Gauge Invariant Overlaps for Classical Solutions in Open String Field Theory

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references:

Kawano-I.K.-Takahashi: arXiv:0804.1541, arXiv:0804.4414,

I.K.: arXiv:0808.0355

I.K.-Takahashi: arXiv:0902.0445, arXiv:0904.1095

Non-perturbative vacuum
in bosonic open string field theory

- Schnabl's solution $\Psi_{\rm Sch}$

Gauge invariants

(1) Action: D-brane tension

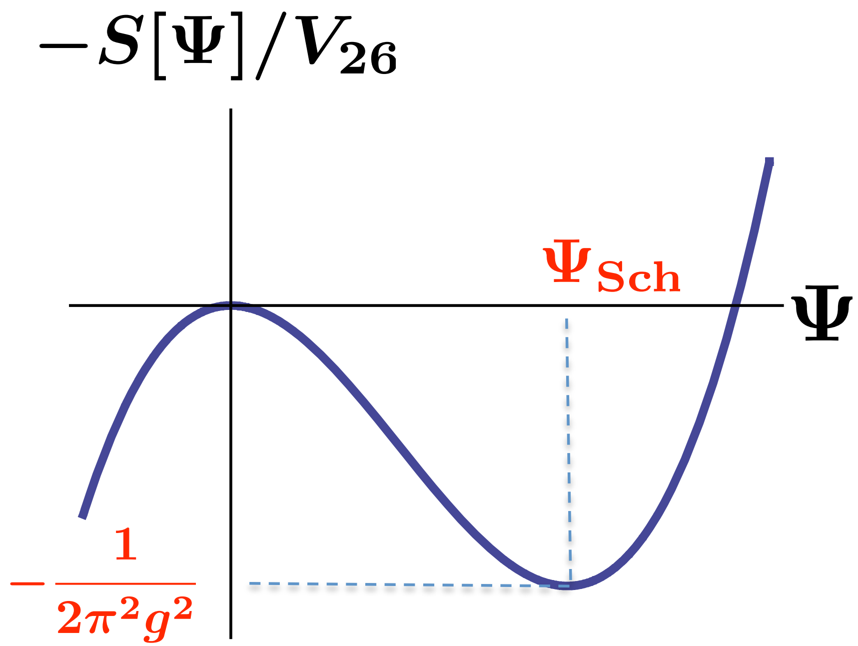
$$S[\Psi_{\rm Sch}]/V_{26} = \frac{1}{2\pi^2 g^2}$$

[Schnabl(2005),Okawa,Fuchs-Kroyter(2006)]

(2) Gauge invariant overlap:

$$\mathcal{O}_V(\Psi_{\rm Sch})/V_{26} = \frac{1}{2\pi}$$

[Ellwood, Kawano-Kishimoto-Takahashi(2008)]

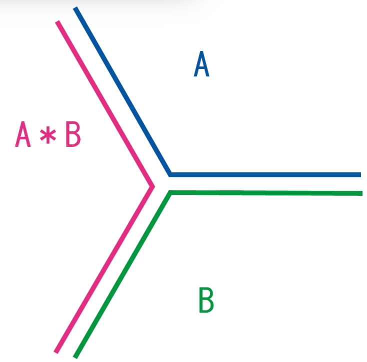


Contents

Bosonic cubic open string field theory

Action: $$S[\Psi] = -\frac{1}{g^2}\left(\frac{1}{2}\langle\Psi, Q\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right)$$

$$Q = \oint \frac{dz}{2\pi i}\left(cT^{\mathrm{m}} + bc\partial c + \frac{3}{2}\partial^2 c\right)$$



Equation of motion: $$Q\Psi + \Psi * \Psi = 0$$

Gauge transformation: $$\delta_\Lambda \Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi$$

$$\rightarrow \quad \delta_\Lambda S[\Psi] = 0$$

Restrict string fields to twist even sector in the universal space:

$$\Psi = (t_1 + t_2 b_{-1} c_{-1} + t_3 L^{(\mathrm{m})}_{-2} + \cdots) c_1|0\rangle + (u_1 b_{-2} + \cdots) c_0 c_1 |0\rangle$$

Gauge invariant overlap

Gauge invariant for on–shell closed string state

$$\mathcal{O}_V(\Psi) = \langle \mathcal{I} | V(i) | \Psi \rangle = \langle \hat{\gamma}(1_c, 2) | \Phi_V \rangle_{1_c} | \Psi \rangle_2$$

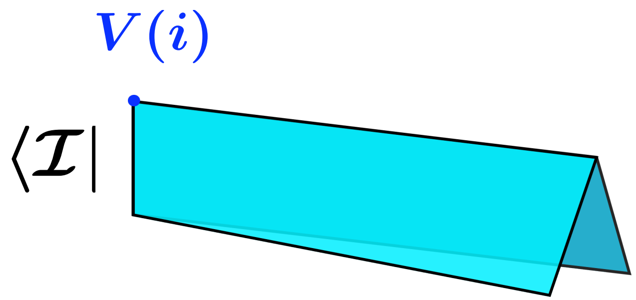
$$|\Phi_V\rangle = c_1 \bar{c}_1 | V_m \rangle$$

V_m :matter primary with (1,1)-dim.

$$\mathcal{O}_V(Q\Lambda) = 0$$

$$\mathcal{O}_V(\Psi * \Lambda) = \mathcal{O}_V(\Lambda * \Psi)$$

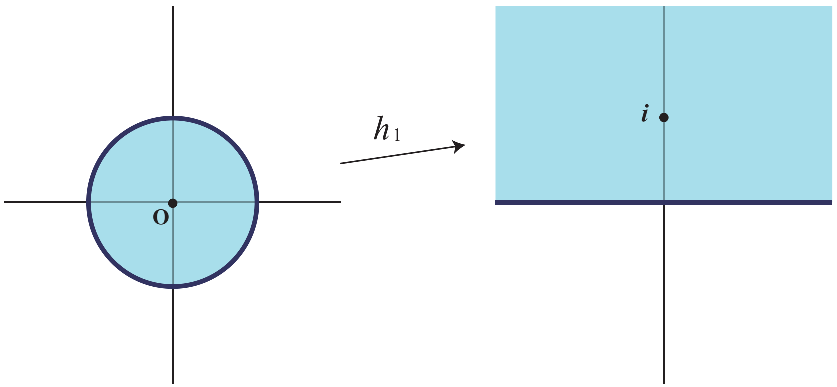
$$\Rightarrow \quad \delta_\Lambda \mathcal{O}_V(\Psi) = 0$$

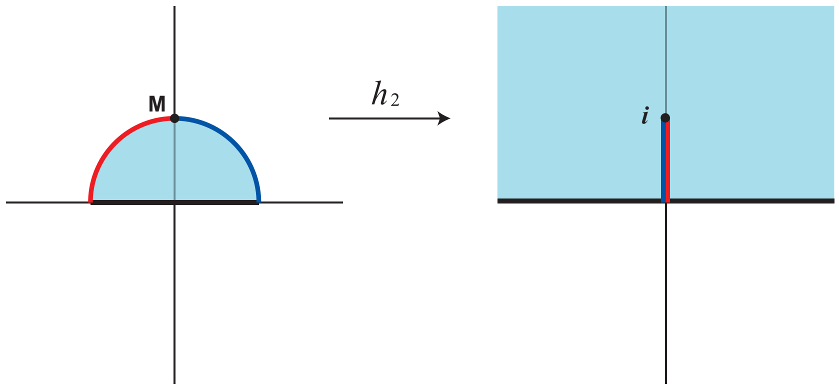


In particular, it vanishes for pure gauge solutions: $\mathcal{O}_V(e^{-\Lambda} Q e^{\Lambda}) = 0$

Shapiro-Thorn's vertex

$$\langle \hat{\gamma}(1_c, 2)|\phi_c\rangle_{1_c}|\psi\rangle_2 = \langle h_1[\phi_c(0)]h_2[\psi(0)]\rangle_{\mathrm{UHP}}$$





$$h_1(w) = -i\frac{w-1}{w+1}$$

closed string

$$h_2(w) = I \circ h_{\mathcal{I}}(w) = \frac{1}{2}\left(w - \frac{1}{w}\right)$$

open string

identity state: $\langle \mathcal{I}|\phi\rangle = \langle h_{\mathcal{I}}[\phi(0)]\rangle_{\mathrm{UHP}}$

Gauge invariant overlap for Schnabl's analytic solution

- Schnabl's solution for tachyon condensation

$$\Psi_{\rm Sch} = \frac{\partial_r}{e^{\partial_r}-1}\psi_r|_{r=0} = \sum_{n=0}^{\infty}\frac{B_n}{n!}\partial_r^n\psi_r|_{r=0}$$

$$= \lim_{N\to+\infty}\left(\psi_{N+1} - \sum_{n=0}^{N}\partial_r\psi_r|_{r=n}\right)$$

$$\psi_r \equiv \frac{2}{\pi}U_{r+2}^\dagger U_{r+2}\left[-\frac{1}{\pi}(\mathcal{B}_0+\mathcal{B}_0^\dagger)\tilde{c}(\frac{\pi r}{4})\tilde{c}(-\frac{\pi r}{4}) + \frac{1}{2}(\tilde{c}(-\frac{\pi r}{4})+\tilde{c}(\frac{\pi r}{4}))\right]|0\rangle \qquad U_r \equiv (2/r)^{\mathcal{L}_0}$$

⇨ $\mathcal{O}_V(\psi_r)$:independent of r

[Ellwood, Kawano-Kishimoto-Takahashi (2008)]

$$\mathcal{O}_V(\Psi_{\rm Sch}) = \mathcal{O}_V(\psi_0) = \lim_{N\to\infty}\mathcal{O}_V(\psi_{N+1})$$

Analytic computation of gauge inv. overlap for Schnabl's solution (1)

- Note: $\psi_r = \frac{2}{\pi} c_1|0\rangle + O(\mathcal{L}_0 - \mathcal{L}_0^\dagger, \mathcal{B}_0 - \mathcal{B}_0^\dagger, c_n + (-1)^n c_{-n})$

ψ_0

does not contribute to the gauge invariant overlap.

$$\langle\hat{\gamma}(1_c,2)|\left((L_n^{(2)} - (-1)^n L_{-n}^{(2)} - (-1)^{\frac{n}{2}}\frac{n}{4}c\,\delta_{n:\mathrm{even}}\right)$$

$$= \langle\hat{\gamma}(1_c,2)|(-2i^n)\sum_{m\geq 0}(-1)^m(\eta_{2m+1}^n - \eta_{2m-1}^n)(L_m^{(1)} + (-1)^n \bar{L}_m^{(1)})$$

$$\langle\hat{\gamma}(1_c,2)|(b_n^{(2)} - (-1)^n b_{-n}^{(2)})$$

$$= \langle\hat{\gamma}(1_c,2)|(-2i^n)\sum_{m\geq 0}(-1)^m(\eta_{2m+1}^n - \eta_{2m-1}^n)(b_m^{(1)} + (-1)^n \bar{b}_m^{(1)})$$

$$\langle\hat{\gamma}(1_c,2)|(c_m^{(2)} + (-1)^m c_{-m}^{(2)})$$

$$\left(\frac{1+x}{1-x}\right)^k = \sum_{n=0}^{\infty}\eta_n^k x^n$$

$$= \langle\hat{\gamma}(1_c,2)|\frac{-i^m}{4}\sum_{n\geq 1}(-1)^n(\eta_{m+1}^{2n} - \eta_{m-1}^{2n} + \delta_{m,1})(c_n^{(1)} + (-1)^m \bar{c}_n^{(1)})$$

Analytic computation of gauge inv. overlap for Schnabl's solution (2)

- Relation to the boundary state

$$\langle\hat{\gamma}(1_c,2)|\psi_0\rangle_2\mathcal{P}_{1_c} = \frac{1}{2\pi}\langle B|c_0^-$$

[Kawano-I.K.-Takahashi(2008)]

⇓ generalization

[Kiermaier-Okawa-Zwiebach(2008)]

$$\begin{aligned}
|B_*(\Psi_{\rm Sch})\rangle &\equiv e^{\frac{\pi^2}{s}(L_0+\bar{L}_0)}\oint_s \mathrm{P}e^{-\int_0^s dt[\mathcal{L}_R(t)+\{\mathcal{B}_R(t),\Psi_{\rm Sch}\}]} \\
&= |B\rangle + \sum_{k=1}^{\infty}|B_*^{(k)}(\Psi_{\rm Sch})\rangle \\
&= 0
\end{aligned}$$

Analytic computation of gauge inv. overlap for Schnabl's solution (3)

$$\begin{aligned}\mathcal{O}_V(\Psi_{\mathrm{Sch}}) &= \mathcal{O}_V(\psi_0) = \langle\hat{\gamma}(1_c,2)|\Phi_V\rangle_{1_c}|\psi_0\rangle_2 \\ &= \frac{1}{2\pi}\langle B|c_0^-|\Phi_V\rangle\end{aligned}$$

$$\langle B| = \langle 0|c_{-1}\bar{c}_{-1}c_0^+ \exp\left(-\sum_{n=1}^{\infty}\left(\frac{1}{n}\alpha_n\cdot\bar{\alpha}_n + c_n\bar{b}_n + \bar{c}_n b_n\right)\right)$$

For the Schnabl solution with a parameter $\lambda \quad (\lambda \neq 1)$:

$$\Psi_\lambda = \frac{\lambda\partial_r}{\lambda e^{\partial_r}-1}\psi_r|_{r=0} = \sum_{n=1}^{\infty}\frac{f_n(\lambda)}{n!}\partial_r^n\psi_r|_{r=0} = -\sum_{n=0}^{\infty}\lambda^{n+1}\partial_r\psi_r|_{r=n}$$

⇨ $\mathcal{O}_V(\Psi_\lambda) = 0$

Gauge invariants for Schnabl's solution

Our result: $(\Psi_{\lambda=1} \equiv \Psi_{\mathrm{Sch}})$

$$\mathcal{O}_V(\Psi_\lambda) = \begin{cases} 1/(2\pi)\langle B|c_0^-|\Phi_V\rangle & (\lambda = 1) \\ 0 & (|\lambda| < 1) \end{cases}$$

is consistent with

$$S[\Psi_\lambda] = \begin{cases} 1/(2\pi^2 g^2) & (\lambda = 1) \\ 0 & (|\lambda| < 1) \end{cases}$$

[Schnabl(2005),Okawa,Fuchs-Kroyter(2006)]

$\lambda = 1$ → Ψ_{Sch} : nontrivial solution

$|\lambda| < 1$ → Ψ_λ : pure gauge solution

Ellwood's proposal

For a *solution* to the equation of motion Ψ

$$\mathcal{O}_V(\Psi) = \mathcal{A}^{\rm disk}_{\Psi}(V) - \mathcal{A}^{\rm disk}_{0}(V)$$ [Ellwood(2008)]

$\uparrow$

Disk amplitude for a closed string vertex V specified by a solution Ψ

$$\lim_{s\to 0}\langle\Phi_V|c_0^-|B_*(\Psi)\rangle - \langle\Phi_V|c_0^-|B\rangle$$

[Kiermaier-Okawa-Zwiebach(2008)]

In particular, $\mathcal{O}_V(\Psi_{\rm Sch}) = 0 - \mathcal{A}^{\rm disk}_{0}(V)$

Gauge invariant overlap for marginal solutions

$$\mathcal{O}_V(\Psi^J_{\lambda_{\rm m}}) = \mathcal{O}_V(\Psi^J_{L,\lambda_{\rm m}}) = \frac{1}{2\pi i}\langle V_{\rm m}(0)\, e^{-\lambda_{\rm m}\oint J}\rangle^{\rm mat}_{\rm disk} - \frac{1}{2\pi i}\langle V_{\rm m}(0)\rangle^{\rm mat}_{\rm disk}$$

[Ellwood(2008)]

Fuchs-Kroyter-Potting/Kiermaier-Okawa's marginal solution (2007)

$$\Psi^J_{L,\lambda_{\rm m}} = \lambda_{\rm m} cJ(0)|0\rangle + \cdots$$

[I.K.(2008)]

Schnabl/Kiermaier-Okawa-Rastelli-Zwiebach's marginal solution (2007)

$$\Psi^J_{\lambda_{\rm m}} = \lambda_{\rm m} cJ(0)|0\rangle + \cdots$$

Gauge invariant overlap for string fields in the universal space

- For string fields in the twist even universal space such as

$$\begin{aligned}\Psi_{\rm univ} &= (t_1 + t_2 b_{-1}c_{-1} + t_3 L^{(\rm m)}_{-2} + t_4 b_{-3}c_{-1} + t_5 b_{-2}c_{-2} + t_6 b_{-1}c_{-3} \\ &\quad + t_7 L^{(\rm m)}_{-2} b_{-1}c_{-1} + t_8 L^{(\rm m)}_{-4} + t_9 (L^{(\rm m)}_{-2})^2 + \cdots)c_1|0\rangle \\ &\quad + (u_1 b_{-2} + u_2 b_{-4} + u_3 b_{-2}b_{-1}c_{-1} + u_4 L^{(\rm m)}_{-2} b_{-2} + u_5 L^{(\rm m)}_{-3} b_{-1} + \cdots)c_0 c_1|0\rangle\end{aligned}$$

⬇

$$\mathcal{O}_V(\Psi_{\rm univ}) = \frac{1}{4}t_1 - \frac{1}{4}t_2 - \frac{3}{4}t_3 + \frac{1}{4}t_5 + \frac{3}{4}t_7 + \frac{3}{2}t_8 + \frac{11}{2}t_9 + \cdots$$

- Here, we take a normalization such as

$$|V_{\rm m}\rangle = \frac{-1}{26}\eta_{\mu\nu}\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1}|0\rangle$$

Zero momentum dilaton state

$$\Phi_\eta = \frac{1}{52\alpha' i}\eta_{\mu\nu}\lim_{\theta\to\frac{\pi}{2}} c(e^{i\theta})\partial X^\mu(e^{i\theta})c(e^{-i\theta})\partial X^\nu(e^{-i\theta})|\mathcal{I}\rangle$$

$$= \left(\frac{1}{4} - \frac{2}{13}\sum_{n,m=1}^{\infty} mn\cos\frac{(m-n)\pi}{2}\,\alpha_{-m}\cdot\alpha_{-n}\right) e^E c_0 c_1|0\rangle\,,$$

$$E = \sum_{n=1}^{\infty}(-1)^n\left(-\frac{1}{2n}\alpha_{-n}\cdot\alpha_{-n} + c_{-n}b_{-n}\right)$$

$$|\Phi_\eta\rangle_3 = \langle\hat{\gamma}(1_{\rm c},2)|c_1\bar{c}_1|V_{\rm m}\rangle_{1_{\rm c}}|R(2,3)\rangle$$

It satisfies $\quad Q|\Phi_\eta\rangle = 0$

$$(L^{\rm mat}_{2n} - L^{\rm mat}_{-2n})|\Phi_\eta\rangle = (-1)^n 3n|\Phi_\eta\rangle$$

$$(L^{\rm mat}_{2n-1} + L^{\rm mat}_{-2n+1})|\Phi_\eta\rangle = 0$$

Level truncation of Schnabl's solution

- Conventional oscillator expression

$$\psi_{r-2} = \left[\prod_{k=1,\leftarrow}^{\infty} e^{u_{2k}(r)L_{-2k}}\right]\left[\frac{1}{\pi}\sin\frac{2\pi}{r}\left(1-\frac{r}{2\pi}\sin\frac{2\pi}{r}\right)\sum_{p\geq -1;p:\mathrm{odd}}\left(\frac{2}{r}\cot\frac{\pi}{r}\right)^{p} c_{-p}|0\rangle\right.$$
$$\left.+\frac{r}{2\pi^2}\left(\sin\frac{2\pi}{r}\right)^2\sum_{s\geq 2;s:\mathrm{even}}\frac{(-1)^{\frac{s}{2}+1}}{s^2-1}\left(\frac{2}{r}\right)^{s}\sum_{p,q\geq -1;p+q:\mathrm{odd}}(-1)^q\left(\frac{2}{r}\cot\frac{\pi}{r}\right)^{p+q} b_{-s}c_{-p}c_{-q}|0\rangle\right]$$

$$u_2(r) = -\frac{r^2-4}{3r^2},\quad u_4(r) = \frac{r^4-16}{30r^4},\quad u_6(r) = -\frac{16(r^2-4)(r^2-1)(r^2+5)}{945r^6},\ldots$$

level L-truncation $(-1 \leq \lambda \leq 1)$

$$\mathcal{O}_\eta(\Psi_{\lambda,L}) = -\sum_{n=0}^{\infty}\lambda^{n+1}\partial_r\langle\Phi_\eta,\psi_{r,L}\rangle|_{r=n}$$

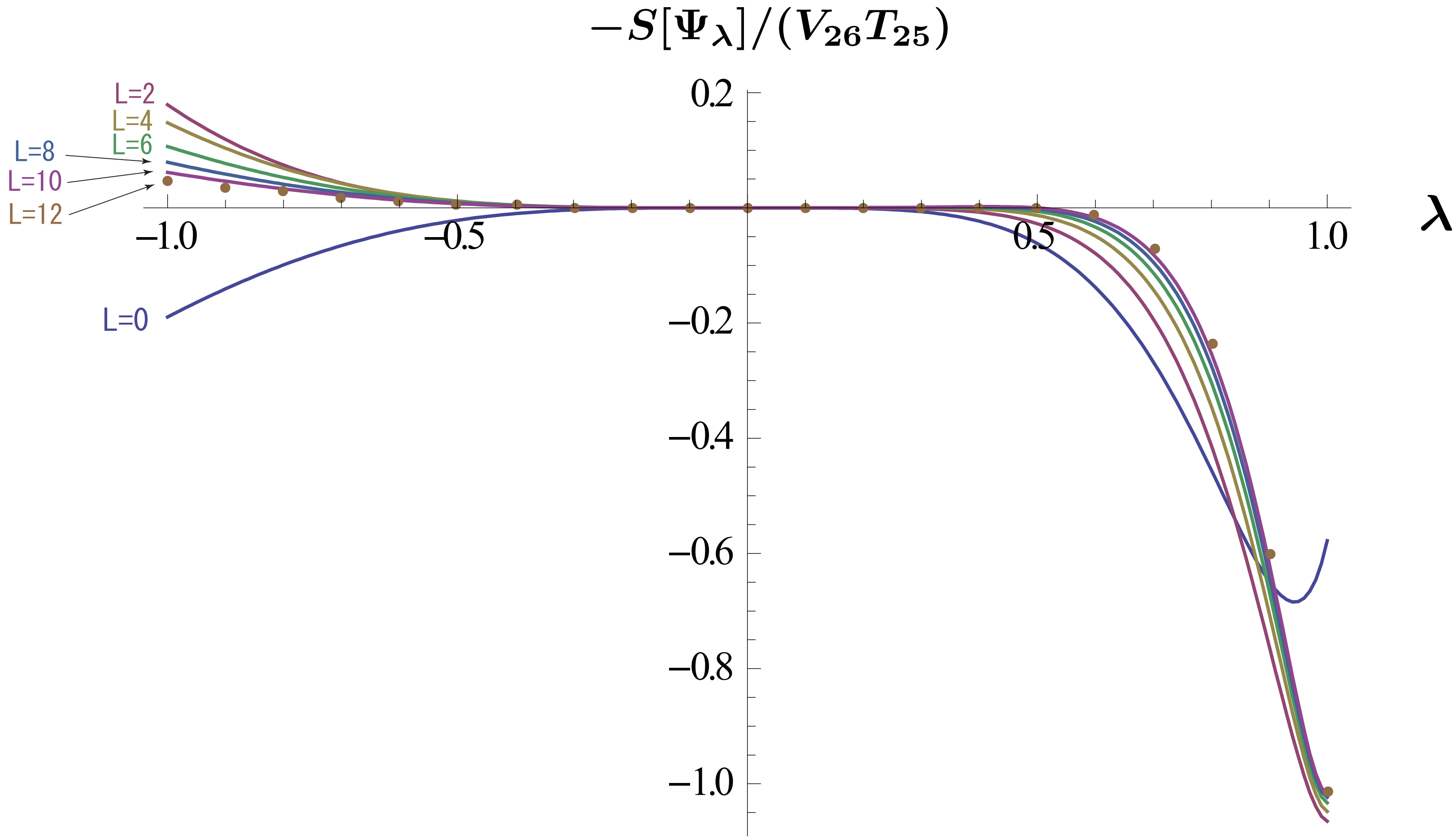
$$\psi_{N+1} = O(N^{-3}) \quad (N\to\infty)$$

Evaluation of the potential height by level truncation

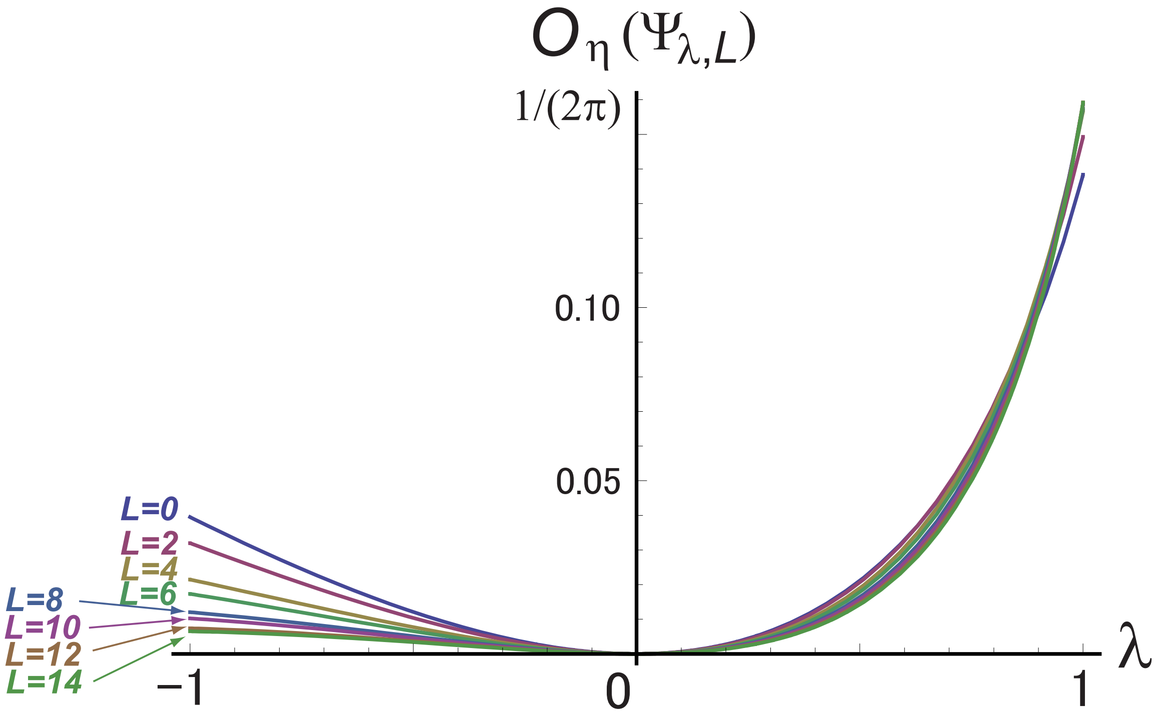
The “phantom” term doesn’t contribute.

[Schnabl(2005),Takahashi(2007)]

$$\Psi_\lambda = -\sum_{n\geq 0} \lambda^{n+1} (\partial_r \psi_r|_{r=n})_L \qquad (-1 \leq \lambda \leq 1)$$



Evaluation of the gauge invariant overlap by level truncation



L	$\mathcal{O}_\eta(\Psi_{\lambda=1,L})$
0	0.13837
2	0.14928
4	0.15686
6	0.15740
8	0.15880
10	0.15877
12	0.15922
14	0.15916

⇒

$$\mathcal{O}_\eta(\Psi_\lambda) = \begin{cases} \dfrac{1}{2\pi} \simeq 0.159155 & (\lambda = 1) \\ 0 & (\lambda \neq 1) \end{cases}$$

Numerical evaluation of gauge invariants for Schnabl's solution by "level truncation"

$S[\Psi_{\rm Sch}\vert_L]/S[\Psi_{\rm Sch}]$	
(2,6)	1.06518
(4,12)	1.04798
(6,18)	1.03287
(8,24)	1.02326
(10,30)	1.01705
(12,36)	1.01287
(14,42)	1.00994

$\mathcal{O}_V(\Psi_{\rm Sch}\vert_L)/\mathcal{O}_V(\Psi_{\rm Sch})$	
L=2	0.937981
L=4	0.985559
L=6	0.988942
L=8	0.997737
L=10	0.997547
L=12	1.00041
L=14	1.00002

Numerical solution by level truncation

- Numerical solution in the Siegel gauge : $b_0|\Psi_{\mathrm{N}}\rangle = 0$

  [...,Sen-Zwiebach(1999),...]

(1) $S[\Psi_{\mathrm{N}}]/S[\Psi_{\mathrm{Sch}}]$

(L,2L)-truncation		(L,3L)-truncation	
(2,4)	0.9485534	(2,6)	0.9593766
(4,8)	0.9864034	(4,12)	0.9878218
(6,12)	0.9947727	(6,18)	0.9951771
(8,16)	0.9977795	(8,24)	0.9979302
(10,20)	0.9991161	(10,30)	0.9991825
(12,24)	0.9997907	(12,36)	0.9998223
(14,28)	1.0001580	(14,42)	1.0001737
(16,32)	1.0003678	(16,48)	1.0003754
(18,36)	1.00049	(18,54)	1.0004937

[Gaiotto-Rastelli(2002)]

(2) $\mathcal{O}_V(\Psi_{\mathrm{N}})/\mathcal{O}_V(\Psi_{\mathrm{Sch}})$

(L,2L)-truncation		(L,3L)-truncation	
(2,4)	0.8783238	(2,6)	0.8898618
(4,8)	0.9294792	(4,12)	0.9319524
(6,12)	0.9501746	(6,18)	0.9510789
(8,16)	0.9606165	(8,24)	0.9611748
(10,20)	0.9677900	(10,30)	0.9681148
(12,24)	0.9723211	(12,36)	0.9725595
(14,28)	0.9760046	(14,42)	0.9761715
(16,32)	0.9785442	(16,48)	0.9786768

[Kawano-Kishimoto-Takahashi(2008)]
and the latest result

Evidence of gauge equivalence: $\Psi_{\mathrm{N}} \sim \Psi_{\mathrm{Sch}}$

Numerical solutions in *a*-gauges

- Asano-Kato's *a*-gauge $$(b_0 M + a\, b_0 c_0 \tilde{Q})|\Psi_a\rangle = 0$$ $$Q = \tilde{Q} + c_0 L_0 + b_0 M$$

$a = 0$ ⇒ Siegel gauge: $b_0|\Psi_0\rangle = 0$

$a = \infty$ ⇒ Landau gauge: $b_0 c_0 \tilde{Q}|\Psi_\infty\rangle = 0$

(1) For *a*-gauge solution, (6,18)-trucation $S[\Psi_a]/S[\Psi_{\mathrm{Sch}}]$

$a = \infty$	⇒	0.9609438
$a = 4.0$		0.9244886
$a = 0.5$		1.0045858
$a = -2.0$		0.9798943
⋮		⋮

[Asano-Kato(2006)]

(and higher level?)

(2) $\mathcal{O}_V(\Psi_a)/\mathcal{O}_V(\Psi_{\mathrm{Sch}})$ (?)

⇒ our computation

Asano-Kato's *a*-gauge

In the worldsheet ghost number 1 sector,

$$(b_0 M + a b_0 c_0 \tilde{Q})\Phi_1 = 0$$

$$M = -2\sum_{n=1}^{\infty} n c_{-n} c_n$$

$$\tilde{Q} = \sum_{n\neq 0} c_{-n} L_n^{(\mathrm{m})} - \frac{1}{2}\sum_{n,m,m+n\neq 0} (m-n) c_{-m} c_{-n} b_{m+n}$$

Note: $a = 1$ ⟹ $b_0 c_0 Q \Phi_1 = 0$

Under the gauge transformation in the free level $\Phi_1 \mapsto \Phi_1 + Q\Lambda_0$
this condition cannot fix the gauge.

⟹ $a \neq 1$ *perturbatively*

On the *a*-gauge

- The *a*-gauge condition conserves the level.

  ⟹ suitable to the level truncation

- The *a*-gauge condition is compatible with the twist even sector in the universal space.

dimension of the truncated space in the *a*-gauge:

L	0	2	4	6	8	10	12	14	16	18
dim.	1	3	9	26	69	171	402	898	1925	3985

the same as that of the Siegel gauge

Asano-Kato's gauge fixed action

$$S_{\mathrm{GF}} = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\langle\Phi_n, Q\Phi_{2-n}\rangle - \frac{g}{3}\sum_{l+n+m=3}\langle\Phi_l, \Phi_m * \Phi_n\rangle + \sum_{n=-\infty}^{\infty}\langle(\mathcal{O}_a\mathcal{B})_{3-n}, \Phi_n\rangle$$

$\Phi_n, \mathcal{B}_n$: worldsheet ghost number *n*

$$(\mathcal{O}_a\mathcal{B})_n = (b_0 M^{n-1} + ac_0 b_0 M^{n-2}\tilde{Q})\mathcal{B}_{3-n}$$

$$(\mathcal{O}_a\mathcal{B})_{3-n} = (b_0 W_{n-2} + ac_0 b_0 W_{n-1}\tilde{Q})\mathcal{B}_n \qquad (n \geq 2)$$

$$W_n = \sum_{i=0}^{\infty}\frac{(-1)^i(n+i-1)!}{i!(n-1)!((n+i)!)^2}M^i(M^-)^{n+i} \qquad M^- = -\sum_{n=1}^{\infty}\frac{1}{2n}b_{-n}b_n$$

integrate out $\mathcal{B}_n$

$$b_0(M^{n-1} + ac_0\tilde{Q}M^{n-2})\Phi_{3-n} = 0$$

$$b_0(W_{n-2} + ac_0\tilde{Q}W_{n-1})\Phi_n = 0 \qquad (n \geq 2)$$

gauge fixing condition

Massless part

Let us consider "level 1" part of the string fields:

$$\begin{aligned}\Phi &= \gamma(x)|0\rangle + (A_\mu(x)\alpha^\mu_{-1}c_1 + \beta(x)c_0)|0\rangle \\ &\quad +(\bar{\gamma}(x)c_{-1}c_1 + u_\mu(x)\alpha^\mu_{-1}c_0c_1)|0\rangle + v(x)c_{-1}c_0c_1|0\rangle\end{aligned}$$

$$\mathcal{B} = \beta_\chi(x)c_0|0\rangle + \beta_\mu(x)\alpha^\mu_{-1}c_0c_1|0\rangle + \beta_v(x)c_{-1}c_0c_1|0\rangle$$

$$\begin{aligned}S_{\mathrm{GF}}|_{\mathrm{quad.}} &= \int d^{26}x\Big(-\frac{\alpha'}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}(-\sqrt{2}i\beta + \sqrt{\alpha'}\partial_\mu A^\mu)^2 \\ &\quad - \alpha'\bar{\gamma}\partial_\mu\partial^\mu\gamma - i\sqrt{2\alpha'}u_\mu\partial^\mu\gamma \\ &\quad + \frac{1}{2}\beta_v v + \beta_\mu(u^\mu + a\sqrt{\alpha'/2}i\partial^\mu\bar{\gamma}) - \sqrt{2}i\beta_\chi(-\sqrt{2}i\beta + a\sqrt{\alpha'}\partial_\mu A^\mu)\Big)\end{aligned}$$

field redefinition

$$S_{\mathrm{GF}}|_{\mathrm{quad.}} = \int d^{26}x\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + B\partial_\mu A^\mu + \frac{\alpha}{2}B^2 + i\bar{c}\partial_\mu\partial^\mu c - \frac{1}{2}\tilde{\chi}^2 + \frac{1}{2}\tilde{\beta}_\mu\tilde{u}^\mu + \frac{1}{2}\beta_v v\Big)$$

$$\alpha = \frac{1}{(a-1)^2}$$

Construction of numerical solutions

$$\Psi_{(0)} = \frac{64}{81\sqrt{3}} c_1|0\rangle$$: nontrivial solution for (0,0)-truncation

$$(b_0 M + a\, b_0 c_0 \tilde{Q})\Psi_{(n+1)} = 0$$: *a*-gauge condition

$$\mathcal{P}(Q_{\Psi_{(n)}}\Psi_{(n+1)} - \Psi_{(n)} * \Psi_{(n)}) = 0$$ **: linear equations!**

$$\mathcal{P} = c_0 b_0$$: a projection to solve equations

$$Q_{\Psi_{(n)}}\Phi \equiv Q\Phi + \Psi_{(n)} * \Phi - (-1)^{|\Phi|}\Phi * \Psi_{(n)}$$

: "BRST operator" around $\Psi_{(n)}$

➡ We can define $\Psi_{(n)} \mapsto \Psi_{(n+1)}$

$$\Psi_{(n+1)} \simeq (Q_{\Psi_{(n)}})^{-1}(\Psi_{(n)} * \Psi_{(n)})$$ [Gaiotto-Rastelli(2002)]

On the equation of motion

If the iteration converges for $n \longrightarrow \infty$

$$(b_0 M + a\, b_0 c_0 \tilde{Q})\Psi_{(\infty)} = 0$$: *a*-gauge condition

$$\mathcal{P}(Q\Psi_{(\infty)} + \Psi_{(\infty)} * \Psi_{(\infty)}) = 0$$: projected part of eq. of motion

We check the remaining part of the equation of motion
for the resulting configuration:

$$\underbrace{(1-\mathcal{P})}_{b_0 c_0}(Q\Psi_{(\infty)} + \Psi_{(\infty)} * \Psi_{(\infty)}) = 0 \quad (?)$$

"BRST invariance" [Hata-Shinohara(2000)]

“Norm” of string fields

Level *L*-truncated string field in the universal space:

$$\Phi = \sum_{k+l \leq L} \sum_{m_k, n_l} t_{k,m_k;l,n_l}\, \varphi_{k,m_k} \otimes \psi_{l,n_l}$$

φ_{k,m_k} : a linear combination of

$$L^{(\mathrm{m})}_{-n_1} L^{(\mathrm{m})}_{-n_2} \cdots L^{(\mathrm{m})}_{-n_q} |0\rangle_{\mathrm{m}} \quad (n_1 \geq n_2 \geq \cdots \geq n_q \geq 2)$$

s.t.

$$\langle \varphi_{k,m_k}, \varphi_{k',m'_{k'}} \rangle = (-1)^k \delta_{k,k'} \delta_{m_k, m'_{k'}}, \quad L^{(\mathrm{m})}_0 |\varphi_{k,m_k}\rangle = k |\varphi_{k,m_k}\rangle$$

$$|\psi_{k,m_k}\rangle = b_{-p_1} b_{-p_2} \cdots b_{-p_r} c_{-q_1} c_{-q_2} \cdots c_{-q_s} c_1 |0\rangle_{\mathrm{gh}}$$

$$p_1 > p_2 > \cdots > p_r \geq 1, \quad q_1 > q_2 > \cdots > q_s \geq 0, \quad \sum_{t=1}^{r} p_t + \sum_{u=1}^{s} q_u = k$$

$$\Longrightarrow \qquad \|\Phi\| = \left(\sum_{k,m_k,l,n_l} |t_{k,m_k;l,n_l}|^2 \right)^{\frac{1}{2}}$$

Convergence of iterations

We continue the iterations until

$$\frac{||\Psi_M - \Psi_{M-1}||}{||\Psi_M||} < 10^{-8}$$

For various *a*, $M < 10$

$-\infty \leqq a \lesssim 0, \ 1 \ll a \leqq \infty$

$$||\Psi_M|| \sim O(1)$$

$$\frac{||c_0 b_0 (Q\Psi_M + \Psi_M * \Psi_M)||}{||\Psi_M||} < 10^{-8}$$

Comments on projection

- If we solve the *a*-gauge condition explicitly and substitute it into the original action, we get

$$\boldsymbol{S[\Psi]\big|_{\Psi:a\text{-gauge}}}$$

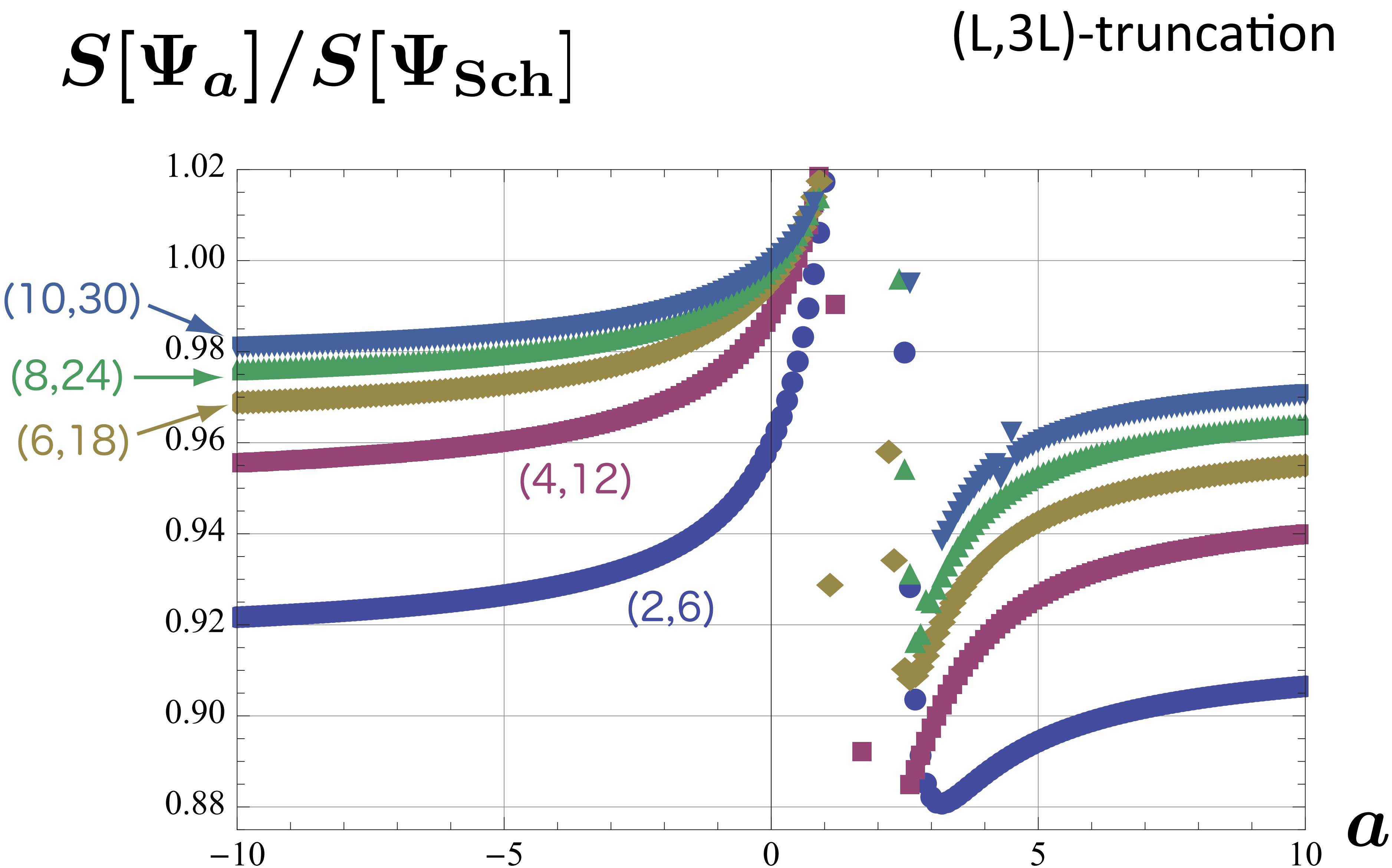
variation

$$\mathbf{bpz}(\mathcal{P}_{\mathbf{GF}})(\boldsymbol{Q\Psi+\Psi*\Psi})=\mathbf{0}$$

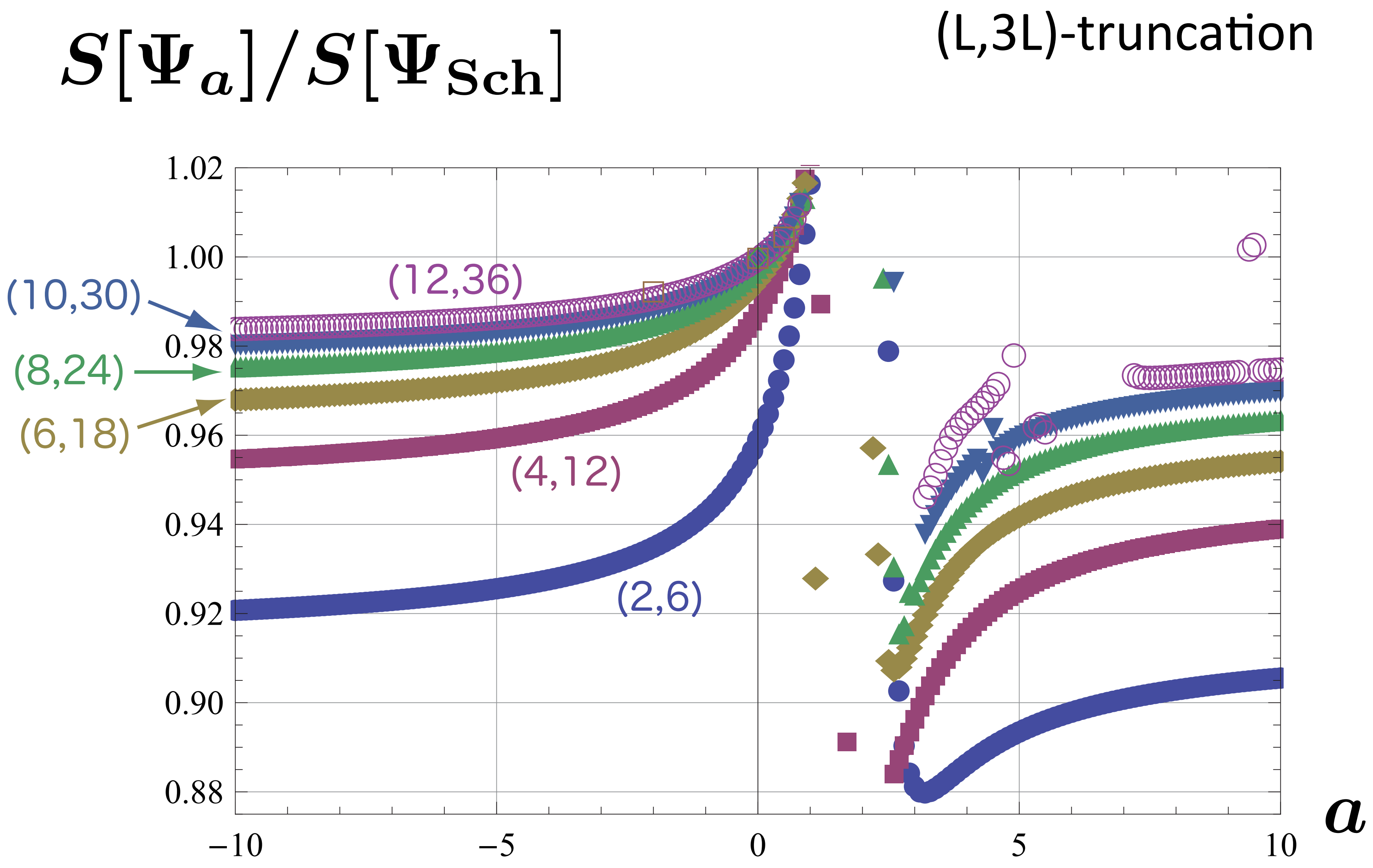
$$\boldsymbol{\mathcal{P}_{\mathrm{GF}}=1+\frac{1}{a-1}\Big(\frac{\tilde{Q}}{L_0}+c_0\Big)(b_0+ab_0c_0W_1\tilde{Q}),}$$

$$\boldsymbol{W_1=\sum_{i=0}^{\infty}\frac{(-1)^i}{\{(i+1)!\}^2}M^i(M^-)^{i+1},\quad M^-=-\sum_{n=1}^{\infty}\frac{1}{2n}b_{-n}b_n.}$$

Complicated projection!



$S[\Psi_a]/S[\Psi_{\rm Sch}]$
(L,3L)-truncation
(10,30)
(8,24)
(6,18)
(4,12)
(2,6)
1.02
1.00
0.98
0.96
0.94
0.92
0.90
0.88
−10
−5
0
5
10
a



$S[\Psi_a]/S[\Psi_{\mathbf{Sch}}]$
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(12,36)
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0.94
0.92
0.90
0.88
−10
−5
0
5
10
a

On fitting of the value of the action

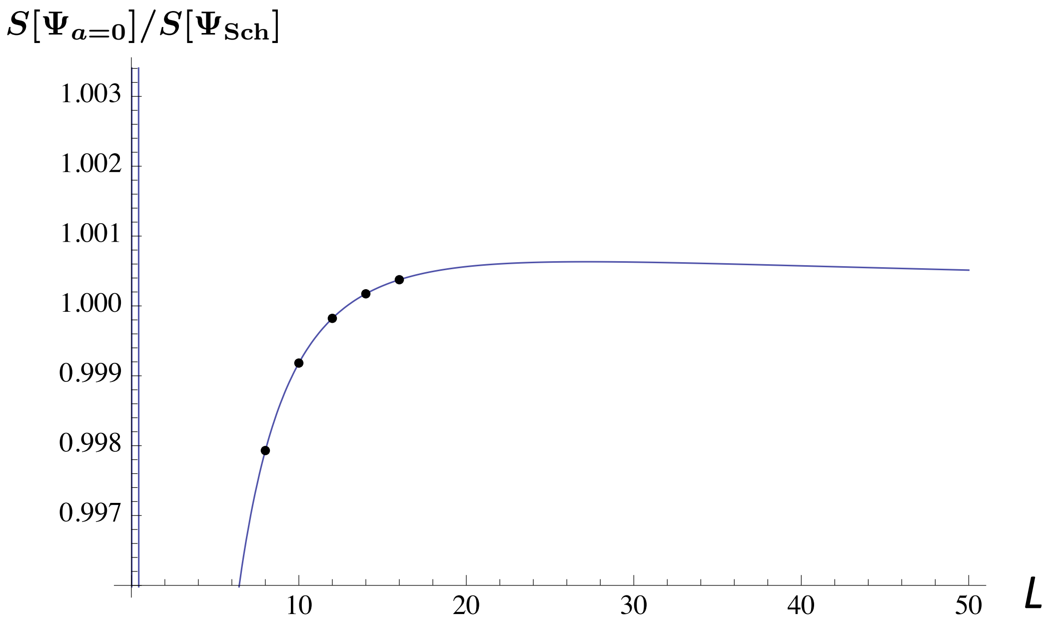
- an extrapolation for the value of the *action*:

$a = 0$ (Siegel gauge) [Gaiotto-Rastelli(2002)]

$$F_N(L) = \sum_{n=0}^{N} \frac{a_n}{(L+1)^n}$$

data for (L,3L)-truncation $(L = 0, 2, 4, 6, 8, 10, 12, 14, 16; N = 9)$

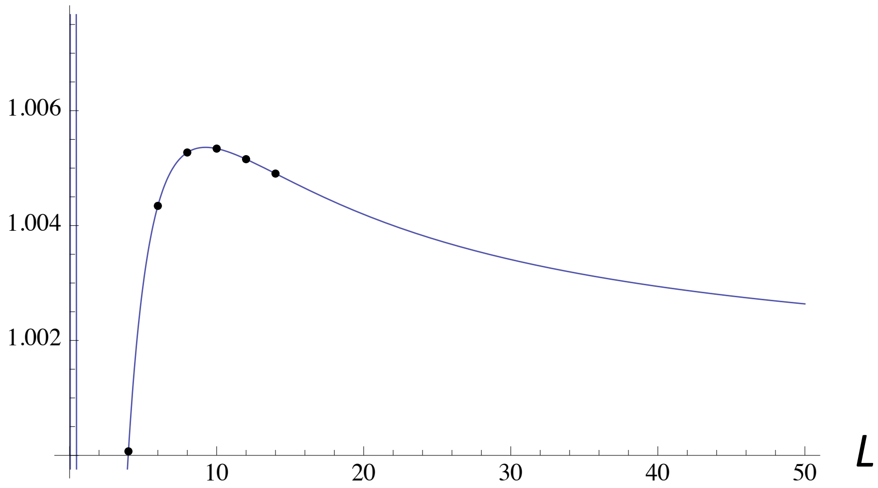
$$F_9(\infty) = 1.00003$$



Extrapolation for the *a*-gauge solutions

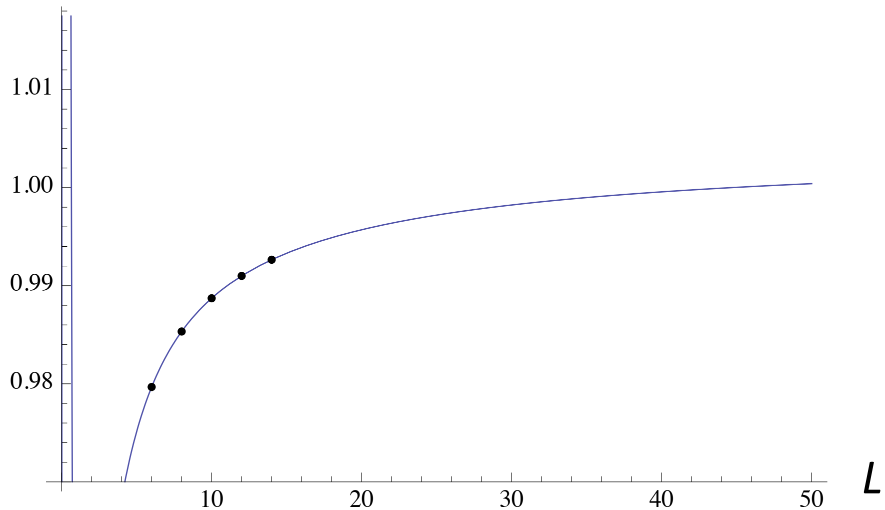
- In the same way, we fit the action for $a(\neq 0)$ -gauge solutions

using data for (L,3L)-truncation. $(L = 0, 2, 4, 6, 8, 10, 12, 14; N = 8)$



$a = 0.5$ -gauge

$F_8(\infty) = 1.00126$

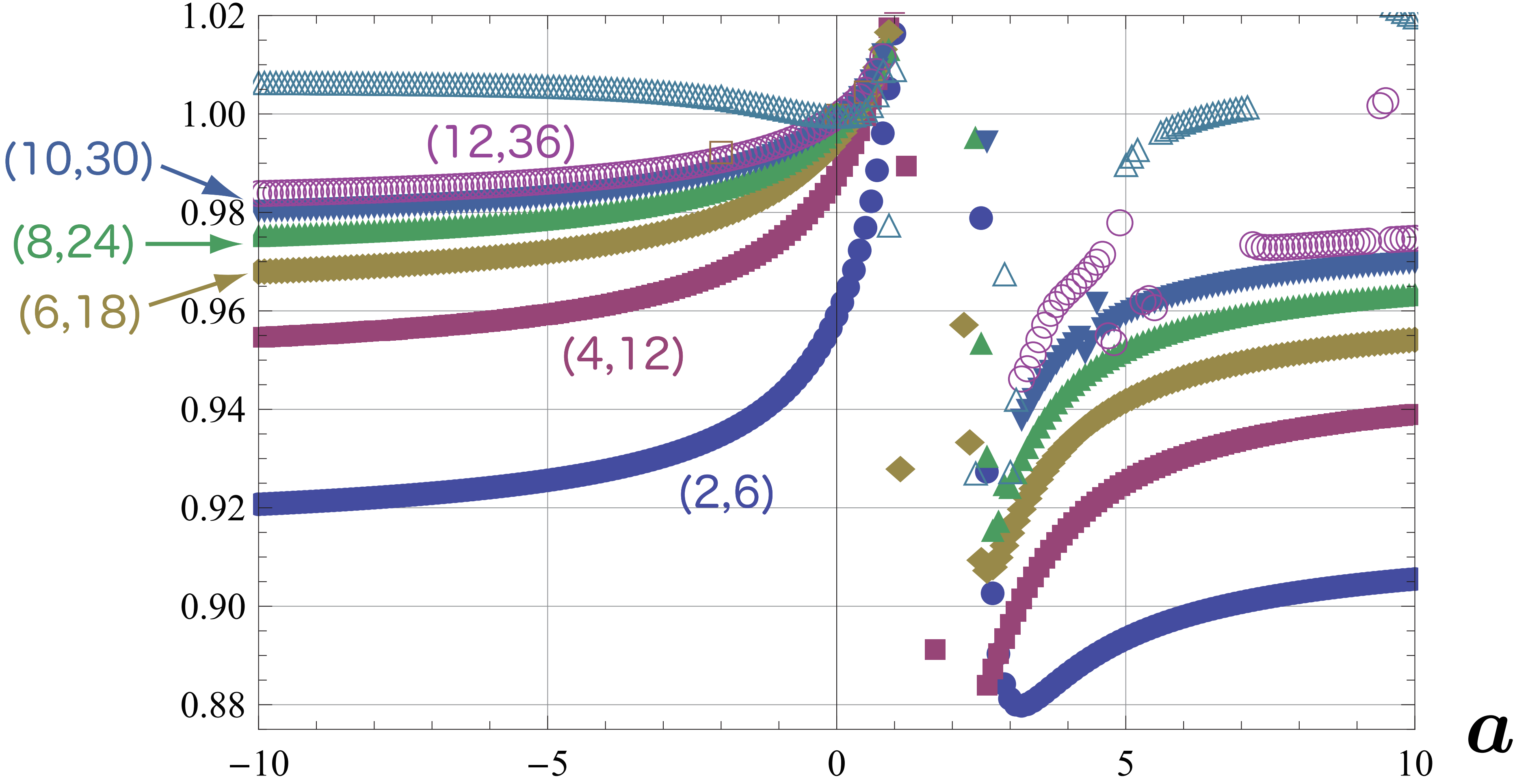


$a = -2$ -gauge

$F_8(\infty) = 1.00408$

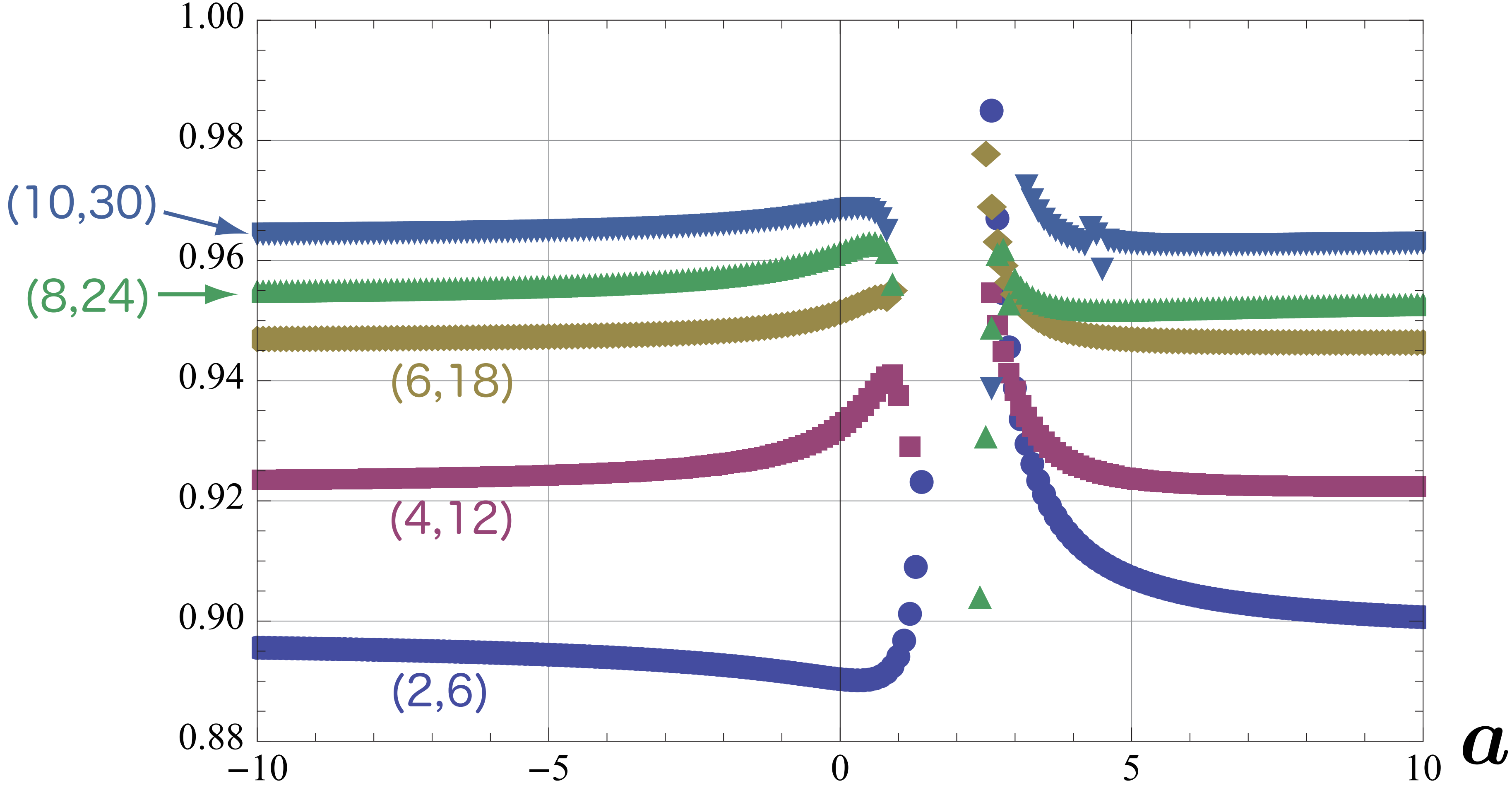
$S[\Psi_a]/S[\Psi_{\mathrm{Sch}}]$

(L,3L)-truncation



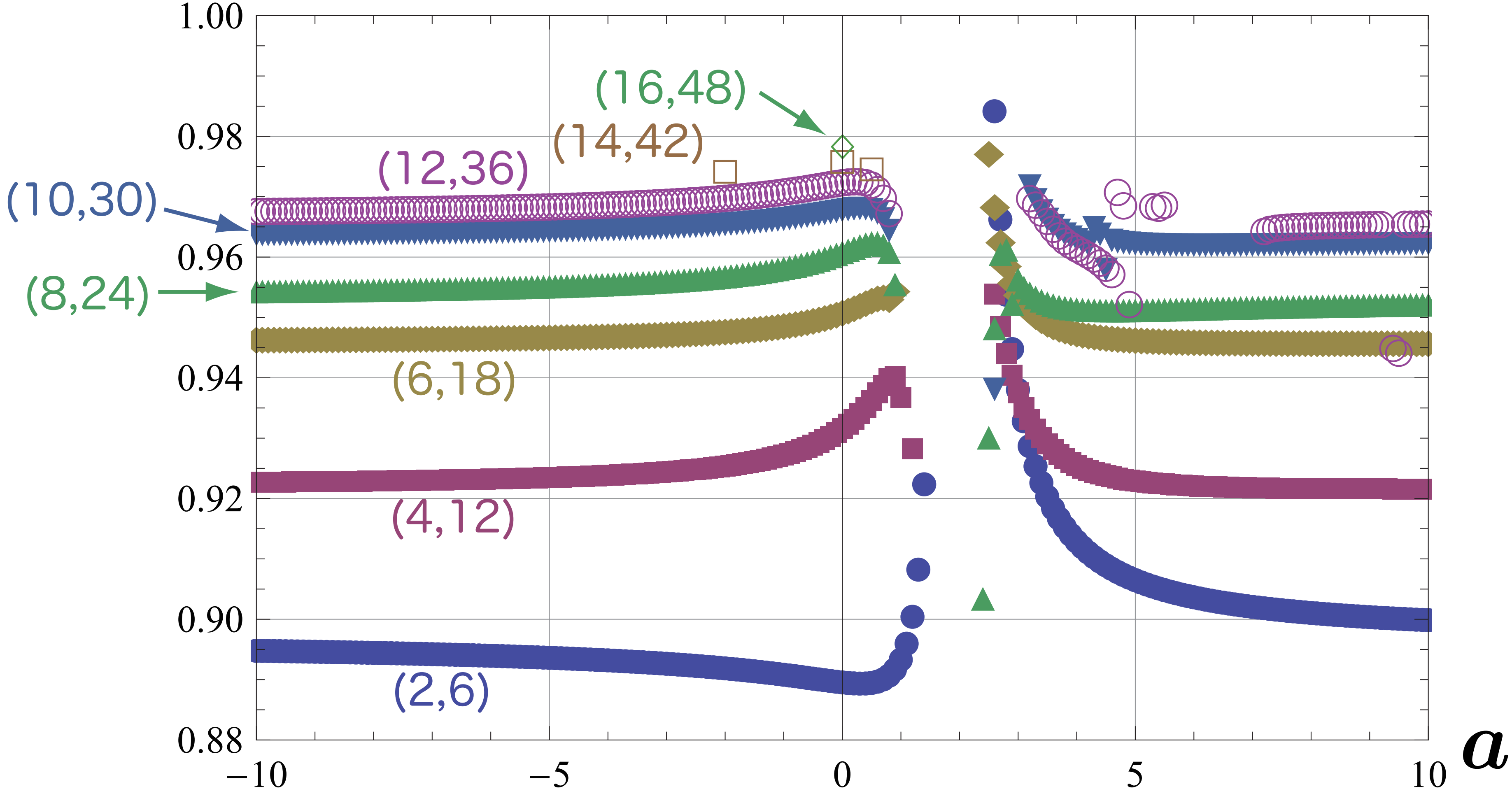
$\mathcal{O}_V(\Psi_a)/\mathcal{O}_V(\Psi_{\mathbf{Sch}})$

(L,3L)-truncation



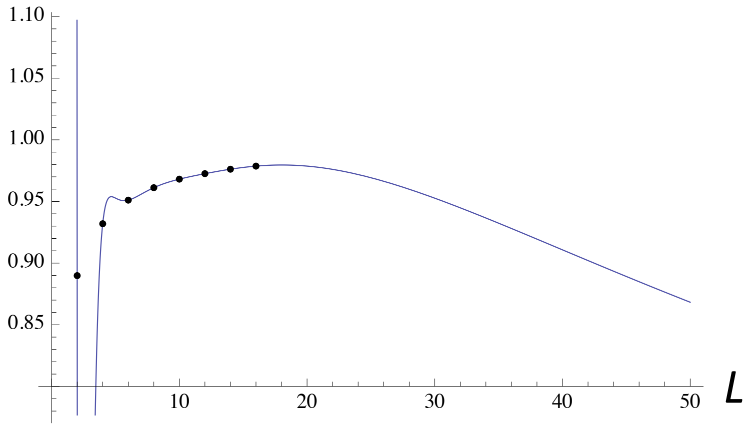
$\mathcal{O}_V(\Psi_a)/\mathcal{O}_V(\Psi_{\mathbf{Sch}})$

(L,3L)-truncation



Extrapolation of the gauge invariant overlap?

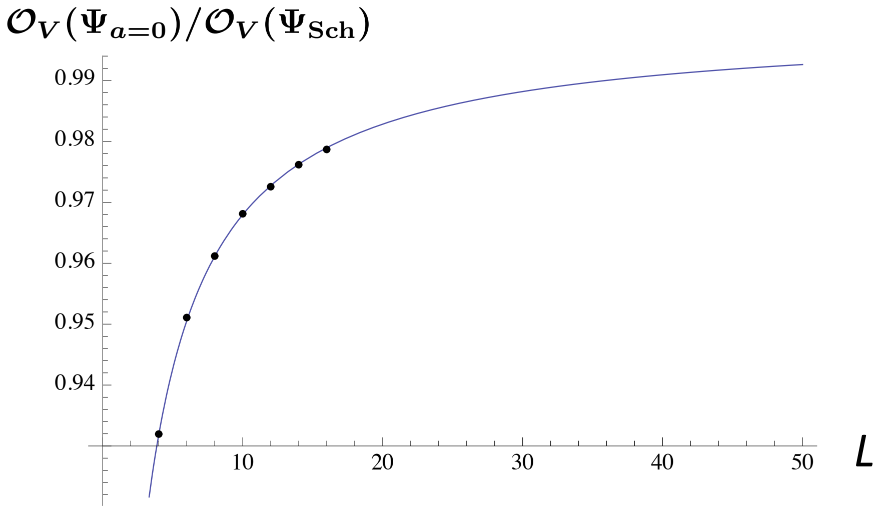
If we use the same fit function in the same way as the action *naively*, we have



$$F_9(\infty) = 0.442107 \quad \text{(Siegel gauge)}$$

The fitting does not work well.

However, if we take a fit function: $$F_{\rm exp}(L) = a_0 \exp\left(-\frac{a_1}{L+1} - \frac{a_2}{(L+1)^2}\right)$$



using data for (L,3L)-truncation

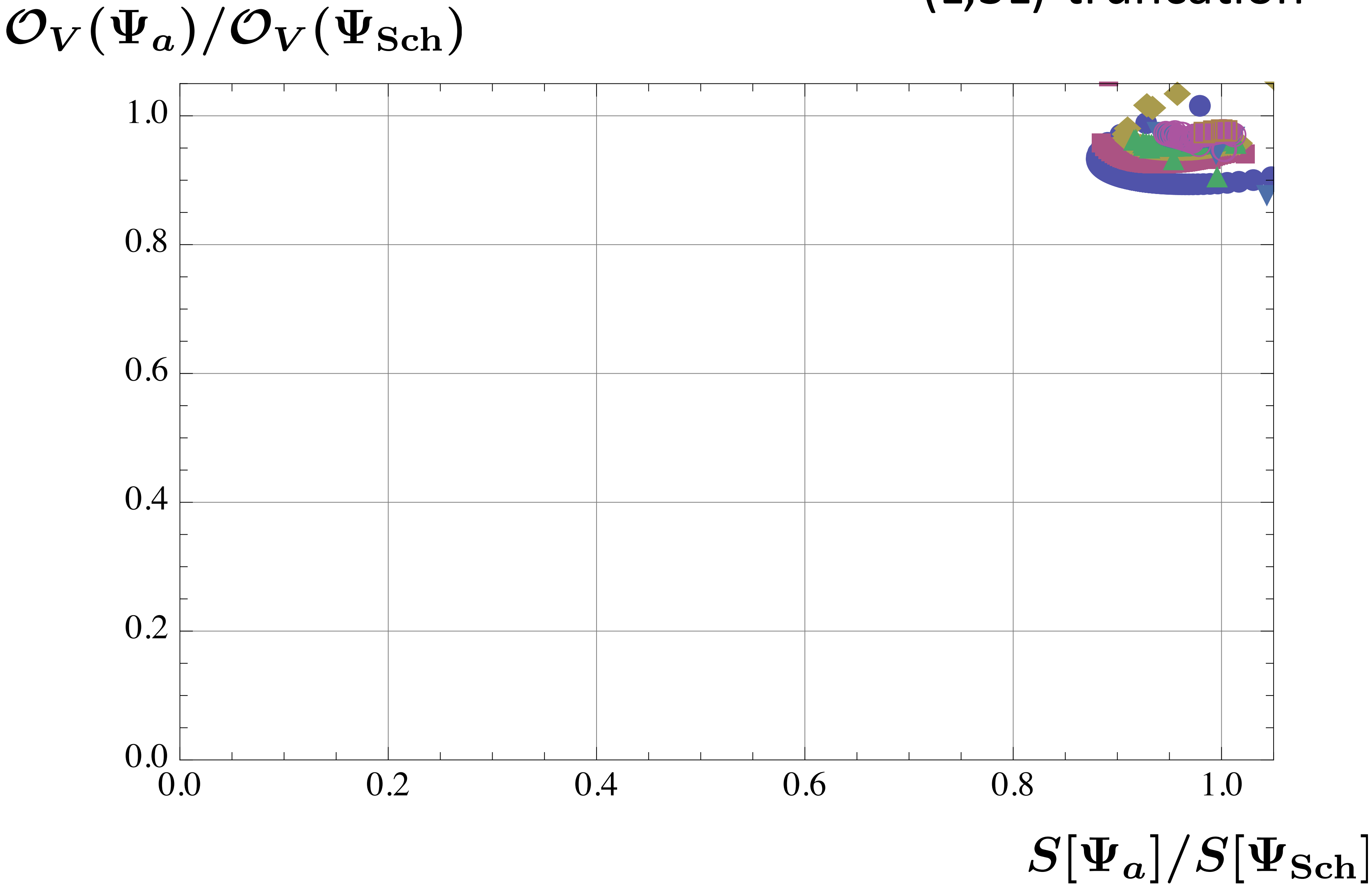
$$(L = 0, 2, 4, 6, 8, 10, 12, 14, 16)$$

$$F_{\rm exp}(\infty) = 0.99954 \quad \text{(Siegel gauge)}$$

A good fitting function (!?)

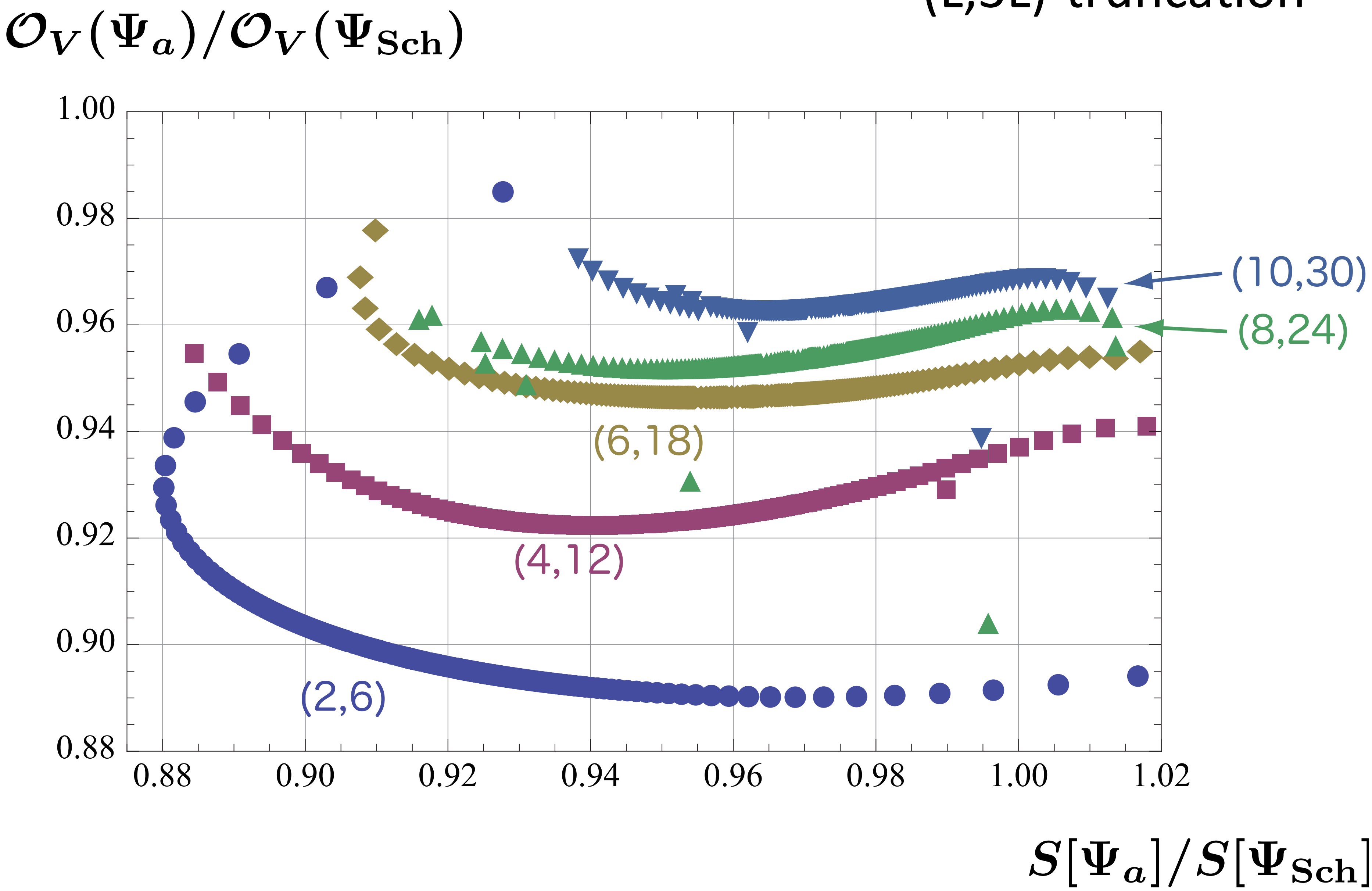
Gauge invariants for various *a*-gauge solutions

(L,3L)-truncation



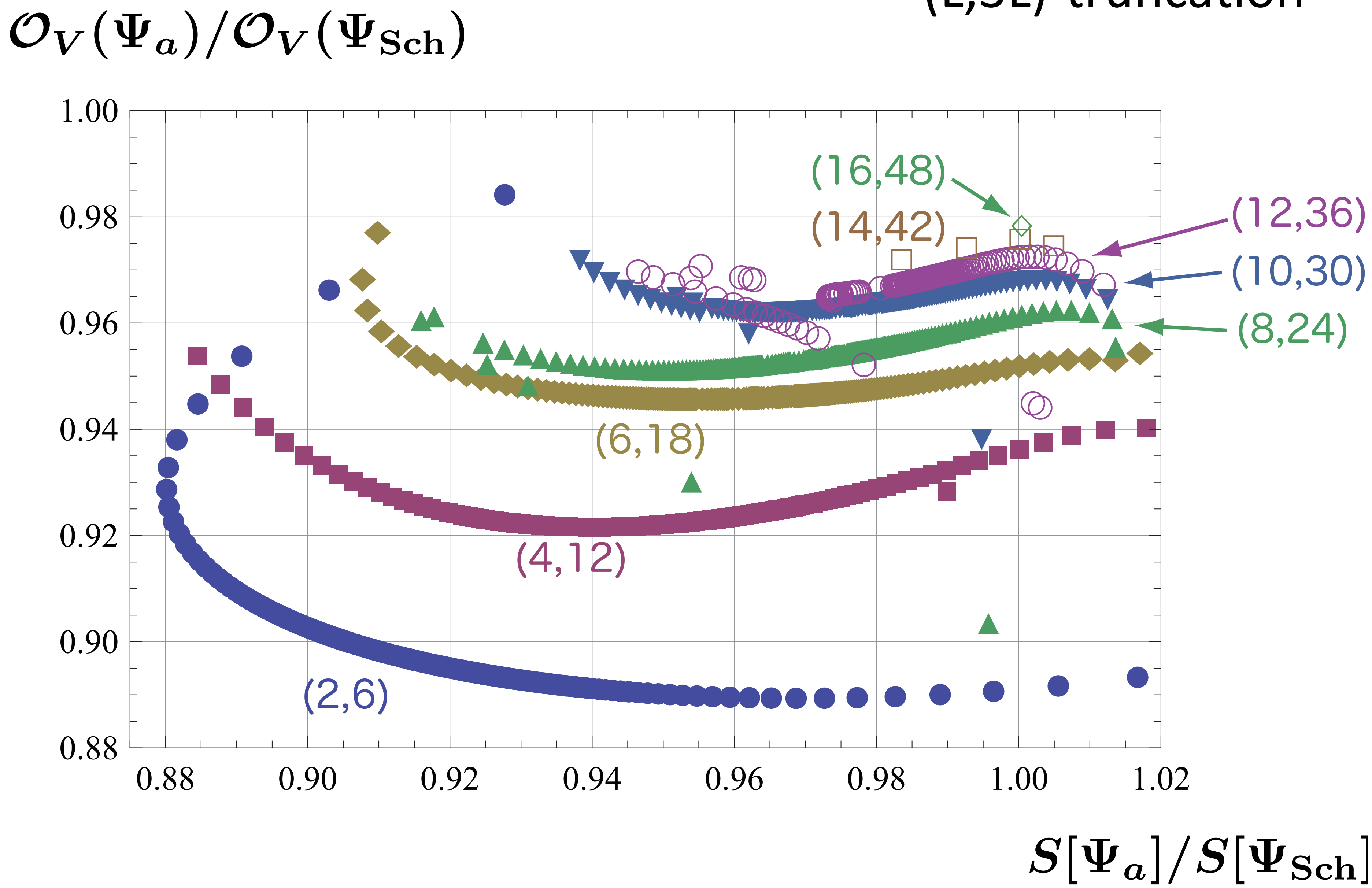
Gauge invariants for various a-gauge solutions

(L,3L)-truncation



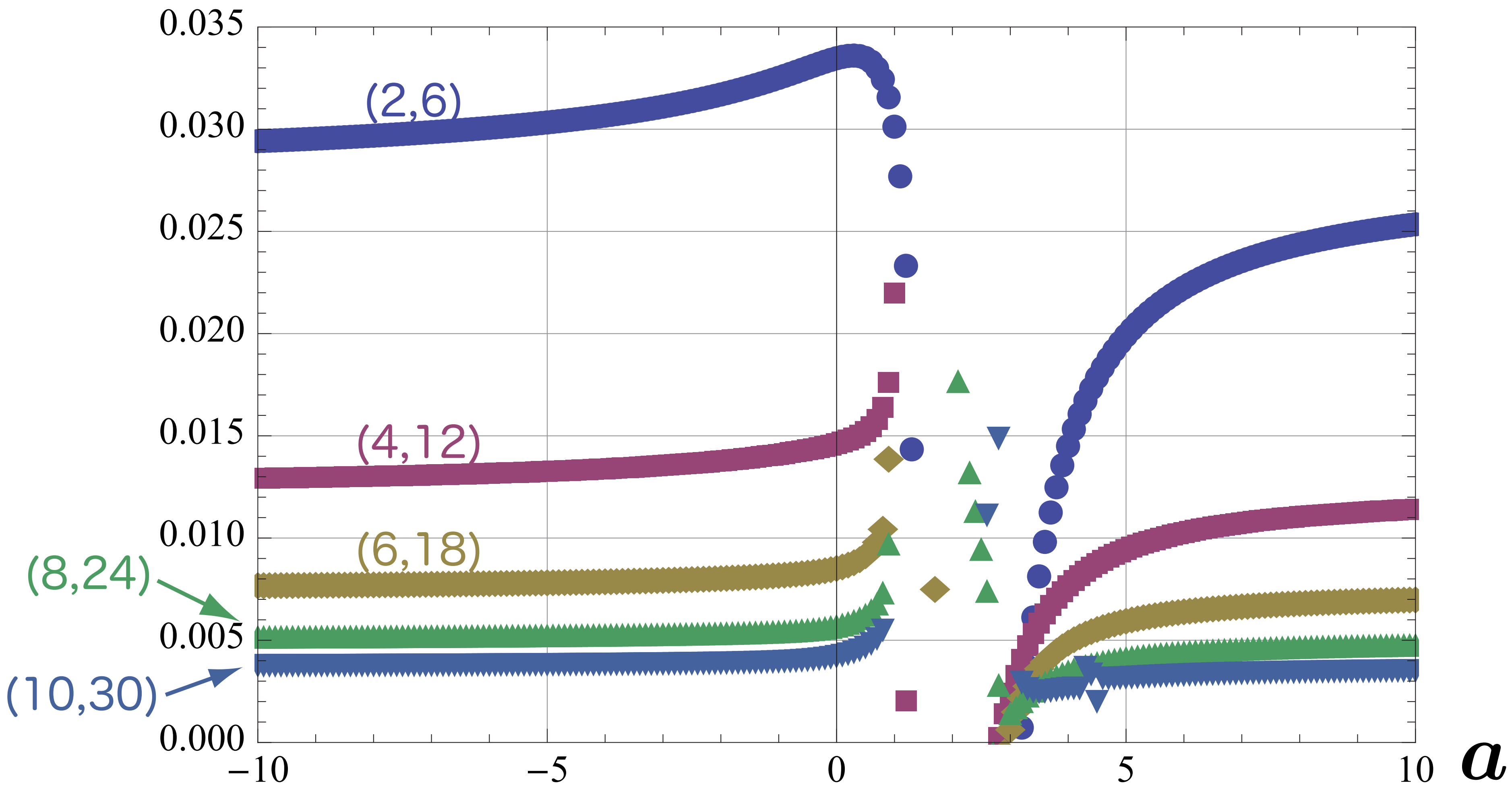
Gauge invariants for various a-gauge solutions

(L,3L)-truncation



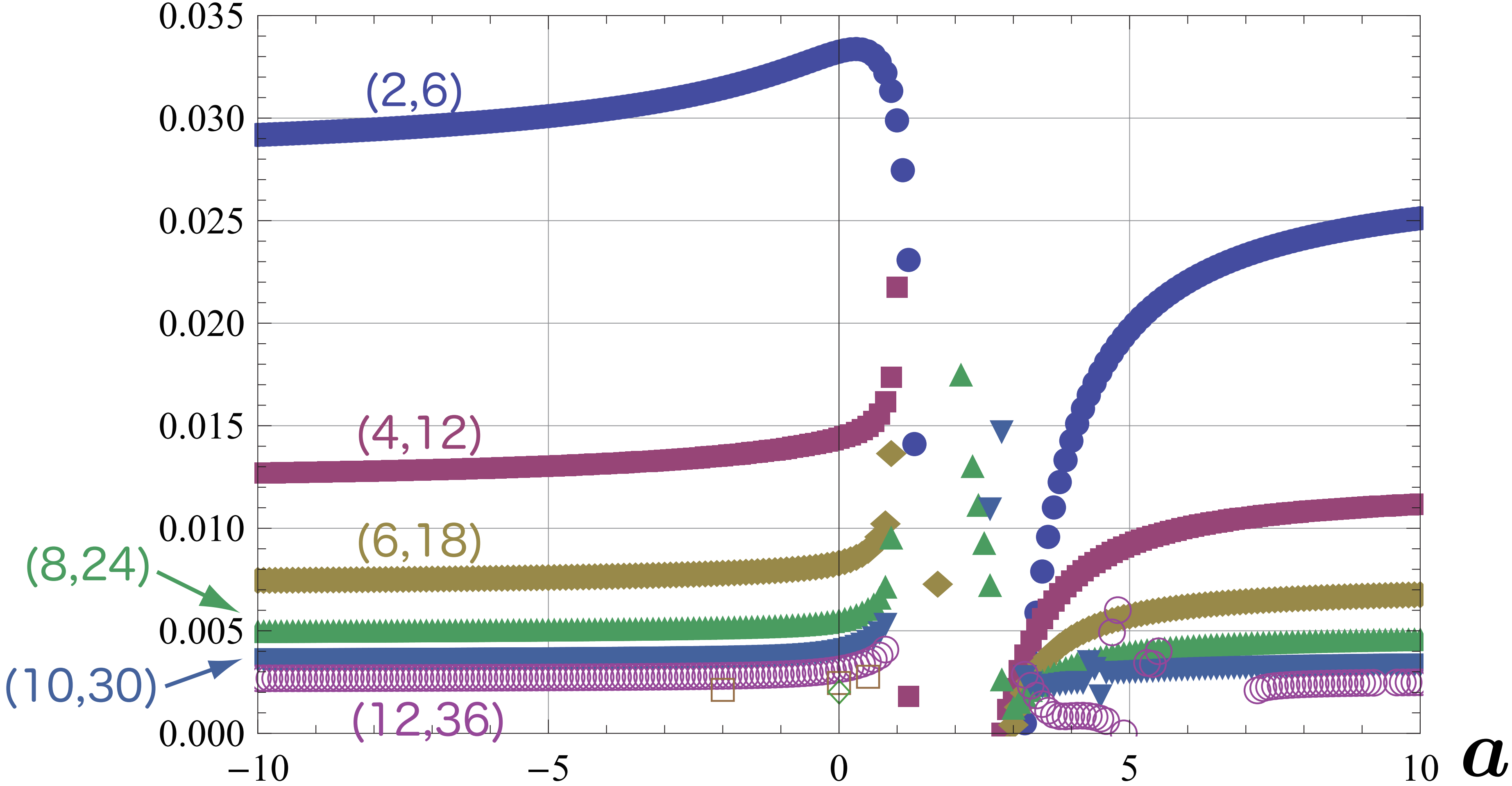
Coefficient of $c_{-2}c_1|0\rangle \in (1-\mathcal{P})(Q\Psi_a+\Psi_a*\Psi_a)$

(L,3L)-truncation



Coefficient of $c_{-2}c_1|0\rangle \in (1-\mathcal{P})(Q\Psi_a+\Psi_a*\Psi_a)$

(L,3L)-truncation



Extrapolation for consistency of EOM

- For the coefficient of $c_{-2}c_1|0\rangle \in (1-\mathcal{P})(Q\Psi_a+\Psi_a*\Psi_a)$

we use $$G_N(L)=\sum_{n=0}^{N}\frac{a_n}{L^n}$$ [Gaiotto-Rastelli(2002)]

as a fitting function using the data $L=2,4,6,\cdots,L_{\max};\,N=L_{\max}/2-1$

Siegel gauge $(a=0)$

$$G_7(\infty)=-0.000026 \qquad L_{\max}=16$$

$a=0.5$ -gauge

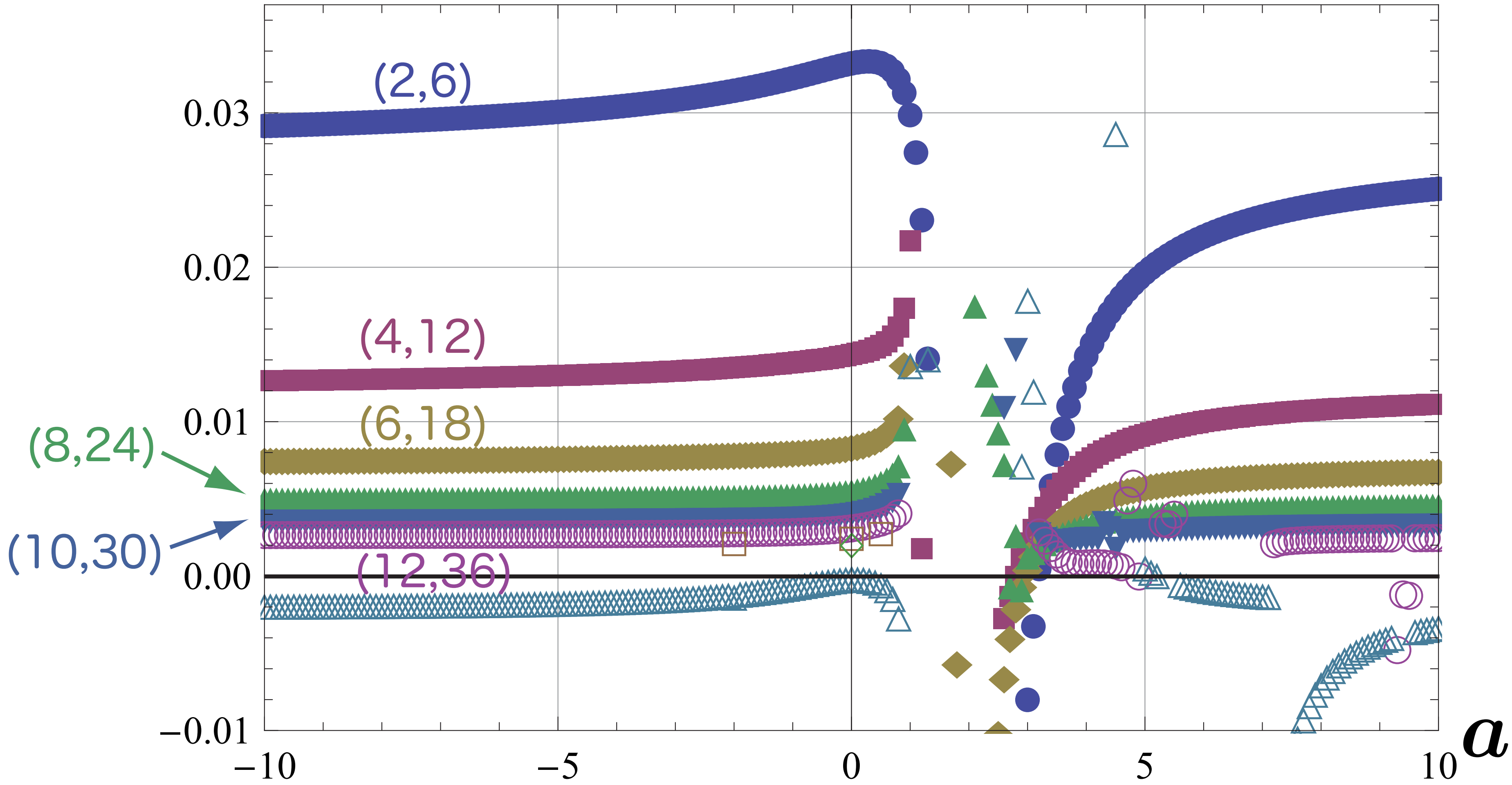
$$G_6(\infty)=-0.000443 \qquad L_{\max}=14$$

$a=-2$ -gauge

$$G_6(\infty)=-0.001239 \qquad L_{\max}=14$$

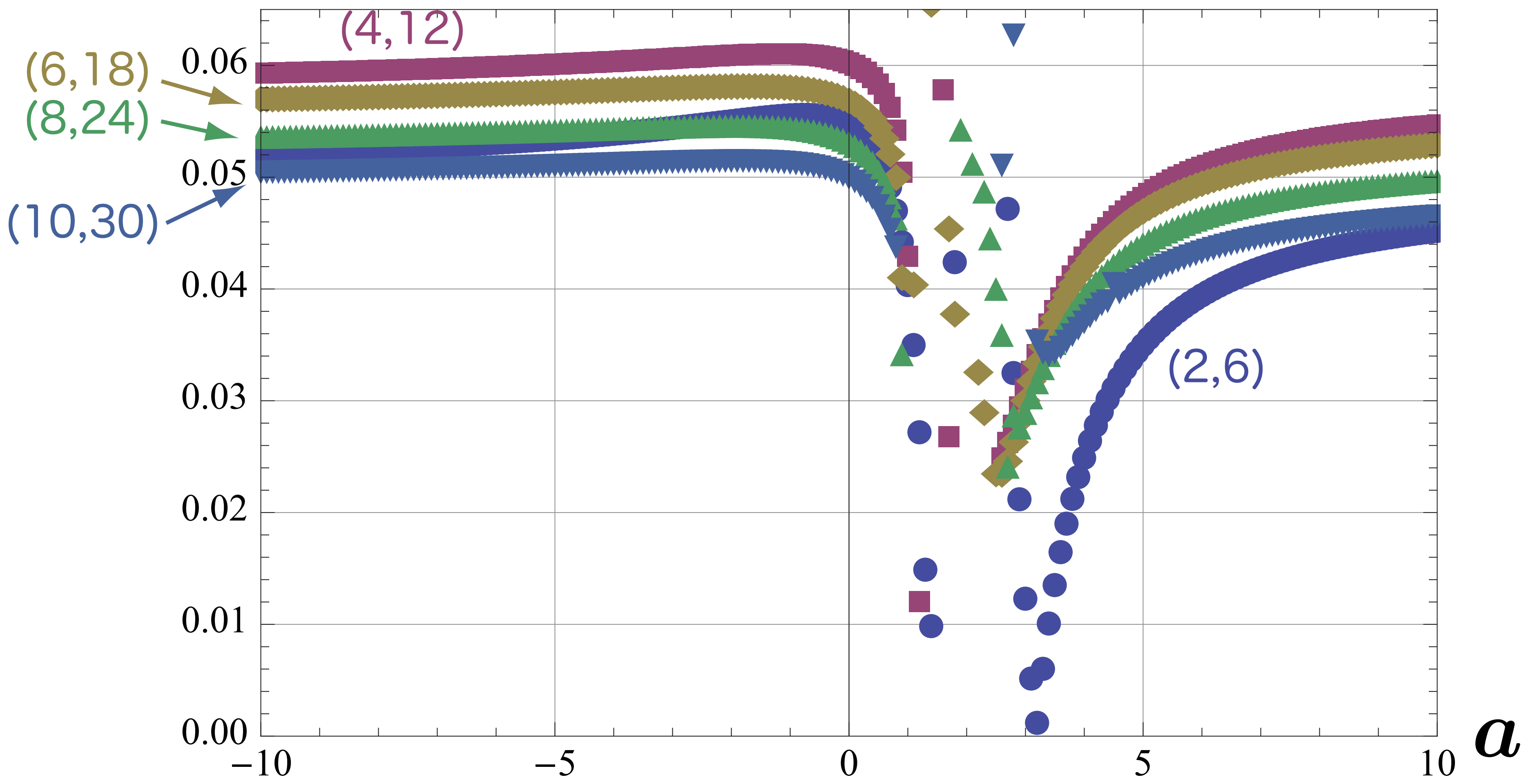
Coefficient of $c_{-2}c_1|0\rangle \in (1-\mathcal{P})(Q\Psi_a + \Psi_a * \Psi_a)$

(L,3L)-truncation



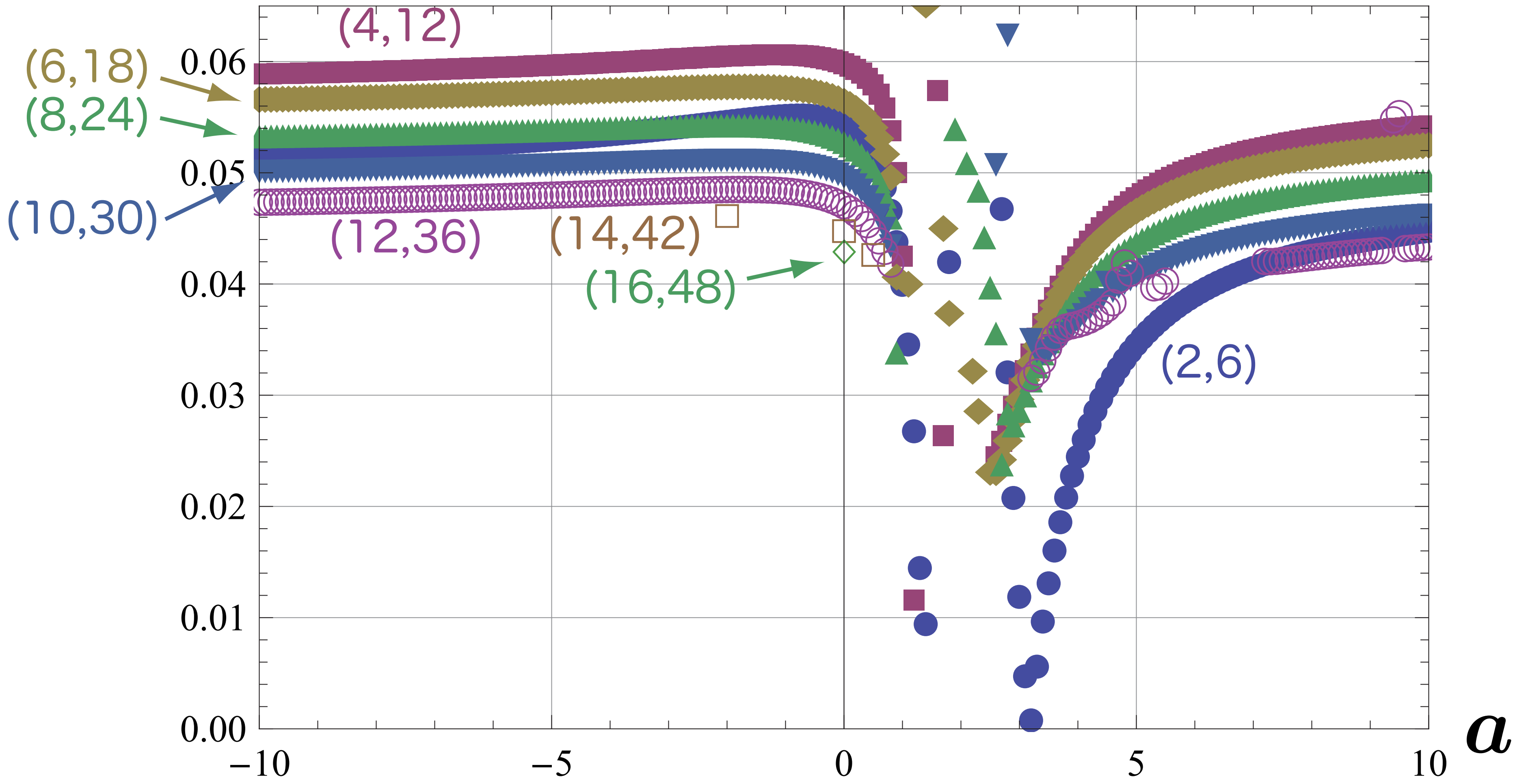
$$\frac{\|(1-\mathcal{P})(Q\Psi_a + \Psi_a * \Psi_a)\|}{\|\Psi_a\|}$$

(L,3L)-truncation



$$\frac{\|(1-\mathcal{P})(Q\Psi_a + \Psi_a * \Psi_a)\|}{\|\Psi_a\|}$$

(L,3L)-truncation



Summary (1)

- We have evaluated gauge invariants (action and gauge invariant overlap) for numerical solutions in *a*-gauges by level truncation ((L,2L) and (L,3L)-method).
- We have checked the consistency of the equation of motion.
- Our numerical results suggest: $-\infty \leqq a \lesssim 0,\ 1 \ll a \leqq \infty$

$$L \to +\infty \qquad \begin{array}{c} S[\Psi_{a,L}]|_L \longrightarrow S[\Psi_{\mathrm{Sch}}] \\ \mathcal{O}_V(\Psi_{a,L}) \longrightarrow \mathcal{O}_V(\Psi_{\mathrm{Sch}}) \end{array}$$

- These are consistent with the gauge equivalence:

$$\Psi_a \sim \Psi_{\mathrm{Sch}}$$

Discussion (1)

- The approaching speed of the overlap to the expected value is slower than that of the action.
- Due to the subtlety of the midpoint(?)

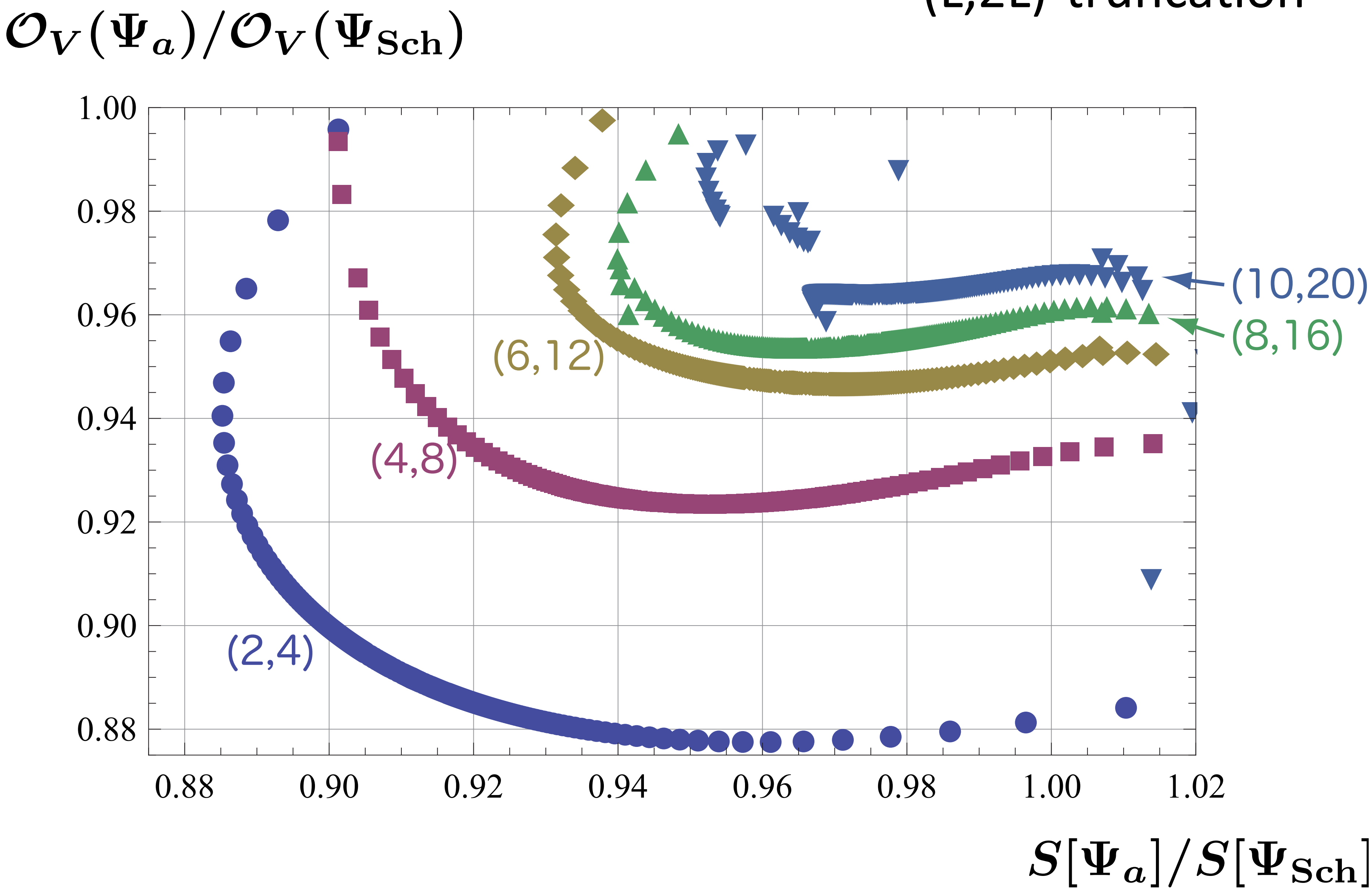
(Suppose that the gauge invariant overlap is always well-defined.)

- If there is a small discrepancy between the gauge invariant overlap for the *a*-gauge solutions and that for the Schnabl solution, they are not gauge equivalent.

If so, they might describe different vacua. (!?)

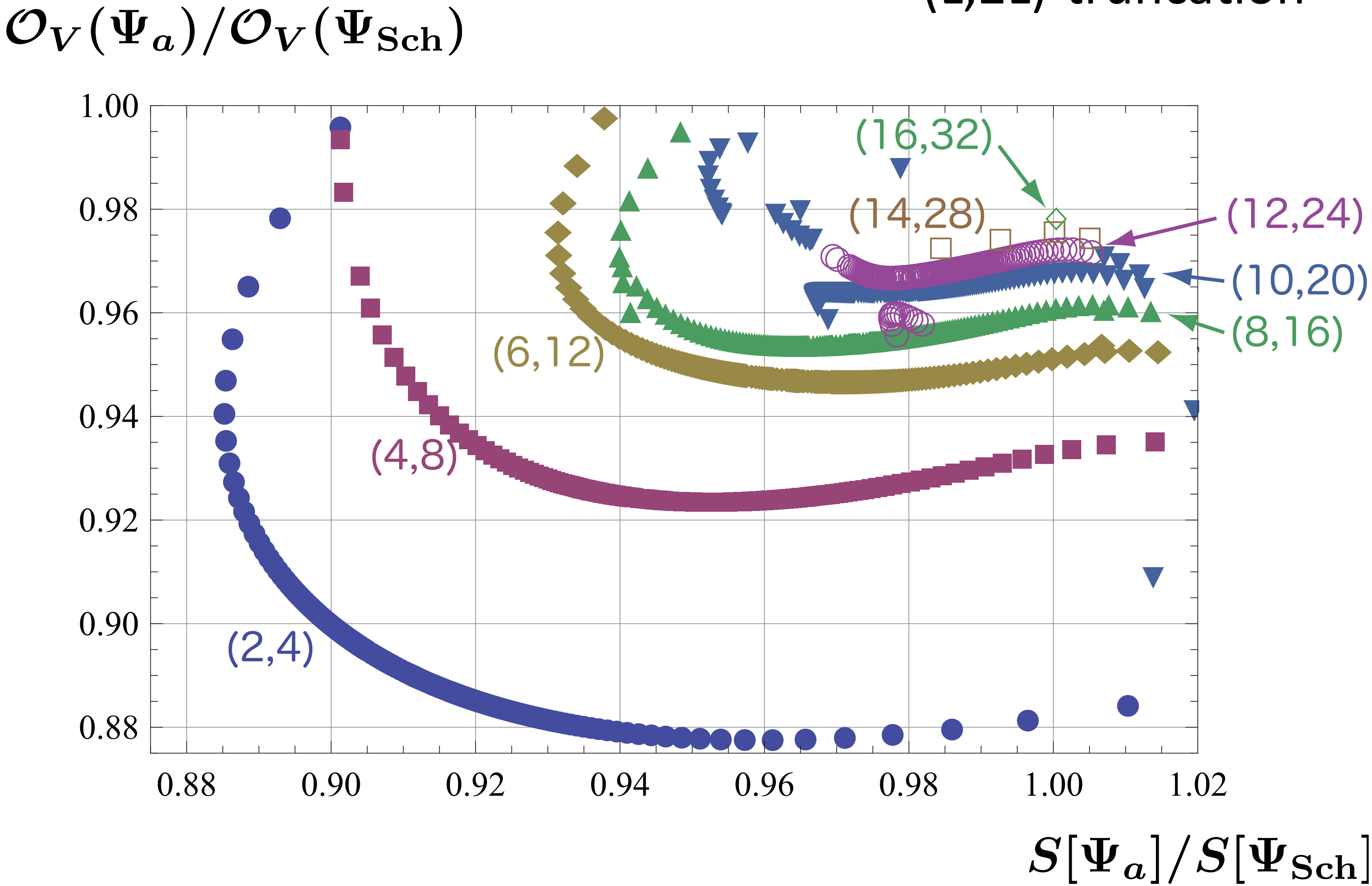
Gauge invariants for various *a*-gauge solutions

(L,2L)-truncation



Gauge invariants for various *a*-gauge solutions

(L,2L)-truncation



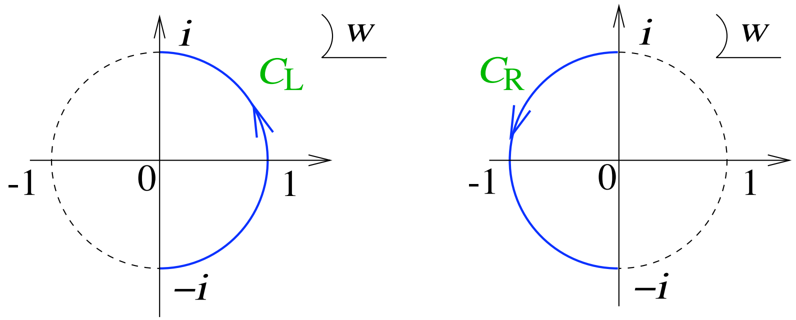
Takahashi-Tanimoto's solution

- "Identity based solution" [Takahashi-Tanimoto(2002)]

$$\Psi_0 = Q_L(e^h - 1)\mathcal{I} - C_L((\partial h)^2 e^h)\mathcal{I}$$

$$Q_L(f) \equiv \int_{C_{\rm left}} \frac{dz}{2\pi i} f(z) j_{\rm B}(z)$$

$$C_L(f) \equiv \int_{C_{\rm left}} \frac{dz}{2\pi i} f(z) c(z)$$



In the following, we take

$$\begin{aligned} h(z) &= \log\left(1+\frac{a}{2}\left(z+\frac{1}{z}\right)^2\right) \\ &= -\log(1-Z(a))^2 - \sum_{n=1}^{\infty}\frac{(-1)^n}{n} Z(a)^n (z^{2n}+z^{-2n}) \end{aligned}$$

$$Z(a) = (1 + a - \sqrt{1+2a})/a \qquad a \geq -1/2$$

On the TT solution

- Formal pure gauge form:

$$\Psi_0 = \exp(q_L(h)\mathcal{I})Q_{\mathrm{B}}\exp(-q_L(h)\mathcal{I})$$

Gauge parameter string field:

$$\exp(\pm q_L(h)\mathcal{I}) = \exp(\pm q_L(h))\mathcal{I}$$

$\exp(\pm q_L(h))$: ill-defined for $a = -1/2$

$$q_L(f) \equiv \int_{C_{\mathrm{left}}} \frac{dz}{2\pi i} f(z) :cb:(z)$$

⇓

Non-trivial solution (?)

However, it is difficult to compute $S[\Psi_0], \quad \mathcal{O}_V(\Psi_0)$

SFT around the TT solution

- Expansion around the TT solution:

$$\begin{aligned} S_a[\Phi] &= S[\Psi_0+\Phi]-S[\Psi_0] \\ &= -\frac{1}{g^2}\left[\frac{1}{2}\langle\Phi, Q'\Phi\rangle+\frac{1}{3}\langle\Phi, \Phi*\Phi\rangle\right] \end{aligned}$$

$$\begin{aligned} Q' &= (1+a)Q_{\mathrm{B}}+\frac{a}{2}(Q_2+Q_{-2})+4aZ(a)c_0-2aZ(a)^2(c_2+c_{-2}) \\ &\quad -2a(1-Z(a)^2)\sum_{n=2}^{\infty}(-1)^n Z(a)^{n-1}(c_{2n}+c_{-2n}) \end{aligned}$$

$$(Q')^2=0$$

$$\delta_\Lambda\Phi = Q'\Lambda+\Phi*\Lambda-\Lambda*\Phi \quad \Longrightarrow \quad \delta_\Lambda S_a[\Phi]=0$$

$$j_{\mathrm{B}}(z)=cT^{\mathrm{m}}(z)+:bc\partial c:+\frac{3}{2}\partial^2 c(z)=\sum_{n=-\infty}^{\infty}Q_n z^{-n-1}$$

On the new BRST operator

- cohomology of Q' [I.K.-Takahashi (2002), Takahashi-Zeze(2003)]

$a > -1/2$ the same as the original Q_{B}

⇒ Ψ_0 : pure gauge

$a = -1/2$ no cohomology at ghost number 1 sector

⇒ no open string

Ψ_0 : tachyon vacuum (!?)

Numerical solution in SFT around the TT solution

- We solve the EOM: $Q'\Phi + \Phi * \Phi = 0$

in the Siegel gauge by level truncation

with the iterative algorithm:

$$c_0 b_0 (c_0 L(a)\Phi^{(n+1)} + \Phi^{(n)} * \Phi^{(n+1)} + \Phi^{(n+1)} * \Phi^{(n)} - \Phi^{(n)} * \Phi^{(n)}) = 0$$

$$\begin{aligned} L(a) &= \{b_0, Q'\} \\ &= (1+a)L_0 + \frac{a}{2}(L_2 + L_{-2}) + a(q_2 - q_{-2}) + 4(1 + a - \sqrt{1+2a}) \end{aligned}$$

If it converges $c_0 b_0 (Q'\Phi^{(\infty)} + \Phi^{(\infty)} * \Phi^{(\infty)}) = 0$

We also check $\|b_0 c_0 (Q'\Phi^{(\infty)} + \Phi^{(\infty)} * \Phi^{(\infty)})\| / \|\Phi^{(\infty)}\| \ll 1$

We evaluate the gauge invariants:

(1) potential height: $f_a(\Phi) = 2\pi^2 \left(\frac{1}{2} \langle \Phi, c_0 L(a) \Phi \rangle + \frac{1}{3} \langle \Phi, \Phi * \Phi \rangle \right)$

(2) gauge invariant overlap: $\mathcal{O}_V(\Phi) = 2\pi \langle \hat{\gamma}(1_c, 2) | \Phi_V \rangle_{1_c} | \Phi \rangle_2$

Construction of stable vacuum solution

- The initial configuration for $a = 0 \ (Q' = Q_{\mathrm{B}})$

$$\Phi^{(0)} = \frac{64}{81\sqrt{3}} c_1 |0\rangle \quad \xrightarrow{\text{iteration}} \quad \Phi_1\big|_{a=0}$$

conventional tachyon vacuum solution

- The initial configuration for $a = \epsilon \ (0 < |\epsilon| \ll 1)$

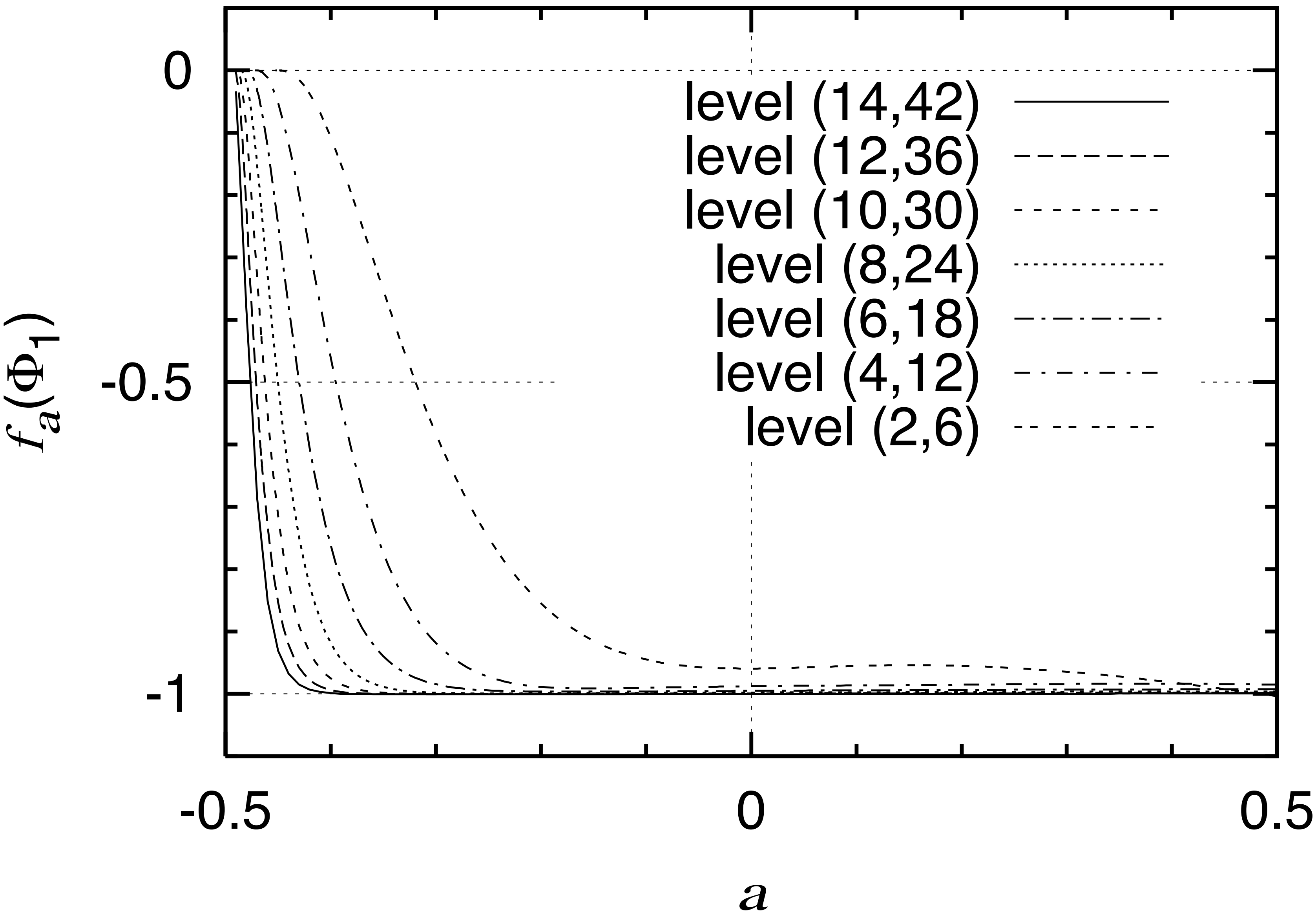
$$\Phi^{(0)} = \Phi_1\big|_{a=0} \quad \xrightarrow{\text{iteration}} \quad \Phi_1\big|_{a=\epsilon}$$

- The initial configuration for $a = 2\epsilon$

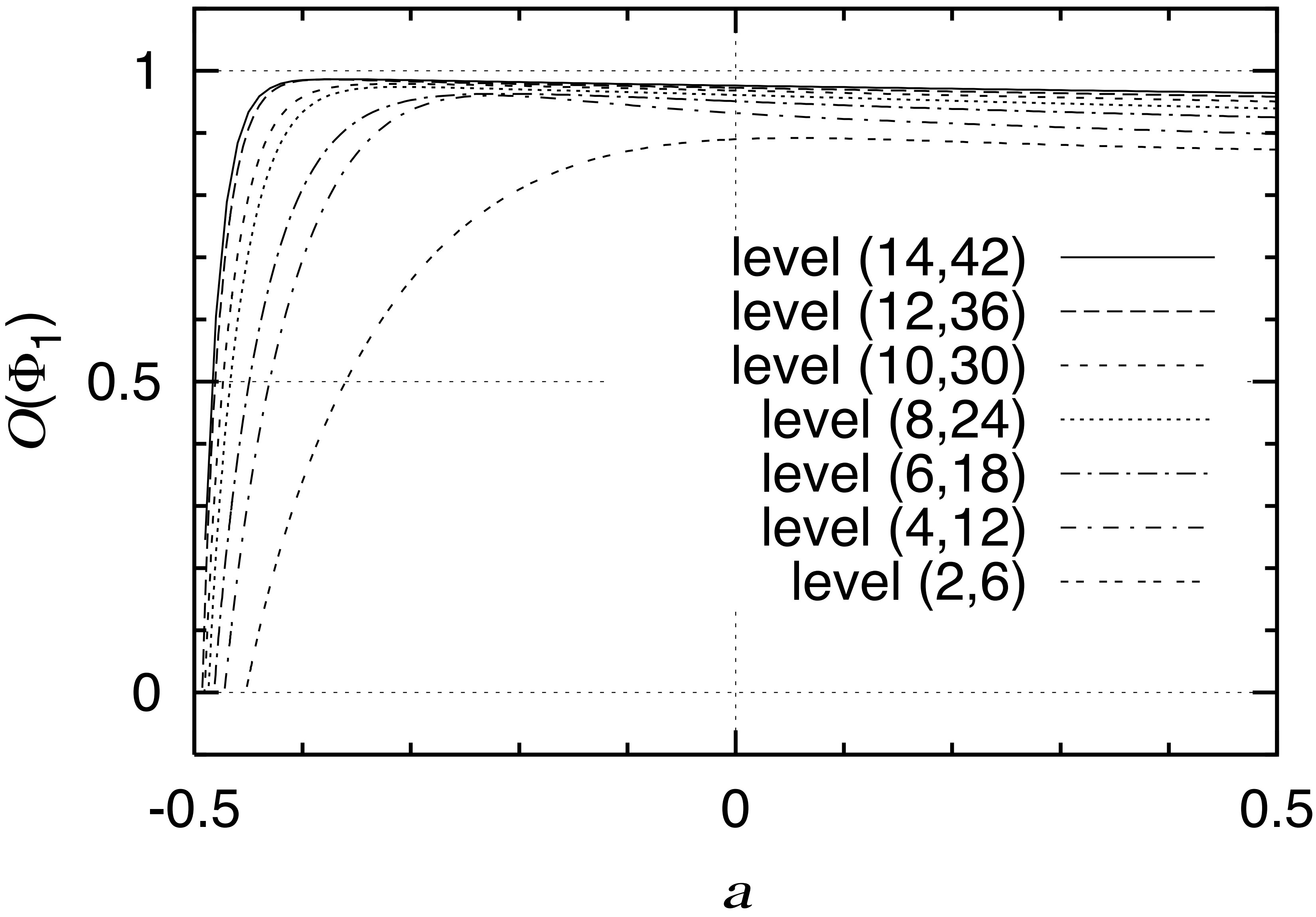
$$\Phi^{(0)} = \Phi_1\big|_{a=\epsilon} \quad \xrightarrow{\text{iteration}} \quad \Phi_1\big|_{a=2\epsilon}$$

$$\vdots$$

Potential height for Φ_1

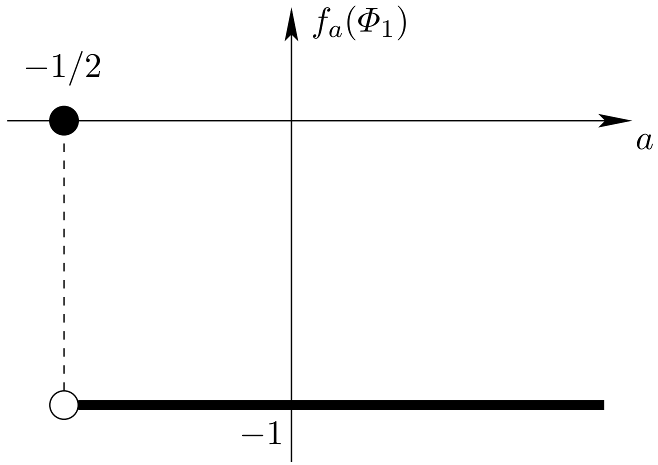


Gauge invariant overlap for Φ_1



Stable vacuum solution

- For $L \to \infty$, numerical results suggest



$$a > -1/2$$

$\mathbf{\Phi_1}$:nontrivial tachyon vacuum

$$a = -1/2$$

$$\mathbf{\Phi_1 = 0}$$

$a > -1/2$ ⇒ $\mathbf{\Psi_0}$: pure gauge

$a = -1/2$ ⇒ $\mathbf{\Psi_0}$: tachyon vacuum (!?)

Construction of unstable vacuum solution

- The initial configuration for $a = -1/2$

$$\Phi^{(0)} = -\frac{32}{9\sqrt{3}} c_1 |0\rangle \xrightarrow{\text{iteration}} \Phi_2|_{a=-1/2}$$

the nontrivial solution for (0,0) truncation

- The initial configuration for $a = -1/2 + \epsilon \ (0 < \epsilon \ll 1)$

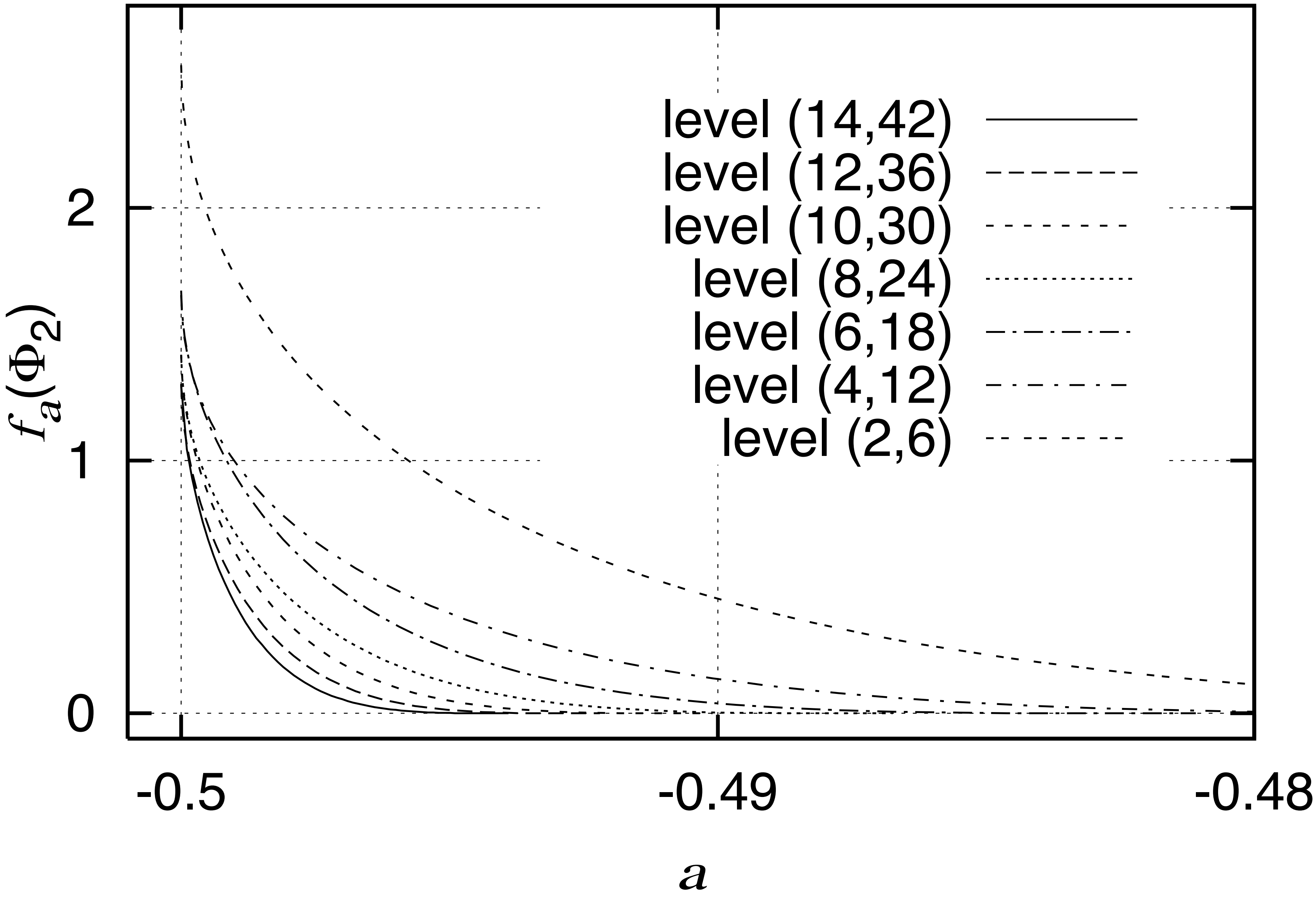
$$\Phi^{(0)} = \Phi_2|_{a=-1/2} \xrightarrow{\text{iteration}} \Phi_2|_{a=-1/2+\epsilon}$$

- The initial configuration for $a = -1/2 + 2\epsilon$

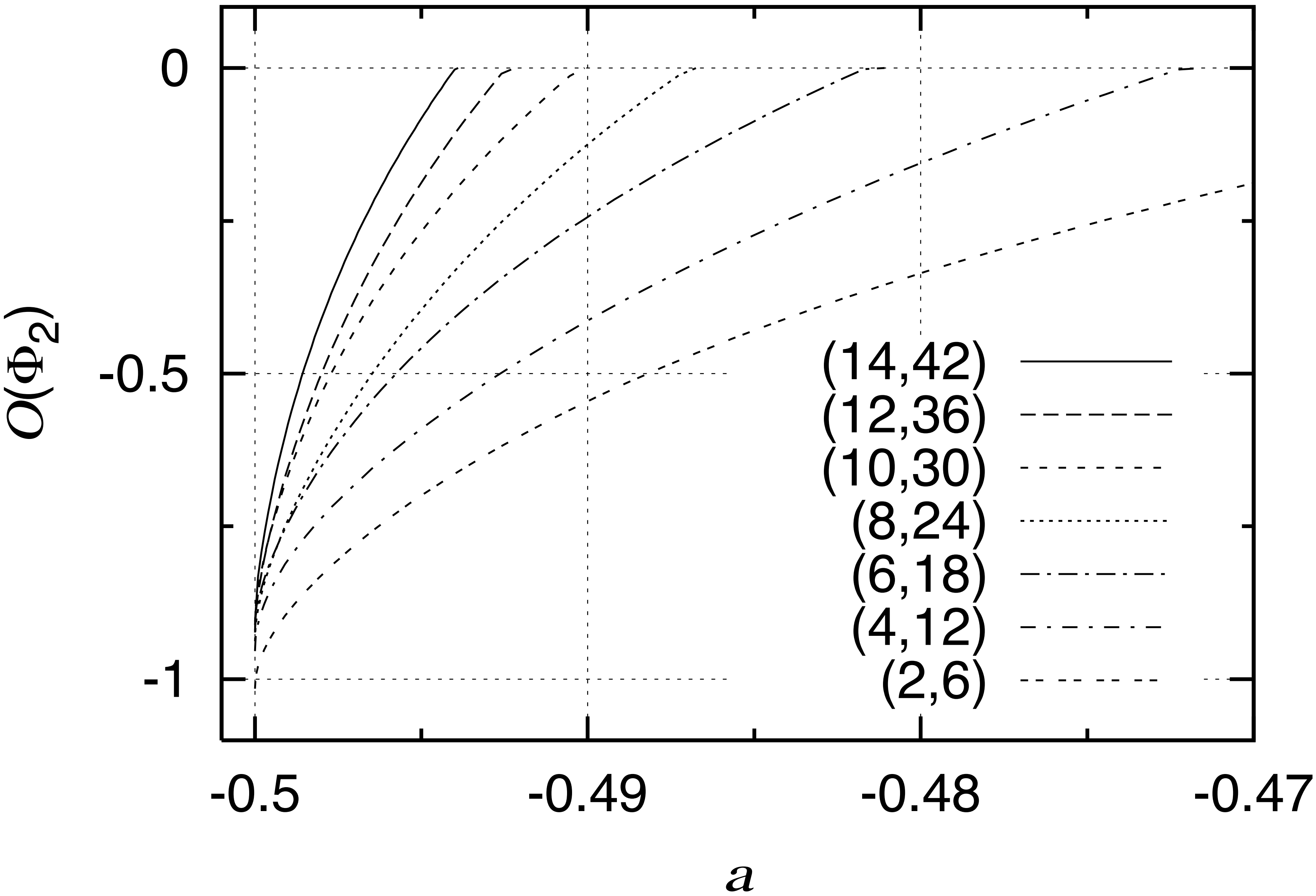
$$\Phi^{(0)} = \Phi_2|_{a=-1/2+\epsilon} \xrightarrow{\text{iteration}} \Phi_2|_{a=-1/2+2\epsilon}$$

$$\vdots$$

Potential height for Φ_2



Gauge invariant overlap for Φ_2



Gauge invariants for $\Phi_2|_{a=-1/2}$

(L,3L)	$f_a(\Phi_2)$	$\mathcal{O}_V(\Phi_2)$
(0,0)	2.3105796	−1.0748441
(2,6)	2.5641847	−1.0156983
(4,12)	1.6550774	−0.9539832
(6,18)	1.6727496	−0.9207572
(8,24)	1.4193393	−0.9377548
(10,30)	1.4168893	−0.9110994
(12,36)	1.3035715	−0.9237917
(14,42)	1.2986472	−0.9056729
(16,48)	1.2357748	−0.9229035

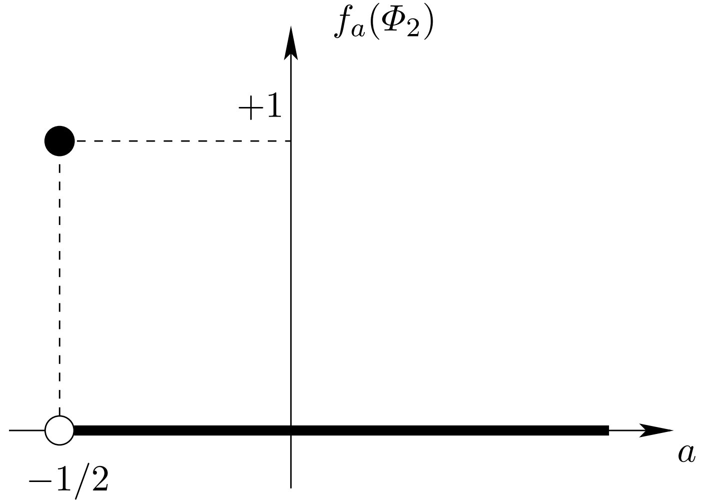
(L,3L)	**Extrapolation of** $f_a(\Phi_2)$
(4∞,12∞)	0.98107
(4∞+2,12∞+6)	0.98146

Fitting function:

$$F_N(L) = \sum_{n=0}^{N} \frac{a_n}{(L+1)^n}$$

Unstable vacuum solution

- For $L \to \infty$, numerical results suggest



$a > -1/2$

$\Phi_2 = 0$

$a = -1/2$

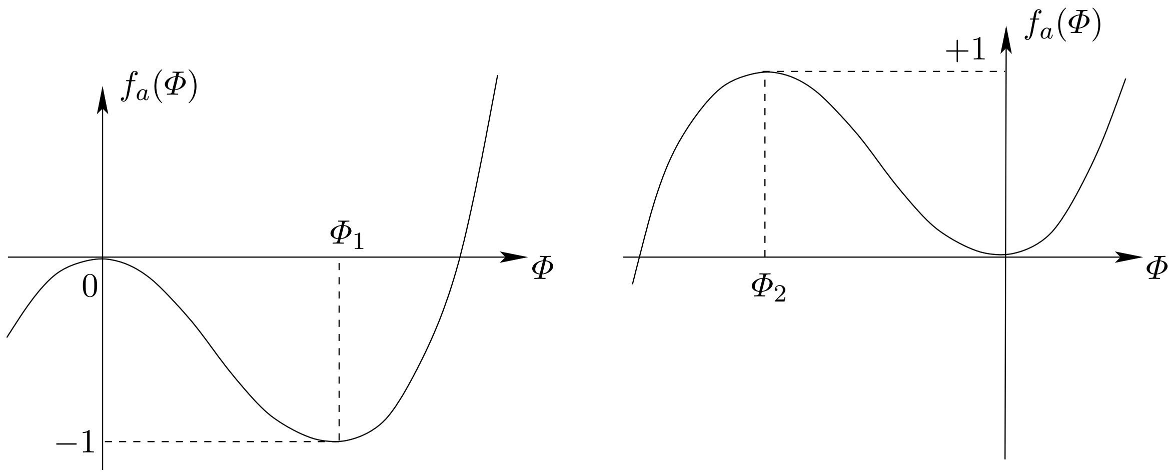
Φ_2 :nontrivial vacuum (perturbative vacuum!?)

$a > -1/2$ ⇒ Ψ_0 : pure gauge

$a = -1/2$ ⇒ Ψ_0 : tachyon vacuum (!?)

Summary (2)

- We constructed stable solution and unstable solution in the expanded theory around TT's identity based solution.
- We evaluated the gauge invariants for the obtained solutions.
- Numerical results suggest the vacuum structure such as



(1) $a > -1/2$ (2) $a = -1/2$

- This is consistent with the expectation that

$\boldsymbol{a > -1/2}$ ⇒ $\boldsymbol{\Psi_0}$: pure gauge

$\boldsymbol{a = -1/2}$ ⇒ $\boldsymbol{\Psi_0}$: tachyon vacuum

Discussion (2)

- Our result on TT solution suggests that the TT solution (a=−1/2) may be "gauge equivalent" to the Schnabl solution (λ=1) and give an alternative approach to the nonperturbative vacuum.

- *Regular* solutions? *Definition* of space of string fields?

- Extension to superstring field theory?