2-s\rangle * \hat{\Phi}_0 * |s\rangle} * e^{\eta_0\Lambda'_{(4)}}\,,
\end{aligned}
$$

where gauge parameters are given by

$$
\begin{aligned}
&\Lambda_{(1)} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}C_{(1)} * (Q_{\rm B}C_{(1)})^{k-1}, \quad C_{(1)} \equiv |r\rangle * \eta_0\hat{\Phi}_0 * \hat{A}^{(r+s-1)} * |2-r\rangle, \quad \Lambda'_{(1)} = -\sum_{k=1}^{\infty}\frac{1}{k}C'_{(1)} * (\eta_0 C'_{(1)})^{k-1}, \quad C'_{(1)} \equiv |r\rangle * \hat{\Phi}_0 * Q_{\rm B}\hat{A}^{(r+s-1)} * |2-r\rangle, \\
&\Lambda_{(2)} = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k}C_{(2)} * (Q_{\rm B}C_{(2)})^{k-1}, \quad C_{(2)} \equiv |2-s\rangle * \eta_0\hat{A}^{(r+s-1)} * \hat{\Phi}_0 * |s\rangle, \quad \Lambda'_{(2)} = \sum_{k=1}^{\infty}\frac{1}{k}C'_{(2)} * (\eta_0 C'_{(2)})^{k-1}, \quad C'_{(2)} \equiv |2-s\rangle * \hat{A}^{(r+s-1)} * Q_{\rm B}\hat{\Phi}_0 * |s\rangle\,, \\
&\Lambda_{(3)} = \sum_{k=1}^{\infty}\frac{1}{k}C_{(3)} * (Q_{\rm B}C_{(3)})^{k-1}, \quad C_{(3)} \equiv |r\rangle * \hat{\Phi}_0 * \eta_0\hat{A}^{(r+s-1)} * |2-r\rangle, \quad \Lambda'_{(3)} = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k}C'_{(3)} * (\eta_0 C'_{(3)})^{k-1}, \quad C'_{(3)} \equiv |r\rangle * Q_{\rm B}\hat{\Phi}_0 * \hat{A}^{(r+s-1)} * |2-r\rangle\,, \\
&\Lambda_{(4)} = -\sum_{k=1}^{\infty}\frac{1}{k}C_{(4)} * (Q_{\rm B}C_{(4)})^{k-1}, \quad C_{(4)} \equiv |2-s\rangle * \hat{A}^{(r+s-1)} * \eta_0\hat{\Phi}_0 * |s\rangle, \quad \Lambda'_{(4)} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}C'_{(4)} * (\eta_0 C'_{(4)})^{k-1}, \quad C'_{(4)} \equiv |2-s\rangle * Q_{\rm B}\hat{A}^{(r+s-1)} * \hat{\Phi}_0 * |s\rangle\,.
\end{aligned}
$$

- In the case of singular super current?
- Generalization to super SFT on a <u>non-BPS</u> D-brane?
- Physical meaning of these solutions?
- ……

• On pure gauge form and induced string field redefinition

Suppose that the original (trivial) solutions can be written as pure gauge form,

$$\hat{\phi} = e^{-\hat{\psi}} * Q_{\mathrm{B}} e^{\hat{\psi}} \qquad\qquad e^{\hat{\Phi}_0} = e^{Q_{\mathrm{B}}\hat{\Lambda}_0} * e^{\eta_0 \hat{\Lambda}_1}$$

Then, our solutions can be written as pure gauge form:

$$\begin{aligned}
\Psi^{(r,s)} &= U^{(r,s)-1} * Q_{\mathrm{B}} U^{(r,s)}, \\
U^{(r,s)} &= 1 + |r\rangle * (e^{\hat{\psi}} - 1) * \frac{1}{1 + A^{(r+s-1)} * \hat{\phi}} * |s\rangle .
\end{aligned}$$

$$e^{\Phi_{(i)}^{(r,s)}} = U_{(i)}^{(r,s)} * V_{(i)}^{(r,s)}, \qquad Q_{\mathrm{B}} U_{(i)}^{(r,s)} = 0, \quad \eta_0 V_{(i)}^{(r,s)} = 0.$$

where

$$\begin{aligned}
U_{(1)}^{(r,s)-1} &= 1 + |r\rangle * (e^{-Q_{\mathrm{B}}\hat{\Lambda}_0} - 1) * \frac{1}{1 - Q_{\mathrm{B}}\hat{A}^{(r+s-1)} * \eta_0 \hat{\Phi}_0} * |s\rangle , \\
V_{(1)}^{(r,s)-1} &= \left[1 - |r\rangle * \frac{1}{1 - \eta_0(\hat{\Phi}_0 * Q_{\mathrm{B}}\hat{A}^{(r+s-1)})} * \hat{\Phi}_0 * |s\rangle\right] * U_{(1)}^{(r,s)}, \\
&\;\vdots
\end{aligned}$$

This implies that around them, the action can be re-expanded as:

$$S[\Psi^{(r,s)} + \Psi] \;=\; S[\Psi^{(r,s)}] + S[U^{(r,s)} * \Psi * U^{(r,s)-1}]$$

$$S[\log(e^{\Phi_{(i)}^{(r,s)}} * e^{\Phi})] \;=\; S[\Phi_{(i)}^{(r,s)}] + S[V_{(i)}^{(r,s)} * \Phi * V_{(i)}^{(r,s)-1}]$$

For light-cone Wilson line solution $\lambda_\mu \lambda_\nu \eta^{\mu\nu} = 0$, similar string field redefinitions are induced in bosonic and super SFT:

$$\hat{\psi} \;=\; U_1^\dagger U_1 i\lambda_\mu X^\mu(0)|0\rangle$$

$$U^{(r,s)} * \Psi * U^{(r,s)-1} \;=\; \Psi + \psi^{(r,s)} * \Psi - \Psi * \psi^{(r,s)} + \mathcal{O}(\lambda^2),$$
$$\psi^{(r,s)} \equiv |r\rangle * \hat{\psi} * |s\rangle\,.$$

$$\hat{\Lambda}_1 = U_1^\dagger U_1 \xi i\lambda_\mu X^\mu(0)|0\rangle, \quad \hat{\Lambda}_0 = U_1^\dagger U_1 c\xi\partial\xi e^{-2\phi} i\lambda_\mu X^\mu(0)|0\rangle$$

$$V_{(i)}^{(r,s)} * \Phi * V_{(i)}^{(r,s)-1} \;=\; \Phi + \phi^{(r,s)} * \Phi - \Phi * \phi^{(r,s)} + \mathcal{O}(\lambda^2),$$
$$\phi^{(r,s)} \equiv |r\rangle * \eta_0 \hat{\Lambda}_1 * |s\rangle, \quad i = 1,2,3,4.$$