Comments on marginal and scalar solutions in open string field theory

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References: I.K., Y.Michishita, “Comments on Solutions for Nonsingular Currents in Open String Field Theories,” arXiv:0706.0409 [hep-th]
I.K., “弦の場の理論における解析解についての最近の進展,” 素粒子論研究114-6, F-13 (2007-3).

Collaboration with Y. Michishita, T. Takahashi, S. Zeze

Introduction

- Witten's bosonic open string field theory (d=26):

$$S[\Psi] = -\frac{1}{g^2}\left(\frac{1}{2}\langle\Psi, Q_{\mathrm{B}}\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right).$$

- There were various attempts to prove Sen's conjecture since around 1999 using the above.
- Numerically, it has been checked with "level truncation approximation." [c.f. … Gaiotto-Ratelli "Experimental string field theory"(2002)]
- Analytically, some solutions have been constructed.
- Here, we generalize "Schnabl's analytical solutions" (2005, 2007) which include "tachyon vacuum solution" in Sen's conjecture and "marginal solutions."

- In Berkovits' WZW-type superstring field theory (d=10) the action in the NS sector is given by

$$S_{\mathrm{NS}}[\Phi] = -\frac{1}{g^2}\int_0^1 dt \langle\langle (\eta_0 \Phi)(e^{-t\Phi} Q_{\mathrm{B}} e^{t\Phi}) \rangle\rangle \,.$$

- There were some attempts to solve the equation of motion.
- Numerically, tachyon condensation was examined using level truncation. [Berkovits(-Sen-Zwiebach)(2000),…]

- Analytically, some solutions have been constructed.
- Recently [April (2007)], Erler / Okawa constructed some solutions, which are generalization of Schnabl / Kiermaier-Okawa-Rastelli-Zwiebach's marginal solution in bosonic SFT.

We consider generalization of their solutions and examined gauge transformations.

Main claim

Suppose that $\hat{\psi}$ is BRST invariant and nilpotent:

$$Q_B\hat{\psi} = 0, \quad \hat{\psi} * \hat{\psi} = 0.$$

Then,

$$\Psi^{(\alpha,\beta)} = P_\alpha * \frac{1}{1 + \hat{\psi} * A^{(\alpha+\beta)}} * \hat{\psi} * P_\beta$$

gives a solution to the EOM: $Q_B\Psi^{(\alpha,\beta)} + \Psi^{(\alpha,\beta)} * \Psi^{(\alpha,\beta)} = 0,$

where

$$Q_B P_\alpha = 0, \quad P_\alpha * P_\beta = P_{\alpha+\beta}, \quad P_{\alpha=0} = I,$$
$$Q_B A^{(\gamma)} = I - P_\gamma.$$

In the case of $|r = \alpha + 1\rangle = P_\alpha$:wedge state, we have $A^{(\gamma)} = \frac{\pi}{2}\int_0^\gamma d\alpha B_1^L P_\alpha.$

$\hat{\psi} = U_1^\dagger U_1 \lambda J(0)|0\rangle,$ $\alpha = \beta = 1/2$: Schnabl/Kiermaier-Okawa-Rastelli-Zwiebach's marginal solution for nonsingular current is reproduced.

$\hat{\psi} = \hat{\lambda} Q_B U_1^\dagger U_1 B_1^L c_1|0\rangle,$ $\alpha = \beta = 1/2, \ \hat{\lambda} = \infty$: Schnabl's tachyon vacuum solution is reproduced.

Suppose that $\hat{\phi}$ satisfies following conditions:

$$\eta_0 Q_{\rm B}\hat{\phi} = 0, \quad \hat{\phi} * \hat{\phi} = 0, \quad \hat{\phi} * \eta_0\hat{\phi} = 0, \quad \hat{\phi} * Q_{\rm B}\hat{\phi} = 0\,.$$

Then,

$$\Phi_{(1)}^{(\alpha,\beta)} = \log(1 + P_\alpha * f_{(1)} * P_\beta), \qquad f_{(1)} = \frac{1}{1 - \eta_0\hat{\phi} * Q_{\rm B}\hat{A}^{(\alpha+\beta)}} * \hat{\phi},$$

$$\Phi_{(2)}^{(\alpha,\beta)} = \log(1 + P_\alpha * f_{(2)} * P_\beta), \qquad f_{(2)} = \hat{\phi} * \frac{1}{1 - \eta_0\hat{A}^{(\alpha+\beta)} * Q_{\rm B}\hat{\phi}},$$

$$\Phi_{(3)}^{(\alpha,\beta)} = -\log(1 - P_\alpha * f_{(3)} * P_\beta), \qquad f_{(3)} = \frac{1}{1 - Q_{\rm B}\hat{\phi} * \eta_0\hat{A}^{(\alpha+\beta)}} * \hat{\phi},$$

$$\Phi_{(4)}^{(\alpha,\beta)} = -\log(1 - P_\alpha * f_{(4)} * P_\beta), \qquad f_{(4)} = \hat{\phi} * \frac{1}{1 - Q_{\rm B}\hat{A}^{(\alpha+\beta)} * \eta_0\hat{\phi}},$$

give solutions to the EOM: $\eta_0(e^{-\Phi_{(i)}^{(\alpha,\beta)}} Q_{\rm B} e^{\Phi_{(i)}^{(\alpha,\beta)}}) = 0\,, \qquad (i = 1, 2, 3, 4)$

where $\eta_0 P_\alpha = 0, \quad Q_{\rm B}P_\alpha = 0, \quad P_\alpha * P_\beta = P_{\alpha+\beta}\,, \quad P_{\alpha=0} = I\,,$

$$\eta_0 Q_{\rm B}\hat{A}^{(\gamma)} = I - P_\gamma\,.$$

In the case of P_α: wedge state, we find $\hat{A}^{(\gamma)} = \int_0^\gamma d\alpha \log\left(\frac{\alpha}{\gamma}\right)\left(\frac{\pi}{2}J_1^{--L} + \alpha\frac{\pi^2}{4}\tilde{G}_1^{-L}B_1^L\right)P_\alpha.$

$$\hat{\phi} = \zeta_a U_1^\dagger U_1 c\xi e^{-\phi}\psi^a(0)|0\rangle\,, \qquad \zeta_a\zeta_b\Omega^{ab} = 0\,, \qquad \alpha = \beta = 1/2$$

: Erler / Okawa's marginal solutions for nonsingular supercurrents are reproduced.

Contents

- Introduction
- Witten’s bosonic SFT and its solutions
  - Marginal solutions
  - Tachyon solutions
- Berkovits’ WZW-type super SFT and its solutions
- Gauge transformations
- Future problems

Witten's bosonic open string field theory

Action:
$$S[\Psi] = -\frac{1}{g^2}\left(\frac{1}{2}\langle\Psi, Q_{\mathrm{B}}\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right)$$

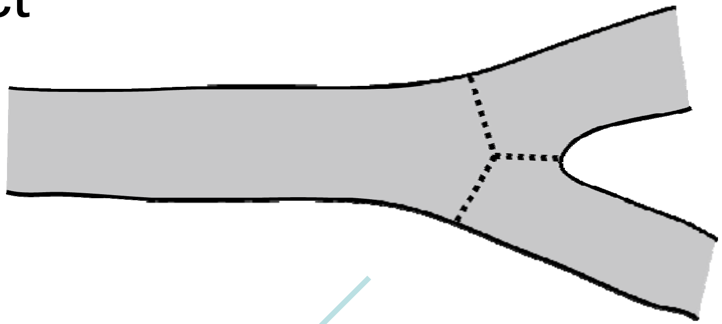
String field: (infinitely many fields are included.)

$$|\Psi\rangle = \phi(x)c_1|0\rangle + A_\mu(x)\alpha^\mu_{-1}c_1|0\rangle + iB(x)c_0|0\rangle + \cdots$$

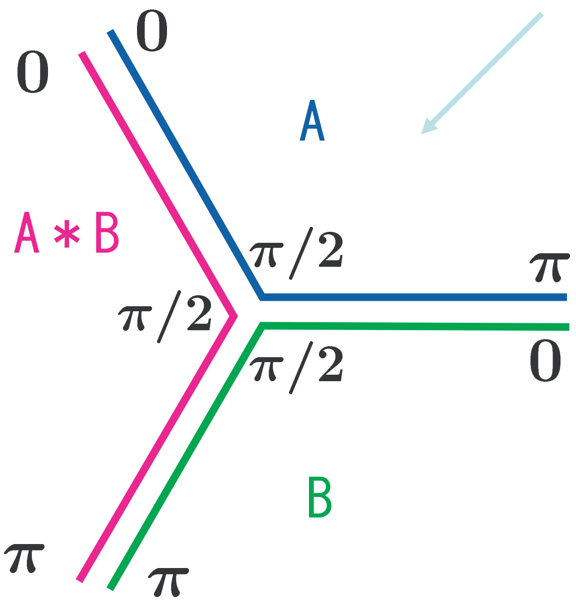
BRST operator:
$$Q_{\mathrm{B}} = \oint \frac{dz}{2\pi i}\left(cT^{\mathrm{m}} + bc\partial c + \frac{3}{2}\partial^2 c\right)$$
(nilpotent for c^m =26.)

Kinetic term:
$$\langle\Psi, Q_{\mathrm{B}}\Psi\rangle = \int d^{26}x\left(\phi(-\alpha'\partial^2 - 1)\phi - \alpha' A_\mu\partial^2 A^\mu + 2\sqrt{2\alpha'}B\partial_\mu A^\mu + 2B^2 + \cdots\right)$$

Interaction term: the Witten star product



$$\int dx\, (\phi(x))^3$$
$$= \int dx_1 dx_2 dx_3 \delta(x_1 - x_2)\delta(x_2 - x_3) \phi(x_1)\phi(x_2)\phi(x_3)$$



$$\langle \Psi, \Psi * \Psi \rangle$$
$$= \langle V_3(1,2,3)|\Psi\rangle_1 |\Psi\rangle_2 |\Psi\rangle_3$$
$$\sim \int \prod_{0 \le \sigma \le \pi/2} \big(\delta(X^{(1)}(\pi - \sigma) - X^{(2)}(\sigma))\delta(X^{(2)}(\pi - \sigma) - X^{(3)}(\sigma))$$
$$\times\, \delta(X^{(3)}(\pi - \sigma) - X^{(1)}(\sigma))(bc \text{ ghost} \cdots)\big)$$
$$\times\, \Psi[X^{(1)}(\sigma), \cdots]\Psi[X^{(2)}(\sigma), \cdots]\Psi[X^{(3)}(\sigma), \cdots]$$

equation of motion: $$Q_{\mathrm{B}}\Psi + \Psi * \Psi = 0$$

gauge transformation: $$\delta_{\Lambda}\Psi = Q_{\mathrm{B}}\Lambda + \Psi * \Lambda - \Lambda * \Psi$$

$$\longrightarrow \quad \delta_{\Lambda} S = 0$$

(※) $$Q_{\mathrm{B}}^2 = 0, \quad \langle A, Q_{\mathrm{B}} B\rangle = -(-1)^{|A|}\langle Q_{\mathrm{B}} A, B\rangle,$$

$$Q_{\mathrm{B}}(A * B) = (Q_{\mathrm{B}} A) * B + (-1)^{|A|} A * (Q_{\mathrm{B}} B),$$

$$\langle A, B\rangle = (-1)^{|A||B|}\langle B, A\rangle, \qquad \langle A, B * C\rangle = \langle A * B, C\rangle,$$

$$(A * B) * C = A * (B * C)$$: associative

Note : $A * B \neq B * A$ in general.

Preliminary

- "sliver frame": $\tilde{z} = \arctan z$ (z :UHP)

  For a primary field ϕ with dim=h,

$$\tilde{\phi}(\tilde{z}) = \left(\frac{dz}{d\tilde{z}}\right)^h \phi(z) = (\cos\tilde{z})^{-2h}\phi(\tan\tilde{z}),$$

$$\tilde{\phi}(\tilde{z}) = \sum_n \tilde{\phi}_n \tilde{z}^{-n-h}, \qquad \phi(z) = \sum_n \phi_n z^{-n-h},$$

$$\tilde{\phi}_n = \oint_0 \frac{d\tilde{z}}{2\pi i}\tilde{z}^{n+h-1}\tilde{\phi}(\tilde{z}) = \oint_0 \frac{dz}{2\pi i}(\arctan z)^{n+h-1}(1+z^2)^{h-1}\phi(z)$$

$$= \sum_{m=n}^{\infty}\phi_m \oint_0 \frac{d\tilde{z}}{2\pi i}\tilde{z}^{n+h-1}(\cos\tilde{z})^{-2h}(\tan\tilde{z})^{-m-h} = \sum_{m=n}^{\infty}\phi_m \oint_0 \frac{dz}{2\pi i}(\arctan z)^{n+h-1}(1+z^2)^{h-1}z^{-m-h},$$

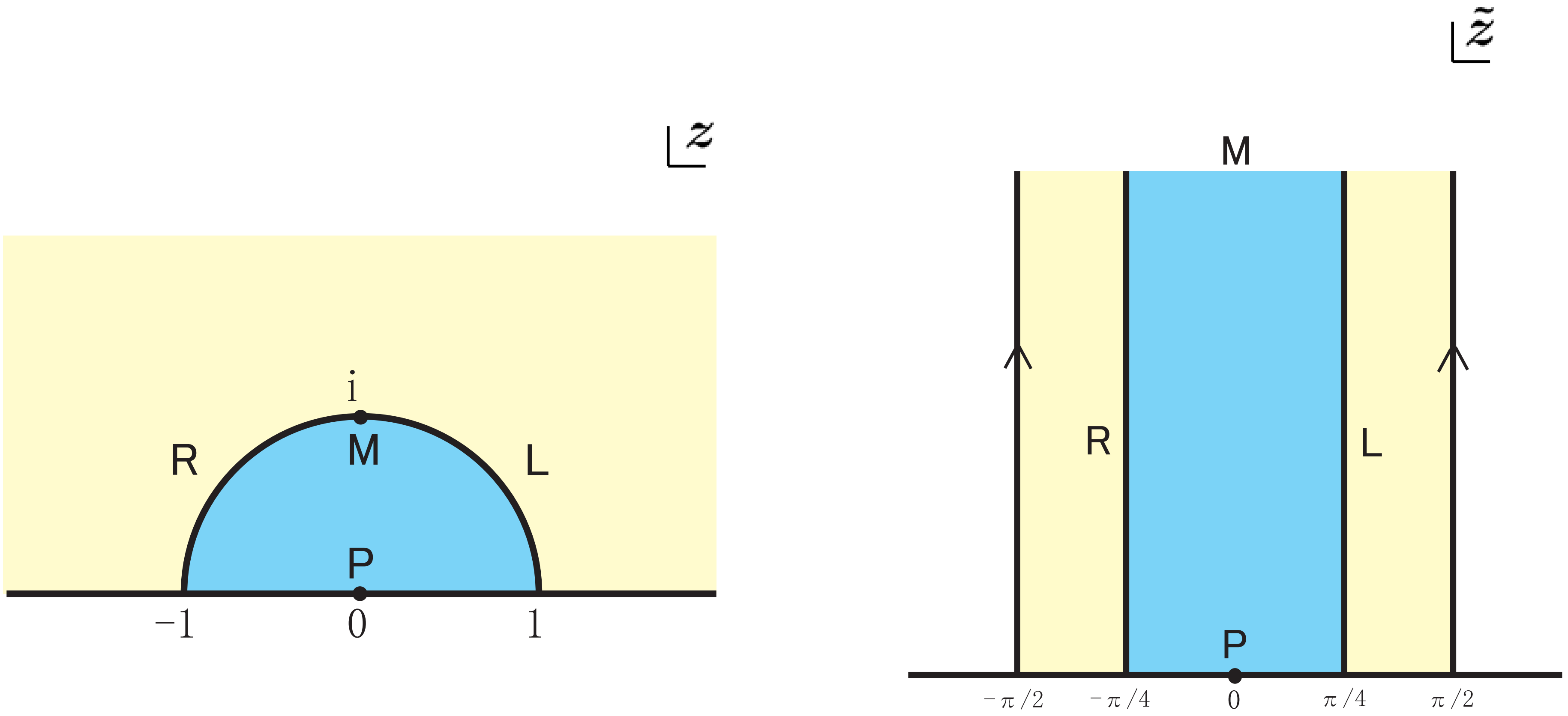
In particular, we often use

$$\mathcal{L}_0 \equiv \tilde{L}_0 = L_0 + \sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{4k^2-1}L_{2k}, \qquad K_1 \equiv \tilde{L}_{-1} = L_1 + L_{-1},$$

$$\mathcal{B}_0 \equiv \tilde{b}_0 = b_0 + \sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{4k^2-1}b_{2k}, \qquad B_1 \equiv \tilde{b}_{-1} = b_1 + b_{-1},$$

and

$$\hat{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_0^\dagger, \quad K_1^{L/R} = \frac{1}{2}K_1 \pm \frac{1}{\pi}\hat{\mathcal{L}}, \quad \hat{\mathcal{B}} = \mathcal{B}_0 + \mathcal{B}_0^\dagger, \quad B_1^{L/R} = \frac{1}{2}B_1 \pm \frac{1}{\pi}\hat{\mathcal{B}}.$$



$$\arctan z = \tilde{z}$$

Using $U_r = \left(\frac{2}{r}\right)^{\mathcal{L}_0} = \left(\frac{2}{r}\right)^{L_0} e^{-\frac{r^2-4}{3r^2}L_2 + \frac{r^4-16}{30r^4}L_4 + \cdots}$ we have a formula for the star product:

$$U_r^\dagger U_r \tilde{\phi}_1(\tilde{x}_1)\cdots\tilde{\phi}_n(\tilde{x}_n)|0\rangle * U_s^\dagger U_s \tilde{\psi}_1(\tilde{y}_1)\cdots\tilde{\psi}_m(\tilde{y}_m)|0\rangle$$
$$= U_{r+s-1}^\dagger U_{r+s-1}\tilde{\phi}_1(\tilde{x}_1 + \frac{\pi}{4}(s-1))\cdots\tilde{\phi}_n(\tilde{x}_n + \frac{\pi}{4}(s-1))\tilde{\psi}_1(\tilde{y}_1 - \frac{\pi}{4}(r-1))\cdots\tilde{\psi}_m(\tilde{y}_m - \frac{\pi}{4}(r-1))|0\rangle$$

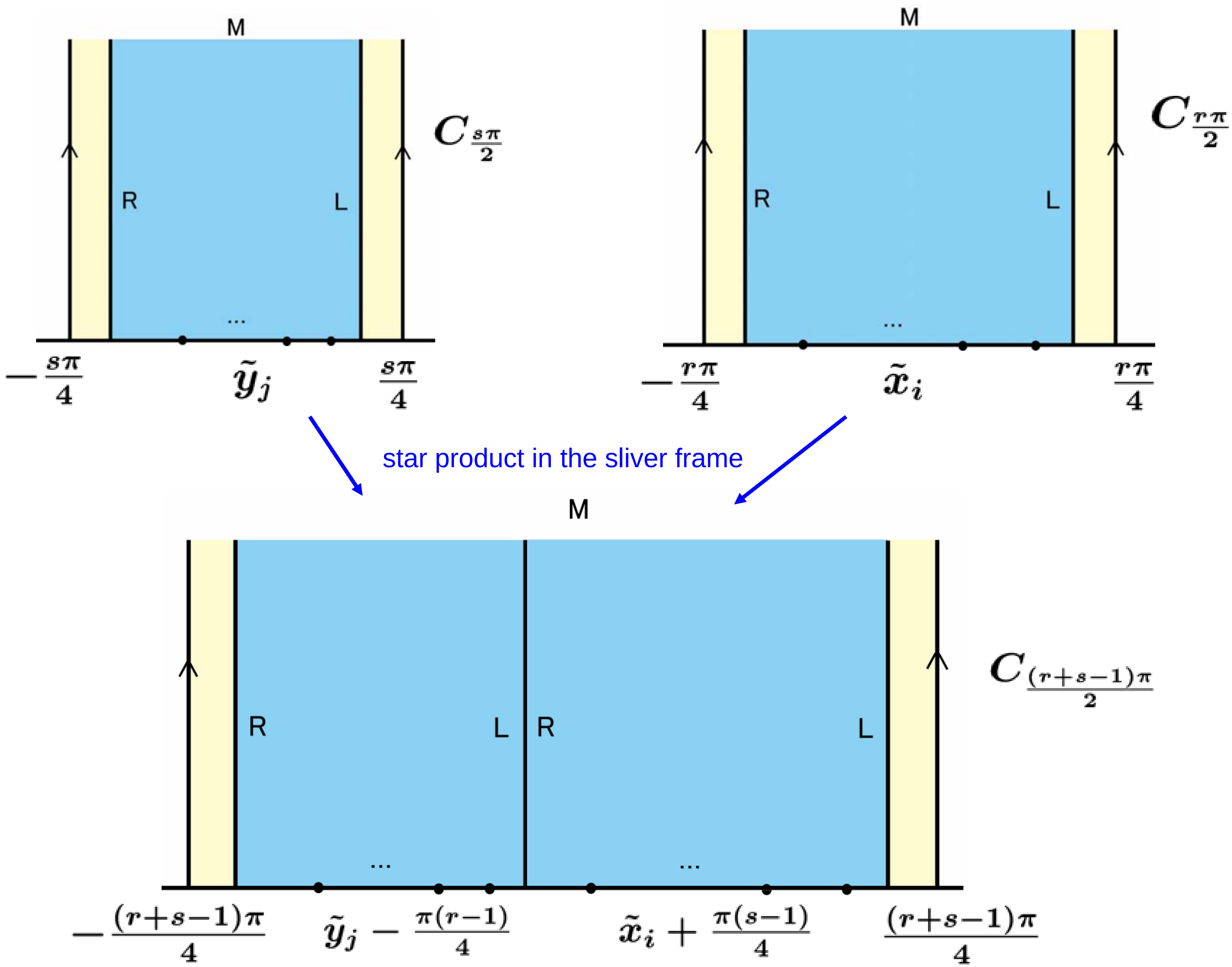
In particular, for the wedge state: $|r = \alpha + 1\rangle = U_{\alpha+1}^\dagger U_{\alpha+1}|0\rangle = P_\alpha$

$$|r\rangle * |s\rangle = |r + s - 1\rangle \longleftrightarrow P_\alpha * P_\beta = P_{\alpha+\beta}$$

$|r = 1\rangle = P_{\alpha=0} = I$: identity state

$|r = 2\rangle = P_{\alpha=1} = |0\rangle$: conformal vacuum

$|r = \infty\rangle = P_\infty$: sliver state


M
R
L
...
$C_{\frac{s\pi}{2}}$
$-\frac{s\pi}{4}$
$\tilde{y}_j$
$\frac{s\pi}{4}$
M
R
L
...
$C_{\frac{r\pi}{2}}$
$-\frac{r\pi}{4}$
$\tilde{x}_i$
$\frac{r\pi}{4}$
star product in the sliver frame
M
R
L
R
L
...
...
$C_{\frac{(r+s-1)\pi}{2}}$
$-\frac{(r+s-1)\pi}{4}$
$\tilde{y}_j - \frac{\pi(r-1)}{4}$
$\tilde{x}_i + \frac{\pi(s-1)}{4}$
$\frac{(r+s-1)\pi}{4}$

Note: the wedge state can be rewritten as

$$|r = \alpha + 1\rangle = e^{-\frac{r-2}{2}\hat{\mathcal{L}}}|0\rangle = P_\alpha = e^{-\alpha\frac{\pi}{2}K_1^L}|I\rangle$$

As a surface state, $r \geq 1$ for the wedge state.

↔ $P_\alpha \quad (\alpha \geq 0)$ (commutative monoid)

However, *if one uses the last expression formally,* the wedge state with "negative angle" $r < 1$, which satisfies $|r\rangle * |s\rangle = |r + s - 1\rangle$, might be considered.

In fact, this algebra can be formally obtained using following properties:

$$A * I = I * A = A, \qquad \forall A,$$
$$K_1^L(A * B) = (K_1^L A) * B, \quad \forall A, B.$$

⇨ $$|r = \alpha + 1\rangle = P_\alpha = \exp\left(-\alpha\frac{\pi}{2}K_1^L I\right).$$

$P_\alpha \quad (\alpha \in \mathbb{R})$ (Abelian group) ??

Wedge state

$|r\rangle$ is defined by

$$f_r(z) = h^{-1}(h(z)^{\frac{2}{r}}) = \tan\left(\frac{2}{r}\arctan z\right),$$

$$h(z) = \frac{1+iz}{1-iz}$$

such as

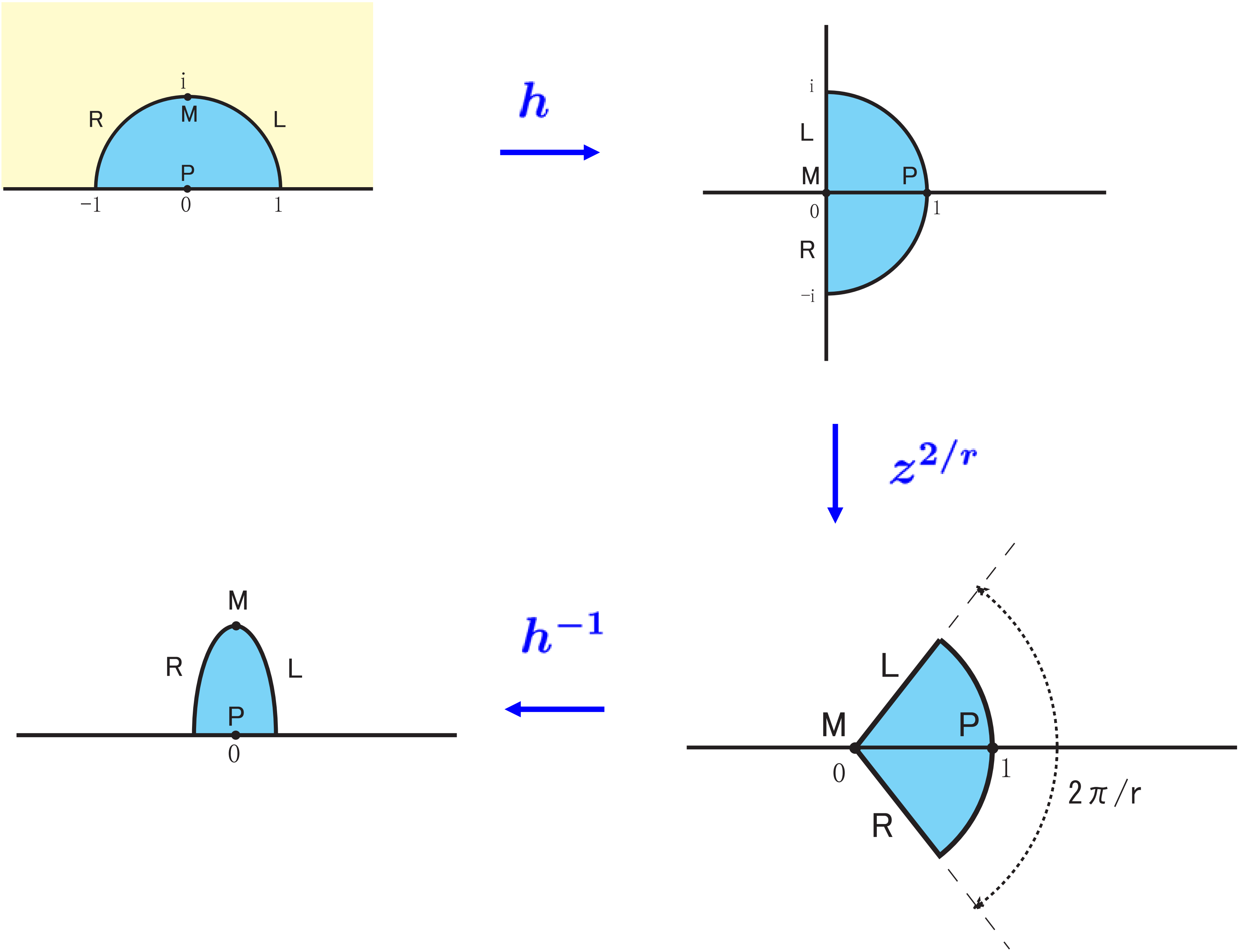
$$\langle r|\phi\rangle = \langle f_r[\phi(0)]\rangle_{\mathrm{UHP}}, \qquad \forall\phi(z)$$

expressed as

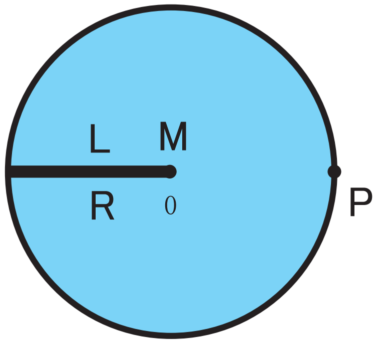
$$\langle r| = \langle 0|U_r, \qquad U_r = \left(\frac{2}{r}\right)^{\mathcal{L}_0},$$

$$|r\rangle = U_r^\dagger|0\rangle, \qquad U_r^\dagger = \left(\frac{2}{r}\right)^{\mathcal{L}_0^\dagger},$$

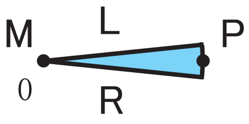
$$\mathcal{L}_0^\dagger = L_0 + \sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{4k^2-1}L_{-2k}.$$


i
R
M
L
P
−1
0
1
h
i
L
M
P
0
1
R
−i
$z^{2/r}$
M
R
L
P
0
h^{-1}
L
M
P
0
1
R
2π/r

For the identity state



For the sliver state



Noting

$$U_r = \left(\frac{2}{r}\right)^{\mathcal{L}_0} = \left(\frac{2}{r}\right)^{L_0} e^{-\frac{r^2-4}{3r^2}L_2 + \frac{r^4-16}{30r^4}L_4 + \cdots}$$

we have

$$\langle\infty| = \lim_{r\to\infty}\langle 0|U_r = \langle 0|U_{\arctan} = \langle 0|U^{-1}_{\tan},$$

$$\lim_{r\to\infty}\frac{r}{2}f_r(z) = \lim_{r\to\infty}\frac{r}{2}\tan\left(\frac{2}{r}\arctan z\right) = \arctan z\,.$$

- Associated with the wedge states, we have

$$A^{(\gamma)} = \frac{\pi}{2}\int_0^{\gamma} d\alpha\, B_1^L P_\alpha$$ such as $$Q_{\rm B} A^{(\gamma)} = I - P_\gamma$$.

↑

$$\{Q_{\rm B}, B_1^L\} = K_1^L$$

With BRST invariant and nilpotent $\hat{\psi}$: $Q_{\rm B}\hat{\psi} = 0, \quad \hat{\psi} * \hat{\psi} = 0,$

we have solution to the equation of motion

$$\begin{aligned} \Psi^{(\alpha,\beta)} &= P_\alpha * \frac{1}{1 + \hat{\psi} * A^{(\alpha+\beta)}} * \hat{\psi} * P_\beta \\ &= \sum_{k=0}^{\infty} (-1)^k P_\alpha * (\hat{\psi} * A^{(\alpha+\beta)})^k * \hat{\psi} * P_\beta \,. \end{aligned}$$

$$
\because \quad Q_{\mathrm{B}}\Psi^{(\alpha,\beta)} = P_\alpha * Q_{\mathrm{B}}\left(\frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}}\right) * P_\beta
$$

$$
= -P_\alpha * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * (Q_{\mathrm{B}}(I+\hat{\psi}*A^{(\alpha+\beta)})) * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * \hat{\psi} * P_\beta
$$

$$
= P_\alpha * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * \hat{\psi} * (Q_{\mathrm{B}}A^{(\alpha+\beta)}) * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * \hat{\psi} * P_\beta
$$

$$
= P_\alpha * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * \hat{\psi} * (I - P_{\alpha+\beta}) * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * \hat{\psi} * P_\beta
$$

$$
= P_\alpha * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * \underbrace{\hat{\psi} * \hat{\psi}}_{=0} * \frac{1}{1+A^{(\alpha+\beta)}*\hat{\phi}} * P_\beta
$$

$$
-P_\alpha * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * \hat{\psi} * P_\beta * P_\alpha * \frac{1}{1+\hat{\psi}*A^{(\alpha+\beta)}} * \hat{\psi} * P_\beta
$$

$$
= -\Psi^{(\alpha,\beta)} * \Psi^{(\alpha,\beta)} .
$$

In general,

$$
Q_{\mathrm{B}}\Psi^{(\alpha,\beta)}(\psi) + \Psi^{(\alpha,\beta)}(\psi) * \Psi^{(\alpha,\beta)}(\psi)
$$

$$
= P_\alpha * \frac{1}{1+\psi*A^{(\alpha+\beta)}} * (Q_{\mathrm{B}}\psi + \psi * \psi) * \frac{1}{1+A^{(\alpha+\beta)}*\psi} * P_\beta .
$$

Note 1.

$\hat{\psi}$ itself is a solution and $\lambda\hat{\psi}$ is also a solution.
$\Psi^{(\alpha,\beta)}$ solution can naturally include 1-parameter.

Note 2.

We can regard $\psi \mapsto \Psi^{(\alpha,\beta)}(\psi) = P_\alpha * \dfrac{1}{1+\psi * A^{(\alpha+\beta)}} * \psi * P_\beta$

as a map *from general solution to solution*.

$$Q_{\rm B}\psi + \psi * \psi = 0$$
$$\to \quad Q_{\rm B}\Psi^{(\alpha,\beta)}(\psi) + \Psi^{(\alpha,\beta)}(\psi) * \Psi^{(\alpha,\beta)}(\psi) = 0$$

Composition of maps forms a commutative monoid:

$$\Psi^{(\alpha,\beta)}(\Psi^{(\alpha',\beta')}(\psi)) = \Psi^{(\alpha+\alpha',\beta+\beta')}(\psi), \quad (\alpha,\beta,\alpha',\beta' \geq 0)$$

$$\Psi^{(0,0)}(\psi) = \psi\,.$$

- Example of BRST invariant and nilpotent $\hat{\psi}$

$$\hat{\psi} = \lambda_{\rm s}\hat{\psi}_{\rm s} + \lambda_{\rm m}\hat{\psi}_{\rm m}\,,$$
$$\hat{\psi}_{\rm s} = Q_{\rm B}\hat{\Lambda}_0\,, \qquad \hat{\Lambda}_0 \equiv U_1^\dagger U_1 B_1^L c_1|0\rangle\,,$$
$$\hat{\psi}_{\rm m} = U_1^\dagger U_1 cJ(0)|0\rangle\,.$$

where $J(z) = \zeta_a J^a(z)$ is “nonsingular” matter primary with dimension 1:

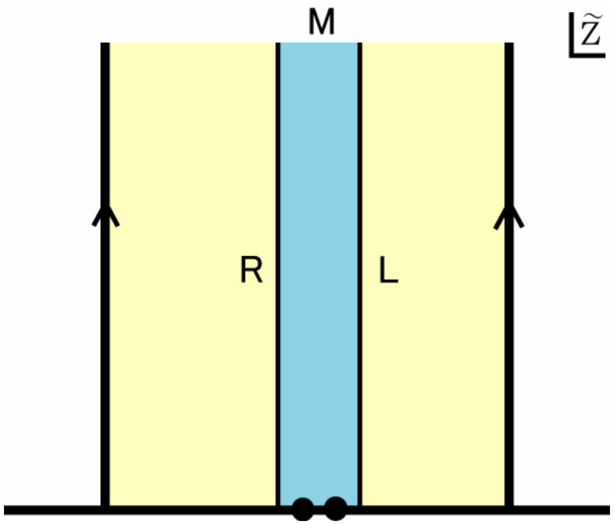
$$\zeta_a\zeta_b g^{ab} = 0\,,$$
$$J^a(y)J^b(z) \sim \frac{g^{ab}}{(y-z)^2} + \frac{1}{y-z} i f^{ab}{}_c J^c(z) + \cdots\,.$$

In particular,

$\lambda_{\rm s} = 0$ ⟹ marginal solution

$\lambda_{\rm m} = 0$ ⟹ tachyon solution

Due to the nonsingular condition for the current, we find nilpotency with respect to the star product: $\hat{\psi}_{\mathrm{m}} * \hat{\psi}_{\mathrm{m}} = 0$.



$$c\zeta_a J^a(\epsilon)\, c\zeta_b J^b(0) \sim 0$$

Marginal solution

From a BRST invariant, nilpotent $\hat{\psi}_{\rm m} = U_1^\dagger U_1 cJ(0)|0\rangle$ which satisfies

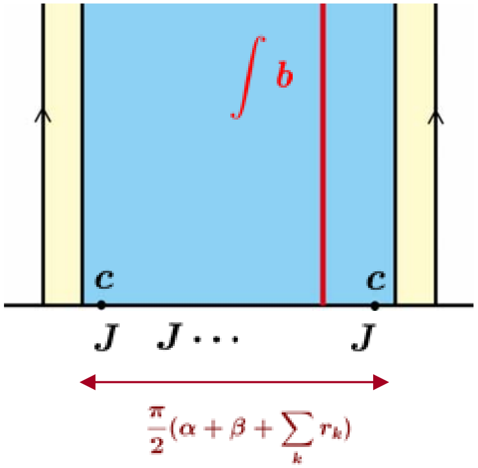
$(\mathcal{B}_0 - \mathcal{B}_0^\dagger)\hat{\psi}_{\rm m} = 0$, we can generate a solution

$$\Psi^{(\alpha,\beta)} = \sum_{k=0}^{\infty}(-1)^k \lambda_{\rm m}^{k+1} P_\alpha * (\hat{\psi}_{\rm m} * A^{(\alpha+\beta)})^k * \hat{\psi}_{\rm m} * P_\beta = \sum_{n=1}^{\infty} \lambda_{\rm m}^n \psi_{{\rm m},n} ,$$

$$\psi_{{\rm m},1} = U^\dagger_{\alpha+\beta+1} U_{\alpha+\beta+1} \tilde{c}\tilde{J}(\frac{\pi}{4}(\beta-\alpha))|0\rangle ,$$

$$\psi_{{\rm m},k+1} = \left(-\frac{\pi}{2}\right)^k \int_0^{\alpha+\beta} dr_1 \cdots \int_0^{\alpha+\beta} dr_k \, U^\dagger_{\alpha+\beta+1+\sum_{l=1}^k r_l} U_{\alpha+\beta+1+\sum_{l=1}^k r_l} \prod_{m=0}^{k} \tilde{J}\left(\frac{\pi}{4}(\beta-\alpha-\sum_{l=1}^{m} r_l + \sum_{l=m+1}^{k} r_l)\right)$$
$$\times\left[-\frac{1}{\pi}\hat{\mathcal{B}}\tilde{c}(\frac{\pi}{4}(\beta-\alpha+\sum_{l=1}^k r_l))\tilde{c}(\frac{\pi}{4}(\beta-\alpha-\sum_{l=1}^k r_l)) + \frac{1}{2}\left(\tilde{c}(\frac{\pi}{4}(\beta-\alpha+\sum_{l=1}^k r_l)) + \tilde{c}(\frac{\pi}{4}(\beta-\alpha-\sum_{l=1}^k r_l))\right)\right]|0\rangle.$$

$$\Psi^{(\alpha,\beta)} \sim \sum \lambda_{\rm m}^n \int dr_k$$



They satisfy $Q_{\rm B}\psi_{{\rm m},1} = 0\,,\quad \mathcal{B}^{(\alpha,\beta)}\psi_{{\rm m},1} = 0\,,\quad \psi_{{\rm m},k+1} = -\frac{\mathcal{B}^{(\alpha,\beta)}}{\mathcal{L}^{(\alpha,\beta)}}\sum_{l=1}^{k}\psi_{{\rm m},l}*\psi_{{\rm m},k-l+1}\,,$

where

$$\mathcal{B}^{(\alpha,\beta)} = \frac{1}{2}(\alpha+\beta-1)\hat{\mathcal{B}} + \mathcal{B}_0 + \frac{\pi}{4}(\alpha-\beta)B_1\,,$$
$$\mathcal{L}^{(\alpha,\beta)} \equiv \{Q_{\rm B},\mathcal{B}^{(\alpha,\beta)}\} = \frac{1}{2}(\alpha+\beta-1)\hat{\mathcal{L}} + \mathcal{L}_0 + \frac{\pi}{4}(\alpha-\beta)K_1\,.$$

In particular, this solution satisfies a “generalized Schnabl gauge”: $\mathcal{B}^{(\alpha,\beta)}\Psi^{(\alpha,\beta)} = 0.$

At each order, they satisfy the equation of motion: $Q_{\rm B}\psi_{{\rm m},k+1} + \sum_{l=1}^{k}\psi_{{\rm m},l}*\psi_{{\rm m},k-l+1} = 0\,.$

Note 1:

In the case of $\alpha = \beta = 1/2$, the above formula reproduces the marginal solution by Schnabl / Kiermaier-Okawa-Rastelli-Zwiebach.

Note 2:

As examples of nonsingular current, we can take

$J = :e^{X^0}:$ rolling tachyon

$$\Psi^{(\alpha,\alpha)} = \left[\lambda_{\rm m}e^{X^0} - \frac{64\cot^3\frac{\pi(2\alpha+1)}{2(4\alpha+1)}}{3(4\alpha+1)^3}\lambda_{\rm m}^2 e^{2X^0} + \cdots + (\sim\alpha^{-k^2-2k}\ \text{for}\ \alpha\gg 1)\lambda_{\rm m}^{k+1}e^{(k+1)X^0}\right]c_1|0\rangle + \cdots$$

$J = i\partial X^+$ light-like deformation $\Psi^{(\alpha,\alpha)} = \left[\lambda_{\rm m}\alpha_{-1}^+ - \frac{4\cot\frac{\pi(2\alpha+1)}{2(4\alpha+1)}}{4\alpha+1}\lambda_{\rm m}^2\alpha_{-1}^+\alpha_{-1}^+ + \cdots\right]c_1|0\rangle + \cdots$

Tachyon solution

- From a BRST invariant, nilpotent $\hat{\psi}_{\rm s} = Q_{\rm B} U_1^\dagger U_1 B_1^L c_1|0\rangle$ which satisfies $(\mathcal{B}_0 - \mathcal{B}_0^\dagger)\hat{\psi}_{\rm s} = 0$, we can generate a solution:

$$\Psi^{(\alpha,\beta)} = \sum_{k=0}^{\infty}(-1)^k \lambda_{\rm s}^{k+1} P_\alpha * \hat{\psi}_{\rm s} * (A^{(\alpha+\beta)} * \hat{\psi}_{\rm s})^k * P_\beta = \sum_{n=1}^{\infty} \lambda_{\rm s}^n \psi_{{\rm s},n} .$$

Each term is computed as

$$\psi_{{\rm s},n} = P_\alpha * (Q_{\rm B}\hat{\Lambda}_0) * P_\beta * (P_\alpha * \hat{\Lambda}_0 * P_\beta - I)^{n-1} = -\sum_{l=0}^{n-1} \frac{(-1)^{n-1-l}(n-1)!}{l!(n-1-l)!} \partial_t \psi_{t,l}^{(\alpha,\beta)}|_{t=0} ,$$

$$\begin{aligned}\psi_{t,n}^{(\alpha,\beta)} = {} & \frac{2}{\pi} U^\dagger_{n(\alpha+\beta)+t+\alpha+\beta+1} U_{n(\alpha+\beta)+t+\alpha+\beta+1} \Big[\\ & -\frac{1}{\pi}\hat{\mathcal{B}}\tilde{c}(\frac{\pi}{4}(\beta-\alpha+t+n(\alpha+\beta)))\tilde{c}(\frac{\pi}{4}(\beta-\alpha-t-n(\alpha+\beta))) \\ & +\frac{1}{2}\Big\{\tilde{c}(\frac{\pi}{4}(\beta-\alpha+t+n(\alpha+\beta))) + \tilde{c}(\frac{\pi}{4}(\beta-\alpha-t-n(\alpha+\beta)))\Big\}\Big]|0\rangle .\end{aligned}$$

Then, we can re-sum the above as

$$\Psi^{(\alpha,\beta)} = -\sum_{l=0}^{\infty} \lambda_S^{l+1} \partial_t \psi_{t,l}^{(\alpha,\beta)}|_{t=0} .$$

Here, expansion parameter is redefined as

$$\lambda_S \equiv \frac{\lambda_{\rm s}}{\lambda_{\rm s}+1} .$$

The solution can be rewritten as $\Psi^{(\alpha,\beta)} = e^{\frac{\pi}{4}(\beta-\alpha)K_1}(\alpha+\beta)^{\frac{D}{2}}\Psi^{(1/2,1/2)}$,

where $K_1 = L_1 + L_{-1}, \quad D = \mathcal{L}_0 - \mathcal{L}_0^\dagger$ are BPZ odd and derivations w.r.t. $*$,

and $\Psi^{(1/2,1/2)}$ is the Schnabl's solution for tachyon condensation at

$$\lambda_S = 1 \quad \leftrightarrow \quad \lambda_s = \infty \,.$$

By <u>regularizing</u> it as

$$\Psi^{(\alpha,\beta)}|_{\lambda_S=1} \;=\; \lim_{N\to\infty}\left(\frac{1}{\alpha+\beta}\psi^{(\alpha,\beta)}_{t=0,N} - \sum_{n=0}^{N}\partial_t\psi^{(\alpha,\beta)}_{t,n}|_{t=0}\right),$$

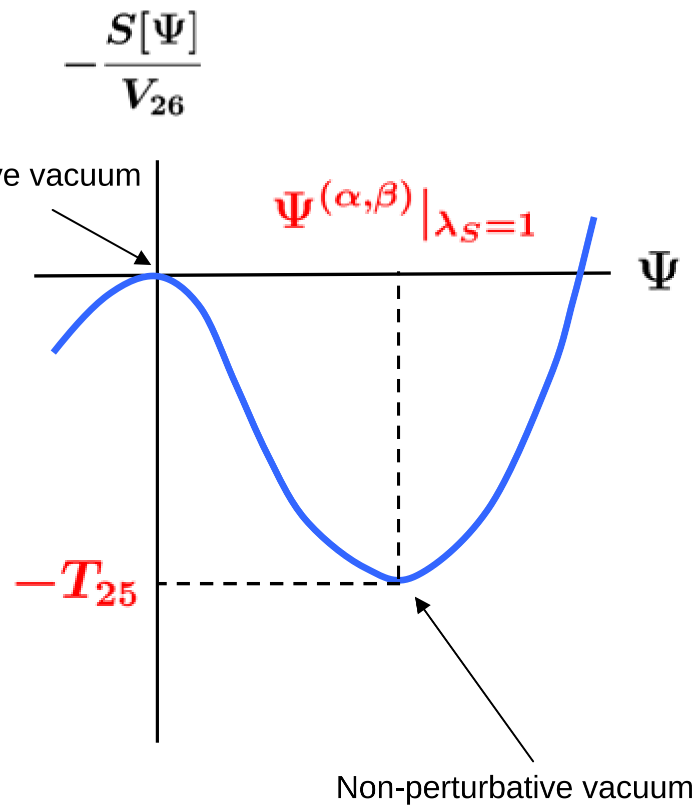
the new BRST operator around the solution Q'_{B} satisfies

$$Q'_{\mathrm{B}}A^{(\alpha+\beta)} \equiv Q_{\mathrm{B}}A^{(\alpha+\beta)} + \Psi^{(\alpha,\beta)}|_{\lambda_S=1} * A^{(\alpha+\beta)} + A^{(\alpha+\beta)} * \Psi^{(\alpha,\beta)}|_{\lambda_S=1} \;=\; I\,,$$

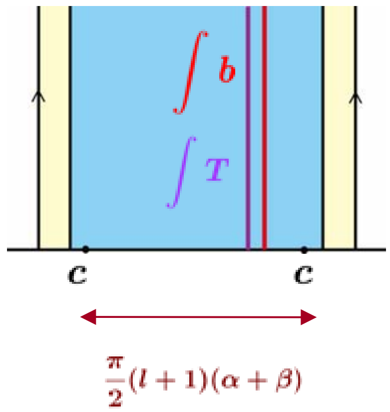
whic implies vanishing cohomology and

$$S[\Psi^{(\alpha,\beta)}|_{\lambda_S=1}]/V_{26} \;=\; \frac{1}{2\pi^2 g^2} = T_{25}.$$

This result is (α,β)-independent.



$$\Psi^{(\alpha,\beta)} \sim \sum \lambda_S^{l+1}$$



Note

We can evaluate the action as $$S[\Psi^{(\alpha,\beta)}]/V_{26} = 0 \quad (|\lambda_S| < 1)\,.$$

In fact, the solution can be rewritten as pure gauge form by taking the infinite summation formally

$$\Psi^{(\alpha,\beta)} = Q_{\rm B}(\lambda_S P_\alpha * \hat{\Lambda}_0 * P_\beta) * \frac{1}{1-\lambda_S P_\alpha * \hat{\Lambda}_0 * P_\beta}\,.$$

Berkovits' WZW-type super SFT

The action for NS sector is given by

$$
\begin{aligned}
&S_{\mathrm{NS}}[\Phi] \\
&= \frac{1}{2g^2}\langle\langle (e^{-\Phi}Q_{\mathrm{B}}e^{\Phi})(e^{-\Phi}\eta_0 e^{\Phi}) - \int_0^1 dt (e^{-t\Phi}\partial_t e^{t\Phi})\{(e^{-t\Phi}Q_{\mathrm{B}}e^{t\Phi}),(e^{-t\Phi}\eta_0 e^{t\Phi})\}\rangle\rangle \\
&= -\frac{1}{g^2}\int_0^1 dt \langle\langle (\eta_0\Phi)(e^{-t\Phi}Q_{\mathrm{B}}e^{t\Phi})\rangle\rangle \\
&= -\frac{1}{g^2}\sum_{M,N=0}^{\infty}\frac{(-1)^M}{(M+N+2)(M+N+1)M!N!}\langle\langle (\eta_0\Phi)\Phi^M(Q_{\mathrm{B}}\Phi)\Phi^N\rangle\rangle\,.
\end{aligned}
$$

String field Φ : ghost number 0, picture number 0, Grassmann even, expressed by matter and ghosts $b, c, \phi, \xi, \eta \quad (\beta = e^{-\phi}\partial\xi, \gamma = \eta e^{\phi})$:

$$Q_{\mathrm{B}} = \oint \frac{dz}{2\pi i}\left(c(T^{\mathrm{m}} - \tfrac{1}{2}(\partial\phi)^2 - \partial^2\phi + \partial\xi\eta) + bc\partial c + \eta e^{\phi}G^{\mathrm{m}} - \eta\partial\eta e^{2\phi}b\right)(z)$$

$$\eta_0 = \oint \frac{dz}{2\pi i}\eta(z)$$

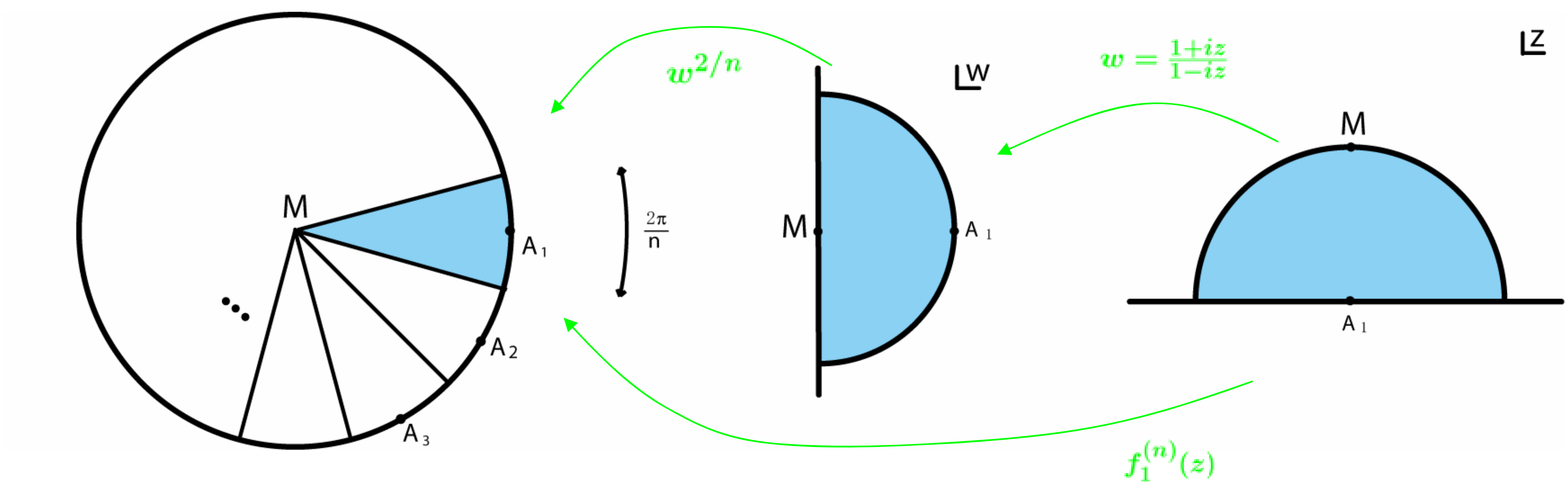
$Q_{\rm B}, \eta_0$ such as $Q_{\rm B}^2 = 0,\ \eta_0^2 = 0,\ \{Q_{\rm B}, \eta_0\} = 0$

are derivations with respect to the star product:

$$Q_{\rm B}(A * B) = Q_{\rm B}A * B + (-1)^{|A|} A * Q_{\rm B}B, \quad \eta_0(A * B) = \eta_0 A * B + (-1)^{|A|} A * \eta_0 B$$

n-string vertex is defined using CFT correlator in the *large* Hilbert space:

$$\langle V_n|A_1\rangle \cdots |A_n\rangle = \langle\langle A_1 \cdots A_n\rangle\rangle := \left\langle f_1^{(n)}[\mathcal{O}_{A_1}] \cdots f_n^{(n)}[\mathcal{O}_{A_n}]\right\rangle$$
$$= \langle A_1|(\cdots(A_2 * A_3) * \cdots * A_{n-1}) * A_n\rangle = \langle A_1|A_2 * \cdots * A_n\rangle$$



Some formulas:

$$\langle\langle A_1 \cdots A_{n-1} \Phi \rangle\rangle = \langle\langle \Phi A_1 \cdots A_{n-1} \rangle\rangle \,,$$
$$\langle\langle A_1 \cdots A_{n-1} (Q_{\rm B} \Phi) \rangle\rangle = - \langle\langle (Q_{\rm B} \Phi) A_1 \cdots A_{n-1} \rangle\rangle \,,$$
$$\langle\langle A_1 \cdots A_{n-1} (\eta_0 \Phi) \rangle\rangle = - \langle\langle (\eta_0 \Phi) A_1 \cdots A_{n-1} \rangle\rangle \,,$$
$$\langle\langle Q_{\rm B} (\cdots) \rangle\rangle = \langle\langle \eta_0 (\cdots) \rangle\rangle = 0 \,.$$

- Variation of the action: $\delta S_{\rm NS} = \frac{1}{g^2} \langle\langle e^{-\Phi} \delta e^{\Phi} \, \eta_0 (e^{-\Phi} Q_{\rm B} e^{\Phi}) \rangle\rangle$
- Equation of motion: $\eta_0 (e^{-\Phi} Q_{\rm B} e^{\Phi}) = 0$
- Gauge transformation: $\delta e^{\Phi} = Q_{\rm B} \Lambda_1 * e^{\Phi} + e^{\Phi} * \eta_0 \Lambda_2$

or equivalently

$$\delta e^{\Phi} = \Xi_1 * e^{\Phi} + e^{\Phi} * \Xi_2 , \quad Q_{\rm B} \Xi_1 = 0, \ \ \eta_0 \Xi_2 = 0.$$

Using the wedge states $|r=\alpha+1\rangle = P_\alpha$ as in bosonic SFT, we have

$$Q_{\rm B} P_\alpha = 0\,, \quad \eta_0 P_\alpha = 0\,, \quad P_\alpha * P_\beta = P_{\alpha+\beta}\,, \quad P_{\alpha=0} = I.$$

Corresponding to the wedge states, we have constructed $\hat{A}^{(\gamma)}$:

$$\hat{A}^{(\gamma)} = \int_0^\gamma d\alpha \log\left(\frac{\alpha}{\gamma}\right)\left(\frac{\pi}{2} J_1^{--L} + \alpha\frac{\pi^2}{4}\tilde{G}_1^{-L} B_1^L\right) P_\alpha\,,$$

such as

$$\eta_0 \hat{A}^{(\gamma)} = -\frac{\pi}{2}\int_0^\gamma d\alpha B_1^L P_\alpha\,, \qquad Q_{\rm B}\hat{A}^{(\gamma)} = -\frac{\pi}{2}\int_0^\gamma d\alpha \tilde{G}_1^{-L} P_\alpha\,,$$

$$\eta_0 Q_{\rm B} \hat{A}^{(\gamma)} = I - P_\gamma\,.$$

Here, $J^{--}(z) = \xi b(z)$, $\tilde{G}^- = [Q_{\rm B}, J^{--}(z)]$

are primary field with dimension 2.

⟹ J_1^{--L}, $\tilde{G}_1^{-L}$ are defined in the same way as B_1^L.

Note:

$$G^{+}(z)=j_B(z)=c(T^{\rm m}+T^{\phi}+T^{\xi\eta})(z)+bc\partial c(z)+\eta e^{\phi}G^{\rm m}(z)-\eta\partial\eta e^{2\phi}b(z)+\partial^2 c(z)+\partial(c\xi\eta)(z)\,,$$
$$\tilde{G}^{+}(z)=\eta(z)\,,\qquad G^{-}(z)=b(z)\,,$$
$$\tilde{G}^{-}(z)=[Q,\xi b(z)]=-\xi T(z)+e^{\phi}G^{\rm m}b(z)+c\partial\xi b(z)+b\partial b\eta e^{2\phi}(z)-\partial^2\xi(z)\,,$$
$$J^{++}(z)=c\eta(z)\,,\qquad J(z)=j_{\rm gh}(z)=-bc(z)-\xi\eta(z)\,,\qquad J^{--}(z)=\xi b(z)\,.$$

Generators of N=4 twisted superconformal algebra. [cf. Berkovits]

In bosonic SFT, roughly

$$\frac{\mathcal{B}_0}{\mathcal{L}_0}(\Psi_1*\Psi_2)\to\Psi_1*A^{(1)}*\Psi_2$$

"$Q_{\rm B}^{-1}$"

In super SFT, roughly

$$\frac{\tilde{\mathcal{G}}_0^{-}}{\mathcal{L}_0}\frac{\mathcal{B}_0}{\mathcal{L}_0}(\Phi_1*\Phi_2)\to\Phi_1*\hat{A}^{(1)}*\Phi_2$$

"$(\eta_0 Q_{\rm B})^{-1}$"

With $\hat{\phi}$ such as : $\eta_0 Q_B \hat{\phi} = 0, \quad \hat{\phi} * \hat{\phi} = 0, \quad \hat{\phi} * \eta_0 \hat{\phi} = 0, \quad \hat{\phi} * Q_B \hat{\phi} = 0,$

$$\Phi^{(\alpha,\beta)}_{(1)} = \log(1 + P_\alpha * f_{(1)} * P_\beta), \qquad f_{(1)} = \frac{1}{1 - \eta_0 \hat{\phi} * Q_B \hat{A}^{(\alpha+\beta)}} * \hat{\phi},$$

$$\Phi^{(\alpha,\beta)}_{(2)} = \log(1 + P_\alpha * f_{(2)} * P_\beta), \qquad f_{(2)} = \hat{\phi} * \frac{1}{1 - \eta_0 \hat{A}^{(\alpha+\beta)} * Q_B \hat{\phi}},$$

$$\Phi^{(\alpha,\beta)}_{(3)} = -\log(1 - P_\alpha * f_{(3)} * P_\beta), \qquad f_{(3)} = \frac{1}{1 - Q_B \hat{\phi} * \eta_0 \hat{A}^{(\alpha+\beta)}} * \hat{\phi},$$

$$\Phi^{(\alpha,\beta)}_{(4)} = -\log(1 - P_\alpha * f_{(4)} * P_\beta), \qquad f_{(4)} = \hat{\phi} * \frac{1}{1 - Q_B \hat{A}^{(\alpha+\beta)} * \eta_0 \hat{\phi}},$$

are solutions to the EOM: $\eta_0 (e^{-\Phi^{(\alpha,\beta)}_{(i)}} Q_B e^{\Phi^{(\alpha,\beta)}_{(i)}}) = 0 \,, \qquad (i = 1,2,3,4)$

We can check them by straightforward computation using derivation property.

<u>Note:</u> $\Phi^{(\alpha,\beta)}_{(2)}$ and $\Phi^{(\alpha,\beta)}_{(3)}$; $\Phi^{(\alpha,\beta)}_{(1)}$ and $\Phi^{(\alpha,\beta)}_{(4)}$ are *gauge equivalent* :

$$e^{\Phi^{(\alpha,\beta)}_{(2)}} = U^{(\alpha,\beta)}_{23} * e^{\Phi^{(\alpha,\beta)}_{(3)}}, \qquad e^{\Phi^{(\alpha,\beta)}_{(1)}} = e^{\Phi^{(\alpha,\beta)}_{(4)}} * V^{(\alpha,\beta)}_{41},$$

$$Q_B U^{(\alpha,\beta)}_{23} = 0 \,, \qquad \eta_0 V^{(\alpha,\beta)}_{41} = 0 \,.$$

Example of $\hat{\phi}$ using nonsingular matter supercurrent:

$$\mathbf{J}^a(z,\theta) = \psi^a(z) + \theta J^a(z)$$

$$\hat{\phi} = \zeta_a U_1^\dagger U_1 c\xi e^{-\phi}\psi^a(0)|0\rangle\,, \qquad \zeta_a\zeta_b\Omega^{ab} = 0\,,$$

where we suppose

$$\begin{aligned}
\psi^a(y)\psi^b(z) &\sim (y-z)^{-1}\Omega^{ab}\,,\\
J^a(y)\psi^b(z) &\sim (y-z)^{-1} i f^{ab}{}_c \psi^c(z)\,,\\
J^a(y)J^b(z) &\sim (y-z)^{-2}\Omega^{ab} + (y-z)^{-1} i f^{ab}{}_c J^c(z)\,.
\end{aligned}$$

More explicitly, on the flat background, we can take

$$\mathbf{J}^\mu(z,\theta) = \psi^\mu(z) + \theta i\partial X^\mu(z), \qquad \zeta_\mu\zeta_\nu\eta^{\mu\nu} = 0\,.$$

Note: $\Phi^{(1/2,1/2)}_{(3)}$ and $\Phi^{(1/2,1/2)}_{(4)}$ are the same as Okawa's solution.

$\Phi^{(1/2,1/2)}_{(3)}$ and $\Phi^{(1/2,1/2)}_{(2)}$ are the same as Erler's solution.

Gauge transformations

- *If* $\{P_\alpha\}_{\alpha\geq 0}$ *can be extended to an abelian group,*

i.e. $P_\alpha^{-1} = P_{-\alpha}$ *, we find gauge transformations:*

$$\Psi^{(\alpha,\beta)} = V^{-1} * \hat{\psi} * V + V^{-1} * Q_{\mathrm{B}} V \,,$$

$$V = (I + \hat{\psi} * A^{(\alpha+\beta)}) * P_\alpha^{-1}, \quad V^{-1} = P_\alpha * \frac{1}{1 + \hat{\psi} * A^{(\alpha+\beta)}} \,,$$

for bosonic SFT and

$$e^{\Phi_{(3)}^{(\alpha,\beta)}} = P_\alpha * \frac{1}{1 - Q_{\mathrm{B}}(\hat{\phi} * \eta_0 \hat{A}^{(\alpha+\beta)})} * e^{\hat{\phi}} * (1 + \eta_0(Q_{\mathrm{B}}\hat{\phi} * \hat{A}^{(\alpha+\beta)})) * P_\alpha^{-1},$$

$$e^{\Phi_{(4)}^{(\alpha,\beta)}} = P_\beta^{-1} * (1 - Q_{\mathrm{B}}(\hat{A}^{(\alpha+\beta)} * \eta_0 \hat{\phi})) * e^{\hat{\phi}} * \frac{1}{1 + \eta_0(Q_{\mathrm{B}}\hat{A}^{(\alpha+\beta)} * \hat{\phi})} * P_\beta \,,$$

for super SFT.

Alternatively, using path-ordering, we found

$$\Psi^{(\alpha,\beta)} = V^{(\alpha,\beta)-1} * \hat{\psi} * V^{(\alpha,\beta)} + V^{(\alpha,\beta)-1} * Q_{\mathrm{B}} * V^{(\alpha,\beta)},$$

$$V^{(\alpha,\beta)} = \mathrm{P}\exp \int_0^1 dt G^{(\alpha,\beta)}(t),$$

$$G^{(\alpha,\beta)}(t) \equiv \frac{-\pi}{2}\left(\alpha(B_1^L P_{t\alpha}) * \frac{1}{1+\hat{\psi} * A^{(t\alpha+t\beta)}} * \hat{\psi} * P_{t\beta} + \beta P_{t\alpha} * \frac{1}{1+\hat{\psi} * A^{(t\alpha+t\beta)}} * \hat{\psi} * B_1^R P_{t\beta}\right),$$

for bosonic SFT.

(In the case of $\alpha = \beta$, this form coincide with Ellwood's one.)

In this sense,

$$\Psi^{(\alpha,\beta)} \quad \sim \quad \hat{\psi}$$

↑ Without the identity state, including Schnabl's marginal and scalar solutions

↑ Based on the identity state, BRST inv. and nilpotent

- Similarly, in super SFT, we have found

$$e^{\Phi_{(3)}^{(\alpha,\beta)}} = W_1 * e^{\hat{\phi}} * W_2 \,, \qquad Q_{\mathrm{B}} W_1 = 0, \qquad \eta_0 W_2 = 0 \,,$$

$$W_1 \equiv \mathrm{P}' \exp \int_0^1 dt G_1^{(\alpha,\beta)}(t) \,, \qquad W_2 \equiv \mathrm{P} \exp \int_0^1 dt G_2^{(\alpha,\beta)}(t) \,,$$

$$G_1^{(\alpha,\beta)}(t) \;\equiv\; -\frac{\pi}{2}\alpha K_1^L I + \frac{\pi}{2}(\alpha+\beta) P_{t\alpha} * \frac{1}{1 - Q_{\mathrm{B}}(\hat{\phi} * \eta_0 \hat{A}^{(t\alpha+t\beta)})} * Q_{\mathrm{B}}(\hat{\phi} * B_1^R P_{t\beta}) \,,$$

$$G_2^{(\alpha,\beta)}(t) \;\equiv\; \frac{\pi}{2}\alpha K_1^L I - \frac{\pi}{2}(\alpha+\beta) P_{t\alpha} * \frac{1}{1 + \eta_0(Q_{\mathrm{B}}\hat{\phi} * \hat{A}^{(t\alpha+t\beta)})} * Q_{\mathrm{B}}\hat{\phi} * B_1^R P_{t\beta} \,,$$

and

$$e^{\Phi_{(1)}^{(\alpha,\beta)}} = W_3 * e^{\hat{\phi}} * W_4 \,, \qquad Q_{\mathrm{B}} W_3 = 0, \qquad \eta_0 W_4 = 0 \,,$$

$$W_3 \equiv \mathrm{P}' \exp \int_0^1 dt G_4^{(\alpha,\beta)}(t) \,, \qquad W_4 \equiv \mathrm{P} \exp \int_0^1 dt G_3^{(\alpha,\beta)}(t) \,,$$

$$G_3^{(\alpha,\beta)}(t) \;\equiv\; \frac{\pi}{2}\alpha K_1^L I - \frac{\pi}{2}(\alpha+\beta) P_{t\alpha} * \frac{1}{1 - \eta_0(\hat{\phi} * Q_{\mathrm{B}} \hat{A}^{(t\alpha+t\beta)})} * \eta_0(\hat{\phi} * \tilde{G}_1^{-R} P_{t\beta}) \,,$$

$$G_4^{(\alpha,\beta)}(t) \;\equiv\; -\frac{\pi}{2}\alpha K_1^L I + \frac{\pi}{2}(\alpha+\beta) P_{t\alpha} * \frac{1}{1 + Q_{\mathrm{B}}(\eta_0\hat{\phi} * \hat{A}^{(t\alpha+t\beta)})} * \eta_0\hat{\phi} * \tilde{G}_1^{-R} P_{t\beta} \,.$$

In this sense, $\Phi^{(\alpha,\beta)}_{(i)} \sim \hat{\phi}$

$$i = 1, 2, 3, 4$$

↑ Without the identity state, including Erler/Okawa's marginal solutions

↑ Based on the identity state,

$$\eta_0 Q_{\rm B} \hat{\phi} = 0, \ \hat{\phi} * \hat{\phi} = 0,$$
$$\hat{\phi} * \eta_0 \hat{\phi} = 0, \ \hat{\phi} * Q_{\rm B} \hat{\phi} = 0.$$

Note:

The gauge equivalence is ***formal*** and might not be well-defined.
Gauge parameter string field might become "singular,"
as well as Schnabl or Takahashi-Tanimoto's tachyon solution.
But they are *almost* gauge equivalent.

If $\hat{\psi}$ *and* $\hat{\phi}$ *are pure gauge:*

$$\hat{\psi} = e^{-\Lambda} Q_{\rm B} e^{\Lambda}, \qquad e^{\hat{\phi}} = e^{Q_{\rm B}\Lambda_1} e^{\eta_0 \Lambda_2},$$

$\Psi^{(\alpha,\beta)}$ *and* $\Phi^{(\alpha,\beta)}_{(i)}$ *are also pure gauge:*

$$\Psi^{(\alpha,\beta)} = U^{(\alpha,\beta)-1} Q_{\rm B} U^{(\alpha,\beta)},$$

$$U^{(\alpha,\beta)} = I + P_\alpha * (e^{\Lambda} - I) * \frac{1}{1 + A^{(\alpha+\beta)} * \hat{\psi}} * P_\beta,$$

for bosonic SFT and

$$e^{\Phi^{(\alpha,\beta)}_{(i)}} = U^{(\alpha,\beta)}_{(i)} * V^{(\alpha,\beta)}_{(i)}, \quad Q_{\rm B} U^{(\alpha,\beta)}_{(i)} = 0, \quad \eta_0 V^{(\alpha,\beta)}_{(i)} = 0,$$

$$V^{(\alpha,\beta)}_{(3)} = V^{(\alpha,\beta)}_{(2)} = I + P_\alpha * (e^{\eta_0\Lambda_2} - I) * \frac{1}{1 - \eta_0 \hat{A}^{(\alpha+\beta)} * Q_{\rm B}\hat{\phi}} * P_\beta, \quad U^{(\alpha,\beta)}_{(3)} = \left[I + P_\alpha * \frac{1}{1 - Q_{\rm B}(\hat{\phi} * \eta_0 \hat{A}^{(\alpha+\beta)})} * \hat{\phi} * P_\beta \right] * V^{(\alpha,\beta)-I}_{(3)},$$

$$U^{(\alpha,\beta)}_{(4)} = U^{(\alpha,\beta)}_{(1)} = I + P_\alpha * \frac{1}{1 - \eta_0\hat{\phi} * Q_{\rm B}\hat{A}^{(\alpha+\beta)}} * (e^{Q_{\rm B}\Lambda_1} - I) * P_\beta, \quad V^{(\alpha,\beta)}_{(4)} = U^{(\alpha,\beta)-1}_{(4)} * \left[I + P_\alpha * \hat{\phi} * \frac{1}{1 + \eta_0(Q_{\rm B}\hat{A}^{(\alpha+\beta)} * \hat{\phi})} * P_\beta \right],$$

$$V^{(\alpha,\beta)}_{(1)} = V^{(\alpha,\beta)}_{(4)} * V^{(\alpha,\beta)}_{41}, \quad U^{(\alpha,\beta)}_{(2)} = U^{(\alpha,\beta)}_{23} * U^{(\alpha,\beta)}_{(3)},$$

for super SFT.

In the case of the above pure gauge form, the actions are re-expanded around the solutions as

$$\begin{aligned} S[\Psi^{(\alpha,\beta)}+\Psi] &= S[\Psi^{(\alpha,\beta)}]+S[U^{(\alpha,\beta)}*\Psi*U^{(\alpha,\beta)-1}]\,, \\ S_{\rm NS}[\log(e^{\Phi^{(\alpha,\beta)}_{(i)}}e^{\Phi})] &= S_{\rm NS}[\Phi^{(\alpha,\beta)}_{(i)}]+S_{\rm NS}[V^{(\alpha,\beta)}_{(i)}*\Phi*V^{(\alpha,\beta)-1}_{(i)}]\,. \end{aligned}$$

The induced string field redefinitions are

$$\begin{aligned} U^{(\alpha,\beta)}*\Psi*U^{(\alpha,\beta)-1} &= \Psi+(P_\alpha*\Lambda*P_\beta)*\Psi-\Psi*(P_\alpha*\Lambda*P_\beta)+\mathcal{O}(\Lambda^2), \\ V^{(\alpha,\beta)}_{(i)}*\Phi*V^{(\alpha,\beta)-1}_{(i)} &= \Phi+(P_\alpha*\eta_0\Lambda_2*P_\beta)*\Phi-\Phi*(P_\alpha*\eta_0\Lambda_2*P_\beta) \\ &\quad+\mathcal{O}(\Lambda_1^2,\Lambda_1\Lambda_2,\Lambda_2^2). \end{aligned}$$

For example, in the case of

$$\begin{aligned} \hat\psi &= \zeta_\mu U_1^\dagger U_1 ci\partial X^\mu(0)|0\rangle, \\ \hat\phi &= \zeta_\mu U_1^\dagger U_1 c\xi e^{-\phi}\psi^\mu(0)|0\rangle, \quad (\zeta_\mu\zeta_\nu\eta^{\mu\nu}=0) \end{aligned}$$

(if we regard X^μ as dimention zero primary field) we have

$$\Lambda=U_1^\dagger U_1 i\zeta_\mu X^\mu(0)|0\rangle\,,$$

$$\Lambda_2=U_1^\dagger U_1\xi i\zeta_\mu X^\mu(0)|0\rangle\,,\quad \Lambda_1=U_1^\dagger U_1 c\xi\partial\xi e^{-2\phi}i\zeta_\mu X^\mu(0)|0\rangle\,.$$

Future problems

How about general (super)currents? Namely, $\zeta_a \zeta_b g^{ab} \neq 0, \quad \zeta_a \zeta_b \Omega^{ab} \neq 0.$

C.f. [KORZ], [Fuchs-Kroyter-Potting]

In bosonic SFT,

$$\psi \mapsto \Psi^{(\alpha,\beta)}(\psi) \equiv P_\alpha * \frac{1}{1 + \psi * A^{(\alpha+\beta)}} * \psi * P_\beta$$

maps general solution to solution:

$$Q_B \psi + \psi * \psi = 0$$

$$\rightarrow \quad Q_B \Psi^{(\alpha,\beta)}(\psi) + \Psi^{(\alpha,\beta)}(\psi) * \Psi^{(\alpha,\beta)}(\psi) = 0$$

Similarly, in super SFT, we found that

$$\phi \mapsto \Phi^{(\alpha,\beta)}(\phi) \equiv \log\left(1 + P_\alpha (e^\phi - 1) \frac{1}{1 - \eta_0 \hat{A}^{(\alpha+\beta)} e^{-\phi} Q_B e^\phi} P_\beta\right)$$

maps general solution to solution:

$$\eta_0 (e^{-\phi} Q_B e^\phi) = 0$$

$$\rightarrow \quad \eta_0 (e^{-\Phi^{(\alpha,\beta)}(\phi)} Q_B e^{\Phi^{(\alpha,\beta)}(\phi)}) = 0$$

On the other hand, in [Takahashi-Tanimoto, Kishimoto-Takahashi] some identity-based solutions for general (super)current were already constructed.

At least formally, $\Psi^{(\alpha,\beta)}(\Psi^{\mathrm{TT}})$ and $\Phi^{(\alpha,\beta)}(\Phi^{\mathrm{KT}})$ with $\alpha,\beta>0$

give solutions which are not based on the identity state!

→ Details : work in progress

So far, various computations seem to be rather formal.

Definition of "regularity" of string fields?

It is very important in order to discuss "regular solutions," gauge transformations among them and cohomology around them.