

Evaluation of Gauge Invariants for Classical Solutions in Open String Field Theory

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Reference:

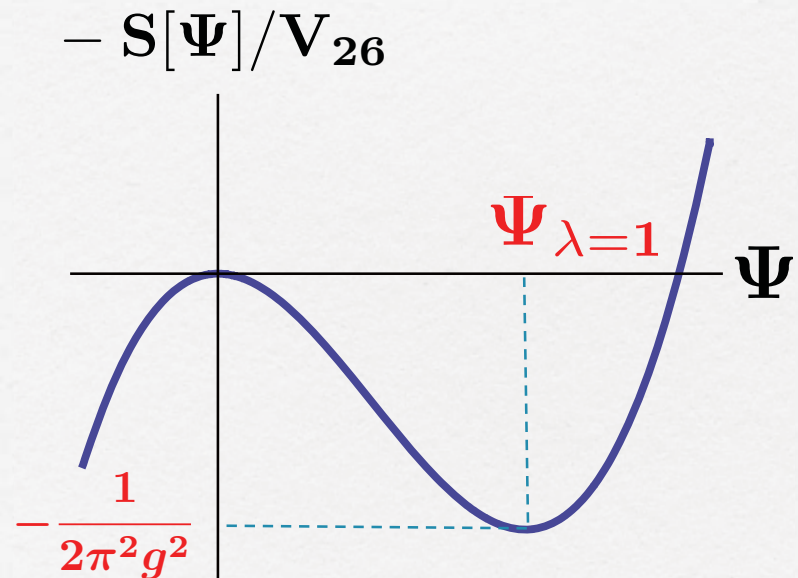
T. Kawano, I. K., T. Takahashi, "Gauge Invariant Overlaps for Classical Solutions in Open String Field Theory," arXiv:0804.1541

Contents

- Introduction
- Gauge invariant overlap
- Evaluation for Schnabl's solution
- Level truncation
- Summary and Discussion

Non-perturbative vacuum in OSFT

- **Schnabl's solution (2005):** $\Psi_{\lambda=1}$
 - Non-perturbative vacuum in OSFT
 - Potential height=D25-brane tension
 - Vanishing cohomology around $\Psi_{\lambda=1}$
[Ellwood-Schnabl(2006)]
- Phantom term (?)
- Closed string?

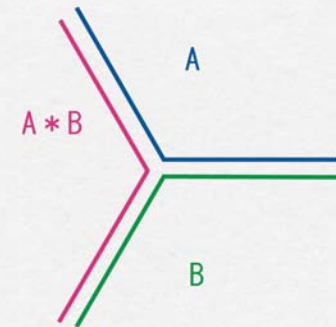


Witten's open string field theory

□ Action:
$$S[\Psi] = -\frac{1}{g^2} \left(\frac{1}{2} \langle \Psi, Q\Psi \rangle + \frac{1}{3} \langle \Psi, \Psi * \Psi \rangle \right)$$

$$|\Psi\rangle = \phi(x)c_1|0\rangle + A_\mu(x)\alpha_{-1}^\mu c_1|0\rangle + iB(x)c_0|0\rangle + \dots$$

$$Q = \oint \frac{dz}{2\pi i} \left(cT^m + bc\partial c + \frac{3}{2}\partial^2 c \right)$$



□ Equation of motion:
$$Q\Psi + \Psi * \Psi = 0$$

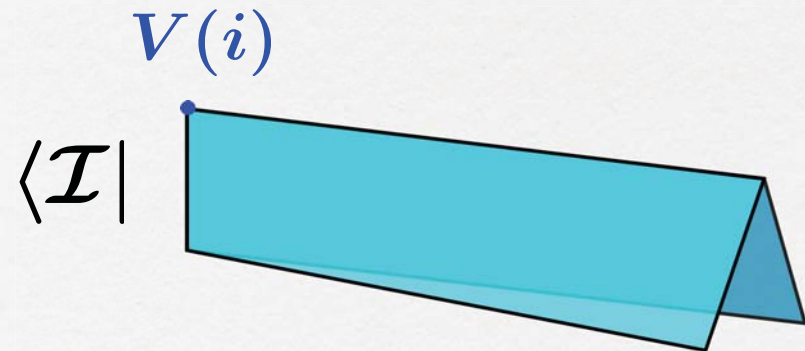
□ Gauge transformation:
$$\delta_\Lambda \Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi$$

Gauge invariant overlap

$$\mathcal{O}_V(\Psi) = \langle \mathcal{I} | V(i) | \Psi \rangle = \langle \Phi_V, \Psi \rangle \quad [\text{Zwiebach(1992),...}]$$

$$V(i) = c(i)c(-i)V_m(i, -i)$$

matter primary, dim (1,1)



On-shell closed string state: $\Phi_V = V(i) | \mathcal{I} \rangle$

Gauge invariance of $\mathcal{O}_V(\Psi)$

$$Q\Phi_V = 0, \quad \langle \Phi_V, \Psi * \Lambda \rangle = \langle \Phi_V, \Lambda * \Psi \rangle$$

on-shell

V : dim (0,0), midpoint

$$\therefore \mathcal{O}_V(\delta_\Lambda \Psi) = 0$$

- In particular, it vanishes for pure gauge solution:

$$\mathcal{O}_V(e^{-\Lambda} Q e^\Lambda) = 0$$

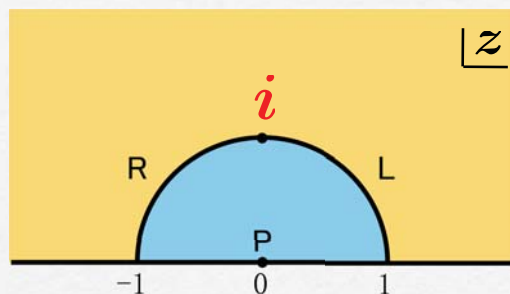
Schnabl's solution

$$\begin{aligned} \Psi_\lambda &= \frac{\lambda \partial_r}{\lambda e^{\partial_r} - 1} \psi_r|_{r=0} = \sum_{n=0}^{\infty} \frac{f_n(\lambda)}{n!} \partial_r^n \psi_r|_{r=0} \\ &= \begin{cases} \lim_{N \rightarrow \infty} \left(\psi_{N+1} - \sum_{n=0}^N \partial_r \psi_r|_{r=n} \right) & (\lambda = 1) \\ - \sum_{n=0}^{\infty} \lambda^{n+1} \partial_r \psi_r|_{r=n} & (\lambda \neq 1) \end{cases} \end{aligned}$$

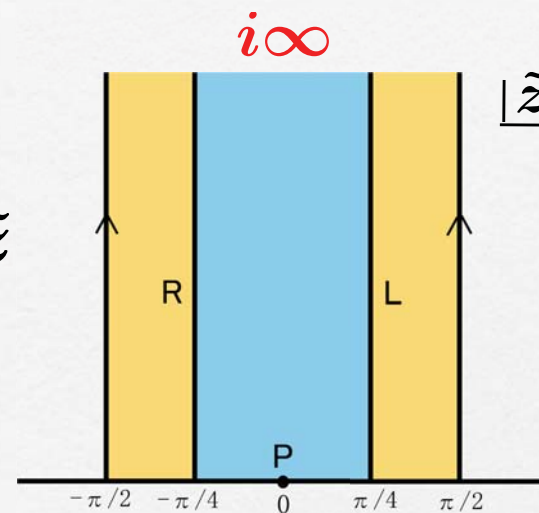
$$\psi_r \equiv \frac{2}{\pi} U_{r+2}^\dagger U_{r+2} \left[-\frac{1}{\pi} (\mathcal{B}_0 + \mathcal{B}_0^\dagger) \tilde{c}\left(\frac{\pi r}{4}\right) \tilde{c}\left(-\frac{\pi r}{4}\right) + \frac{1}{2} \left(\tilde{c}\left(-\frac{\pi r}{4}\right) + \tilde{c}\left(\frac{\pi r}{4}\right) \right) \right] |0\rangle$$

$$U_{r+2} = \left(\frac{2}{r+2} \right)^{\mathcal{L}_0}$$

Sliver frame



$$\arctan z = \tilde{z}$$



$$\tilde{\phi}(\tilde{z}) = (\cos \tilde{z})^{-2h} \phi(\tan \tilde{z}) \quad \text{for primary with dim } h$$

$$\mathcal{B}_0 = b_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2 - 1} b_{2k}$$

$$\mathcal{L}_0 = \{Q, \mathcal{B}_0\} = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2 - 1} L_{2k}$$

Evaluation of the action

$$S[\Psi_\lambda]/V_{26} = \begin{cases} \frac{1}{2\pi^2 g^2} & (\lambda = 1) \\ 0 & (|\lambda| < 1) \end{cases}$$

- Analytically evaluated using the “regularized” expression of the solution
[Schnabl (2005), Okawa / Fuchs-Kroyter (2006)]
- “Phantom term” gives finite contribution

Analytic evaluation of $\mathcal{O}_V(\Psi)$ for Schnabl's solution

- It is convenient to compute in the sliver frame:

$$\mathcal{O}_V(\Psi_\lambda) = \langle \Phi_V, \Psi_\lambda \rangle = \langle \mathcal{I} | \Phi_V * \Psi_\lambda \rangle$$

Written in terms of sliver frame

Simple formulas in sliver frame

On-shell closed string states in sliver frame

$$\begin{aligned}\Phi_V &= \sum_{m,n} \zeta_{mn} c(i) V_m(i) c(-i) V_n(-i) |\mathcal{I}\rangle \\ &= \sum_{m,n} \zeta_{mn} U_1^\dagger U_1 \tilde{c}(i\infty) \tilde{V}_m(i\infty) \tilde{c}(-i\infty) \tilde{V}_n(-i\infty) |0\rangle\end{aligned}$$

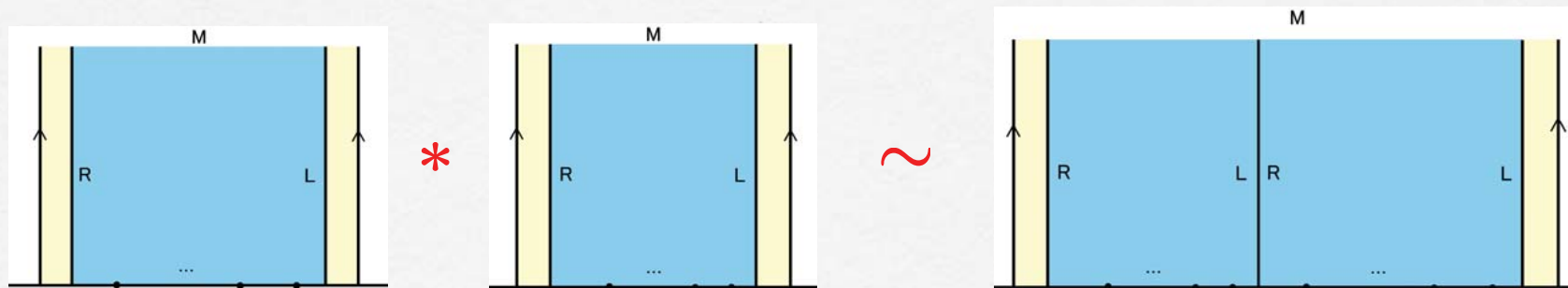
□ “regularization” $\pm i\infty \longrightarrow \pm iM$

$$\Phi_{V,M} \equiv \sum_{m,n} \zeta_{mn} U_1^\dagger U_1 \tilde{c}(iM) \tilde{V}_m(iM) \tilde{c}(-iM) \tilde{V}_n(-iM) |0\rangle$$

Star product formula

- Formal algebraic computation can be performed by

$$\begin{aligned}
 & U_r^\dagger U_r \tilde{\phi}_1(\tilde{x}_1) \cdots \tilde{\phi}_n(\tilde{x}_n) |0\rangle * U_s^\dagger U_s \tilde{\psi}_1(\tilde{y}_1) \cdots \tilde{\psi}_m(\tilde{y}_m) |0\rangle \\
 &= U_{r+s-1}^\dagger U_{r+s-1} \tilde{\phi}_1(\tilde{x}'_1) \cdots \tilde{\phi}_n(\tilde{x}'_n) \tilde{\psi}_1(\tilde{y}'_1) \cdots \tilde{\psi}_m(\tilde{y}'_m) |0\rangle
 \end{aligned}$$



$$((\mathcal{B}_0 + \mathcal{B}_0^\dagger)\Psi_1) * \Psi_2 = (\mathcal{B}_0 + \mathcal{B}_0^\dagger)(\Psi_1 * \Psi_2) + (-1)^{|\Psi_1|} \frac{\pi}{2} \Psi_1 * \mathcal{B}_1 \Psi_2, \dots$$

[Schnabl(2005)]

Computation and result

- OPE: $V_m(y)V_n(z) \sim \frac{v_{mn}}{(y-z)^2} + \text{finite} \quad (y \rightarrow z)$
- Computation for ψ_r

$$\langle \Phi_{V,M}, \psi_r \rangle = \frac{C_V}{2\pi i} \left(\sinh \frac{4M}{r+1} - \frac{4M}{\pi} \sin \frac{\pi}{r+1} \right) \left(\cosh \frac{4M}{r+1} - \cos \frac{\pi}{r+1} \right) \left(\sinh \frac{4M}{r+1} \right)^{-2},$$

$$C_V = \text{mat} \langle 0|0 \rangle_{\text{mat}} \sum_{m,n} \zeta_{mn} v_{mn}.$$

$$\therefore \langle \Phi_V, \psi_r \rangle = \lim_{M \rightarrow +\infty} \langle \Phi_{V,M}, \psi_r \rangle = \frac{C_V}{2\pi i} \quad \text{independent of } r$$

- Result for Schnabl's solution

$$\mathcal{O}_V(\Psi_\lambda) = \sum_{k=0}^{\infty} \frac{f_k(\lambda)}{k!} \partial_r^k \langle \Phi_V, \psi_r \rangle |_{r=0} = f_0(\lambda) \langle \Phi_V, \psi_0 \rangle = \begin{cases} \frac{C_V}{2\pi i} & (\lambda = 1) \\ 0 & (\lambda \neq 1) \end{cases}$$

Subtleties

- Only the phantom term contributes if the regularized expression for $\lambda = 1$ is used:

$$\mathcal{O}_V(\Psi_{\lambda=1}) = \lim_{N \rightarrow \infty} \left(\langle \Phi_V, \psi_{N+1} \rangle - \sum_{n=0}^N \partial_r \langle \Phi_V, \psi_r \rangle |_{r=n} \right)$$

- The order of the limits is not exchangeable:

$$\lim_{N \rightarrow \infty} \left(\lim_{M \rightarrow +\infty} \langle \Phi_{V,M}, \psi_{N+1} \rangle \right) \neq \lim_{M \rightarrow +\infty} \left(\lim_{N \rightarrow \infty} \langle \Phi_{V,M}, \psi_{N+1} \rangle \right) = 0$$

Absorption of the phantom term

- For $\lambda = 1$, the phantom term can be eliminated:

$$\begin{aligned}\Psi_{\lambda=1} &= \lim_{N \rightarrow \infty} \left(\psi_{N+1} - \sum_{n=0}^N \partial_r \psi_r |_{r=n} \right) \\ &= \lim_{N \rightarrow \infty} \left(\psi_0 + \sum_{n=0}^N (\psi_{n+1} - \psi_n - \partial_r \psi_r |_{r=n}) \right) \\ &= \underbrace{\psi_0}_{\text{~~~~~}} + \sum_{n=0}^{\infty} (\psi_{n+1} - \psi_n - \partial_r \psi_r |_{r=n})\end{aligned}$$

On-shell closed tachyon state

$$\begin{aligned}\Phi_k &= \frac{1}{4i} \lim_{\theta \rightarrow \frac{\pi}{2}} c(e^{i\theta})c(e^{-i\theta}) : e^{ik \cdot X(e^{i\theta}, e^{-i\theta})} : |\mathcal{I}\rangle \\ &= \frac{1}{4} e^{E_m + E_{\text{gh}}} c_0 c_1 |0\rangle\end{aligned}$$

$$E_m = - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \alpha_{-n} \cdot \alpha_{-n} - \sum_{n=1}^{\infty} \frac{2i\sqrt{2\alpha'}(-1)^n}{2n-1} k_i \alpha_{-2n+1}^i,$$

$$E_{\text{gh}} = \sum_{n=1}^{\infty} (-1)^n c_{-n} b_{-n}$$

k^i : Dirichlet direction

$$k^i k^i = 4/\alpha' \longleftrightarrow Q|\Phi_k\rangle = 0$$

Zero momentum dilaton state

$$\begin{aligned}\Phi_\eta &= \frac{1}{52\alpha' i} \eta_{\mu\nu} \lim_{\theta \rightarrow \frac{\pi}{2}} c(e^{i\theta}) \partial X^\mu(e^{i\theta}) c(e^{-i\theta}) \partial X^\nu(e^{-i\theta}) |\mathcal{I}\rangle \\ &= \left(\frac{1}{4} - \frac{2}{13} \sum_{n,m=1}^{\infty} mn \cos \frac{(m-n)\pi}{2} \alpha_{-m} \cdot \alpha_{-n} \right) e^E c_0 c_1 |0\rangle, \\ E &= \sum_{n=1}^{\infty} (-1)^n \left(-\frac{1}{2n} \alpha_{-n} \cdot \alpha_{-n} + c_{-n} b_{-n} \right)\end{aligned}$$

- BRST invariance can be checked by explicit computation.

$$Q|\Phi_\eta\rangle = 0$$

Map to get on-shell closed string states

$$|V_c\rangle = c\bar{c}V_m|0\rangle \quad \mapsto \quad \langle \hat{\gamma}(1_c, 2) | V_c \rangle_{1_c} = {}_2 \langle \Phi_V |$$

↑
open-closed string vertex

Examples:

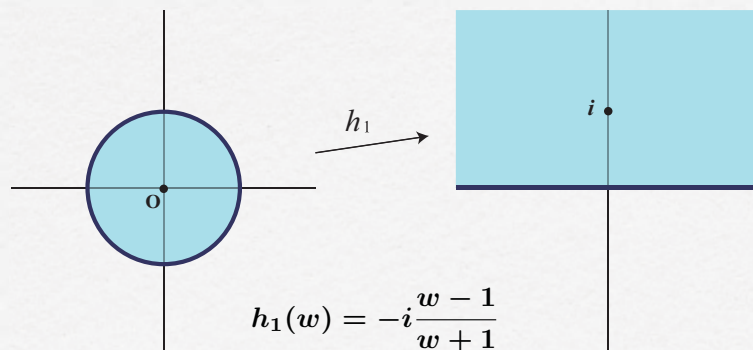
$$V_m = e^{ik_i X^i} \quad \mapsto \quad \Phi_k \quad \text{closed tachyon state}$$

$$V_m = \frac{1}{26} \partial X \cdot \bar{\partial} X \quad \mapsto \quad \Phi_\eta \quad \text{zero momentum dilaton state}$$

Shapiro-Thorn vertex

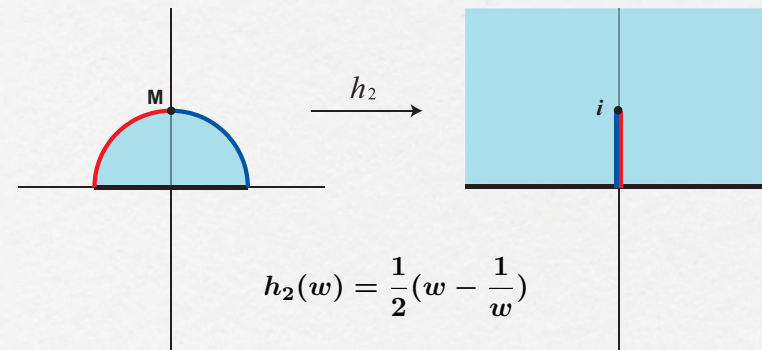
$$\langle \hat{\gamma}(1_c, 2) | \phi_c \rangle_{1_c} | \psi \rangle_2 = \langle h_1[\phi_c(0, 0)] h_2[\psi(0)] \rangle$$

CFT correlator on UHP [LPP]



$$h_1(w) = -i \frac{w - 1}{w + 1}$$

closed



$$h_2(w) = \frac{1}{2} \left(w - \frac{1}{w} \right)$$

open

□ BRST invariance:

$$\langle \hat{\gamma}(1_c, 2) | (Q_c^{(1)} + \bar{Q}_c^{(1)} + Q^{(2)}) = 0$$

□ Relations among Virasoro generators:

[Rastelli-Zwiebach]

$$\begin{aligned} & \langle \hat{\gamma}(1_c, 2) | (L_{2m-1}^{(2)} + L_{-2m+1}^{(2)}) \\ &= \langle \hat{\gamma}(1_c, 2) | \left(4(2m-1)i(-1)^m (L_0^{(1)} - \bar{L}_0^{(1)}) - \sum_{k=2}^{\infty} (f_{2m-1,k} L_{k-1}^{(1)} + \bar{f}_{2m-1,k} \bar{L}_{k-1}^{(1)}) \right) \end{aligned}$$

$$\begin{aligned} & \langle \hat{\gamma}(1_c, 2) | (L_{2m}^{(2)} - L_{-2m}^{(2)}) \\ &= \langle \hat{\gamma}(1_c, 2) | \left((-1)^m m \frac{c}{2} - 8m(-1)^m (L_0^{(1)} + \bar{L}_0^{(1)}) - \sum_{k=2}^{\infty} (f_{2m,k} L_{k-1}^{(1)} + \bar{f}_{2m,k} \bar{L}_{k-1}^{(1)}) \right) \end{aligned}$$

Level expansion of Schnabl's solution

□ Derived from expansion of ψ_r

$$\psi_{r-2} = \left[\prod_{k=1, \leftarrow}^{\infty} e^{u_{2k}(r)L_{-2k}} \right] \left[\frac{1}{\pi} \sin \frac{2\pi}{r} \left(1 - \frac{r}{2\pi} \sin \frac{2\pi}{r} \right) \sum_{p \geq -1; p: \text{odd}} \left(\frac{2}{r} \cot \frac{\pi}{r} \right)^p c_{-p} |0\rangle \right. \\ \left. + \frac{r}{2\pi^2} \left(\sin \frac{2\pi}{r} \right)^2 \sum_{s \geq 2; s: \text{even}} \frac{(-1)^{\frac{s}{2}+1}}{s^2 - 1} \left(\frac{2}{r} \right)^s \sum_{p, q \geq -1; p+q: \text{odd}} (-1)^q \left(\frac{2}{r} \cot \frac{\pi}{r} \right)^{p+q} b_{-s} c_{-p} c_{-q} |0\rangle \right]$$

$$u_2(r) = -\frac{r^2 - 4}{3r^2}, \quad u_4(r) = \frac{r^4 - 16}{30r^4}, \quad u_6(r) = -\frac{16(r^2 - 4)(r^2 - 1)(r^2 + 5)}{945r^6}, \dots$$

From its coefficients, $\psi_{N+1} = O(N^{-3}) \quad (N \rightarrow \infty)$

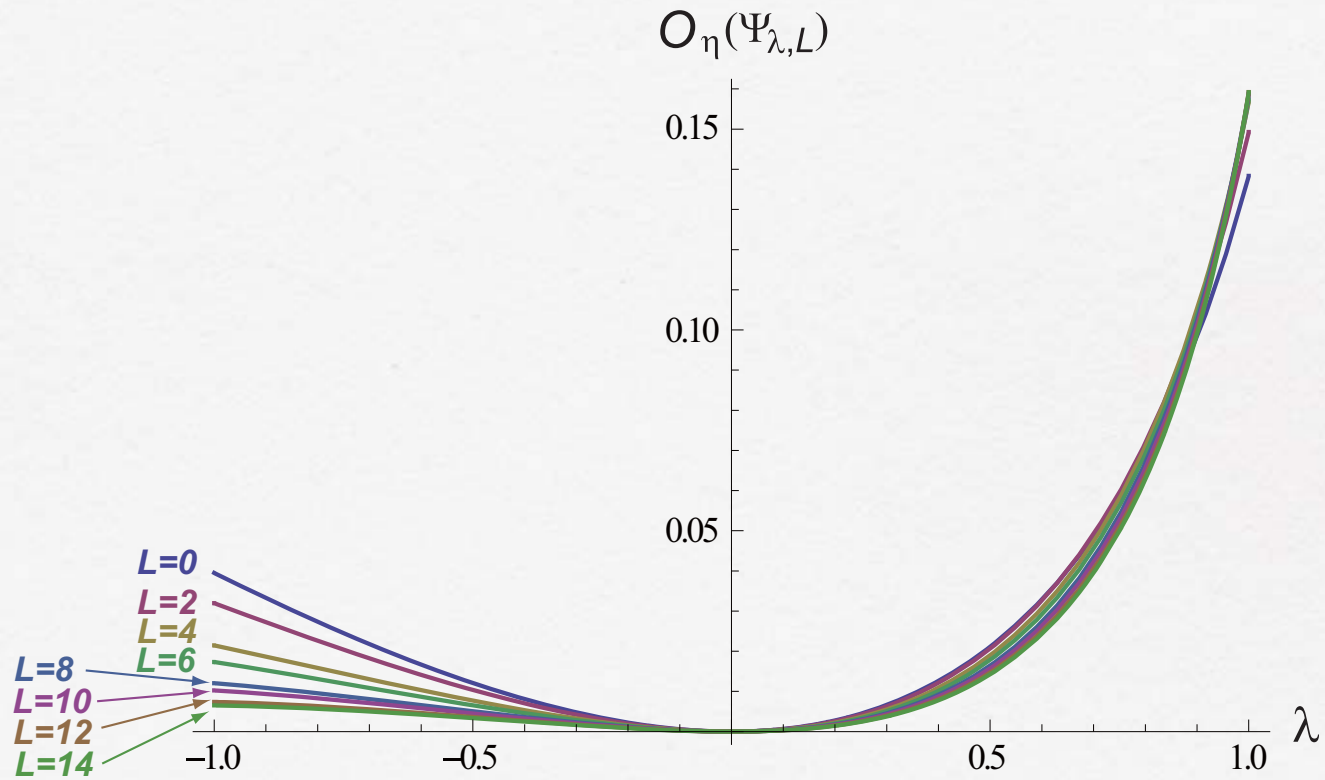
Level truncation for the gauge invariant overlap

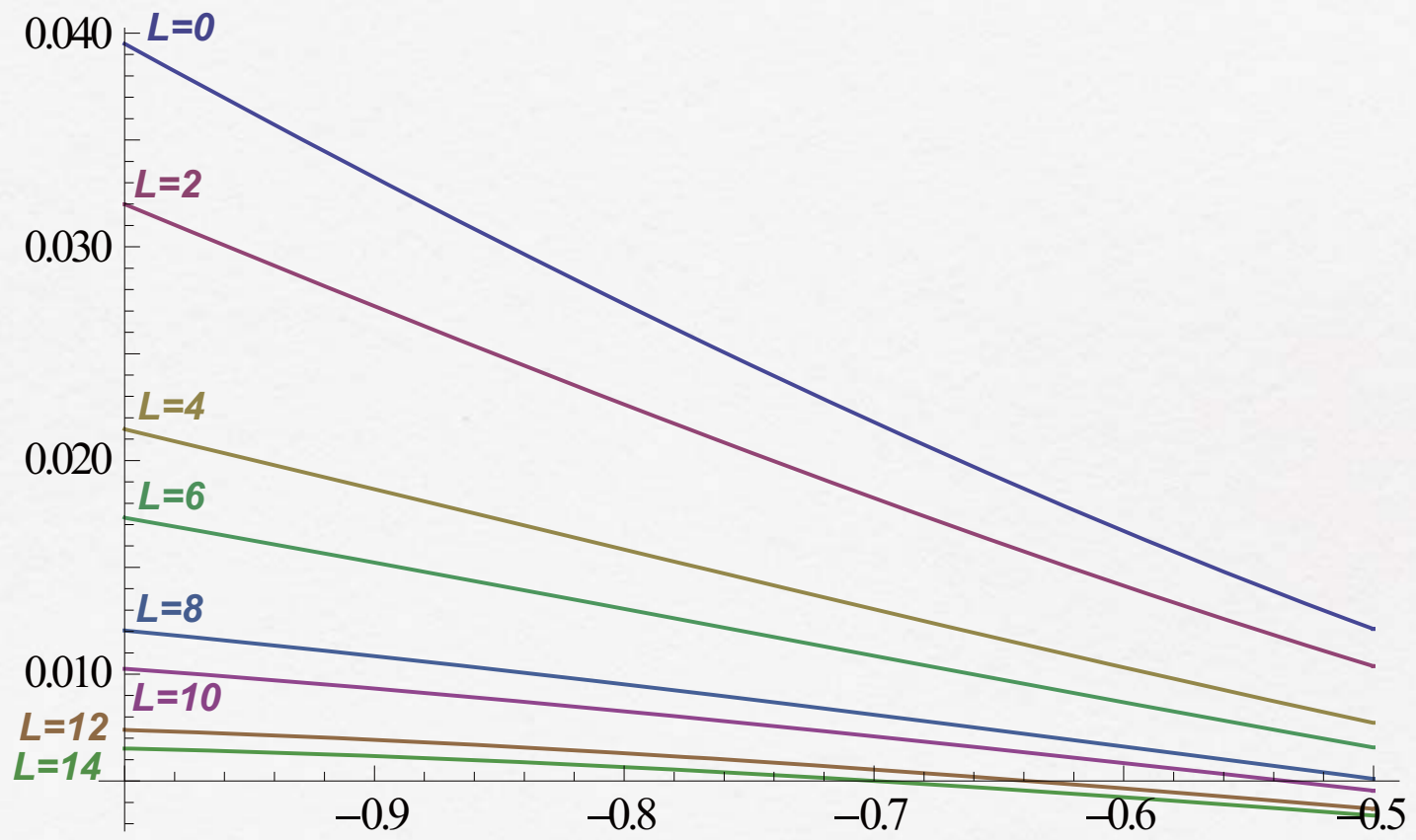
- Up to level L , using *Mathematica*, we have evaluated

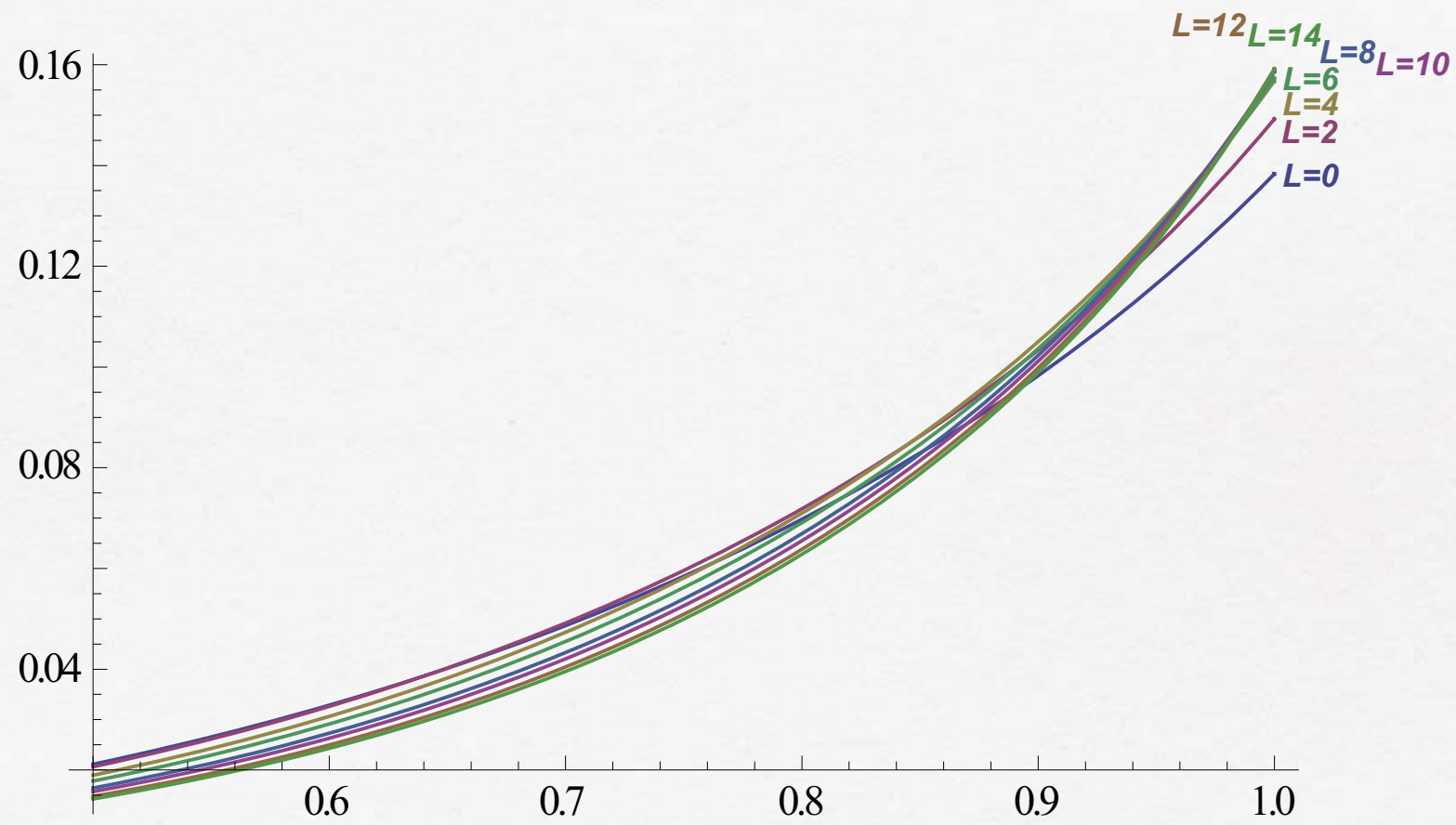
$$\mathcal{O}_\eta(\Psi_{\lambda,L}) = - \sum_{n=0}^{\infty} \lambda^{n+1} \partial_r \langle \Phi_\eta, \psi_{r,L} \rangle |_{r=n}$$

- Convergent for $-1 \leq \lambda \leq 1$
- The phantom term for $\lambda = 1$ doesn't contribute.

Result by level truncation







- Approaches to the analytical result
for $L \rightarrow \infty$

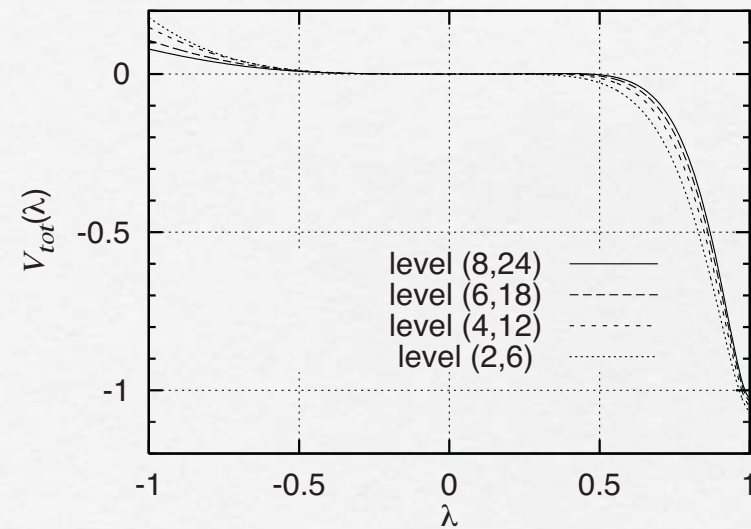
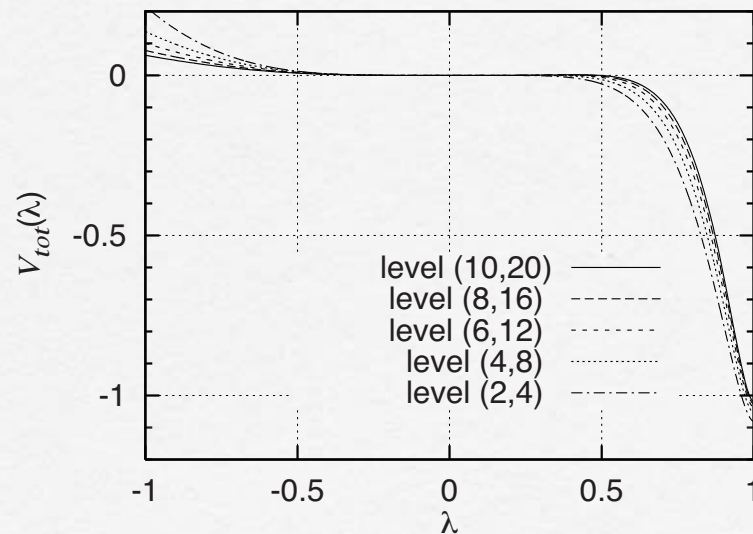
$$\mathcal{O}_\eta(\Psi_\lambda) = \begin{cases} \frac{1}{2\pi} \simeq 0.159155 & (\lambda = 1) \\ 0 & (\lambda \neq 1) \end{cases}$$

L	$\mathcal{O}_\eta(\Psi_{\lambda=1,L})$
0	0.13837
2	0.14928
4	0.15686
6	0.15740
8	0.15880
10	0.15877
12	0.15922
14	0.15916

Evaluation of the action by level truncation

□ Schnabl(2005), Takahashi(2007)

$$V_{tot}(\lambda) = -2\pi^2 g^2 S[\Psi_\lambda] / V_{26}$$



$\lambda = 1$

	$L = 0$	$L = 2$	$L = 4$	$L = 6$	$L = 8$	$L = 10$
$(L, 2L)$	-0.577920	-1.081077	-1.054081	-1.036779	-1.025645	-1.018552
$(L, 3L)$	-0.577920	-1.065177	-1.047979	-1.032868	-1.023261	—

Gauge invariant overlap for the numerical solution in Siegel gauge

$$b_0 |\Psi_N\rangle = 0 \quad [\text{Sen-Zwiebach(1999),...,Gaiotto-Rastelli(2002)}]$$

L	$\mathcal{O}_\eta(\Psi_N)$
0	0.114044
2	0.139790
4	0.147931
6	0.151225
8	0.152887
10	0.154029
12	0.154750

(L,2L) approximation

L	$\mathcal{O}_\eta(\Psi_N)$
0	0.114044
2	0.141626
4	0.148325
6	0.151369
8	0.152976
10	-
12	-

(L,3L) approximation

□ 97.2% of $\frac{1}{2\pi} = 0.159155 \dots$ $\mathcal{O}_\eta(\Psi_N) \simeq \mathcal{O}_\eta(\Psi_{\lambda=1})$

Evaluation of the action for numerical solution

□ **Gaiotto-Rastelli(2002), Table 1**

L	$-2\pi^2 g^2 S[\Psi_N]/V_{26}$
0	-0.6846161
2	-0.9485534
4	-0.9864034
6	-0.9947727
8	-0.9977795
10	-0.9991161
12	-0.9997907
14	-1.0001580
16	-1.0003678
18	-1.00049

(L,2L) approximation

L	$-2\pi^2 g^2 S[\Psi_N]/V_{26}$
0	-0.6846161
2	-0.9593766
4	-0.9878218
6	-0.9951771
8	-0.9979302
10	-0.9991825
12	-0.9998223
14	-1.0001737
16	-1.0003754
18	-1.0004937

(L,3L) approximation

Equivalence of gauge invariant overlaps in universal space

□ For any L_0 -level L ,

$$\mathcal{O}_k(\Psi_\lambda|_L) = \mathcal{O}_\eta(\Psi_\lambda|_L) \quad \mathcal{O}_k(\Psi_N|_L) = \mathcal{O}_\eta(\Psi_N|_L)$$

□ In general, $\mathcal{O}_k(\psi_L) = \mathcal{O}_\eta(\psi_L)$ for

$$\psi_L = \sum_{\substack{p,q \geq 0, n_i \geq 2, j_i \geq 1, k_i \geq 0 \\ n_1 + \dots + n_p + j_1 + \dots + j_l + k_1 + \dots + k_q = L}} c_{n_i, j_i, k_i}^{(L)} L_{-n_1}^{(m)} \dots L_{-n_p}^{(m)} b_{-j_1} \dots b_{-j_q} c_{-k_1} \dots c_{-k_q} c_1 |0\rangle$$

because on-shell closed string states satisfy

$$(L_{2n}^{(m)} - L_{-2n}^{(m)})|\Phi_{c\bar{c}V_m}\rangle = (-1)^n 3n|\Phi_{c\bar{c}V_m}\rangle, \quad (L_{2n-1}^{(m)} + L_{-2n+1}^{(m)})|\Phi_{c\bar{c}V_m}\rangle = 0$$

Summary

- Schnabl's solution Ψ_λ takes nontrivial values for gauge invariants only for $\lambda = 1$:

$$S[\Psi_\lambda]/V_{26} = \begin{cases} \frac{1}{2\pi^2 g^2} & (\lambda = 1) \\ 0 & (|\lambda| < 1) \end{cases} \quad \mathcal{O}_{k/\eta}(\Psi_\lambda) = \begin{cases} \frac{1}{2\pi} & (\lambda = 1) \\ 0 & (-1 \leq \lambda < 1) \end{cases}$$

- The results is consistent with the interpretation that
 - $\Psi_{\lambda=1}$: nontrivial solution, $\Psi_{|\lambda|<1}$: pure gauge solution
- $\Psi_{\lambda=1}$ takes (almost) the same values for gauge invariants as the numerical solution in the Siegel gauge Ψ_N :

$$S[\Psi_{\lambda=1}] \simeq S[\Psi_N] \quad \mathcal{O}_{k/\eta}(\Psi_{\lambda=1}) \simeq \mathcal{O}_{k/\eta}(\Psi_N)$$

- The above suggests the gauge equivalence: $\Psi_{\lambda=1} \sim \Psi_N$

Discussion

- What is the physical meaning of the gauge invariant overlaps?

$$\mathcal{O}_V(\Psi_{\lambda=1}) = \langle \hat{\gamma}(1_c, 2) | \psi_0 \rangle_2 c_1^{(1)} \bar{c}_1^{(1)} | V_m \rangle_{1_c} \propto \langle B_N | c_0^- c_1 \bar{c}_1 | V_m \rangle$$

- More directly,
$$\langle \hat{\gamma}(1_c, 2) | \psi_0 \rangle_2 \mathcal{P}_{1_c} = \frac{1}{2\pi} \langle B_N | c_0^-$$

- Back reaction in open-closed string field theory?

Ellwood proposed [[arXiv:0804.1131](https://arxiv.org/abs/0804.1131)]

$$\mathcal{O}_V(\Psi) = \mathcal{A}_\Psi^{\text{disk}}(V) - \mathcal{A}_0^{\text{disk}}(V)$$