# Gauge invariants for identity-based solutions in string field theories

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# References

This presentation is based on

I. K., Toru Masuda and Tomohiko Takahashi,
[KMT]"Observables for identity-based tachyon vacuum solutions," PTEP **2014**, 103B02 (2014) [arXiv:1408.631**8**].
I. .K., Tomohiko Takahashi,
[KT2014]"Comments on observables for identity-based marginal solutions in Berkovits' superstring field theory," JHEP **1407**, 031 (2014) [arXiv:1404.4427 [hep-th]];
[KT2013]"Gauge invariant overlaps for identity-based marginal solutions," PTEP **2013**, 093B07 (2013) [arXiv:1307.1203 [hep-th]].

(Related paper: N. Ishibashi, arXiv:1408.631**9**)

# Contents

# Summary

We have computed gauge invariant overlaps for identity-based *marginal* solution [KT2013] using "$K'Bc$ algebra" in cubic bosonic open string field theory (SFT). It can be applied to Berkovits' WZW-like open superstring field theory (SSFT) [KT2014].

$\Downarrow$ Extension to the Takahashi-Tanimoto (TT) identity-based scalar solution

In [KMT], we have computed gauge invariants, vacuum energy and gauge invariant overlap (GIO), for the TT identity-based *scalar* solution, using a deformed $K'Bc$ algebra. The result is consistent with the previous one obtained with other methods.

# Introduction

It is *difficult* to evaluate vacuum energy and gauge invariant overlaps for "identity-based" solutions because of singular property of the identity state. They are of the form: $\langle I|(\cdots)|I\rangle$, corresponding to *zero width* in terms of the sliver frame, which gives indefinite quantity at least naively.

Instead of straightforward calculations, there have been various studies about the gauge invariants indirectly for identity-based solutions in bosonic SFT. (2001$\sim$, T. Takahashi and collaborators)

$$\updownarrow$$

On the other hand, gauge invariants for "wedge-based" solutions using $KBc$ algebra and its extention have been evaluated exactly. (2005$\sim$, Schnabl, Okawa, Erler, Ellwood, $\cdots$)

In our paper [KT2013], we have evaluated the gauge invariant overlaps (GIO), $\langle\Psi\rangle_{\mathcal{V}}$, for *identity-based* marginal solution $\Psi_J$ in *bosonic* SFT:

$$\langle\Psi_J\rangle_{\mathcal{V}} = \langle\Psi_T^{\rm ES}\rangle_{\mathcal{V}} - \langle\Psi_T\rangle_{\mathcal{V}}$$

using the relationship to (wedge-based) tachyon vacuum solutions:

$$\Psi_J = \Psi_T^{\rm ES} - \Psi_T + \int_0^1 Q_{\Psi_T^t}\Lambda_t dt.$$

In fact, we can show that $\Psi_T^{\rm ES}$ [Erler-Schnabl(2009)] and the sum of $\Psi_J$ [TT(2001)] and $\Psi_T$ [Inatomi-KT(2012)] are gauge equivalent:

$$\Psi_J + \Psi_T = g^{-1}\Psi_T^{\rm ES} g + g^{-1}Q_{\rm B}g, \qquad g = {\rm P}\exp\left(\int_0^1 \Lambda_t dt\right)$$

$$\to \qquad S[\Psi_J;Q_{\rm B}] = S[\Psi_T^{\rm ES};Q_{\rm B}] - S[\Psi_T;Q_{\Psi_J}] = 0$$

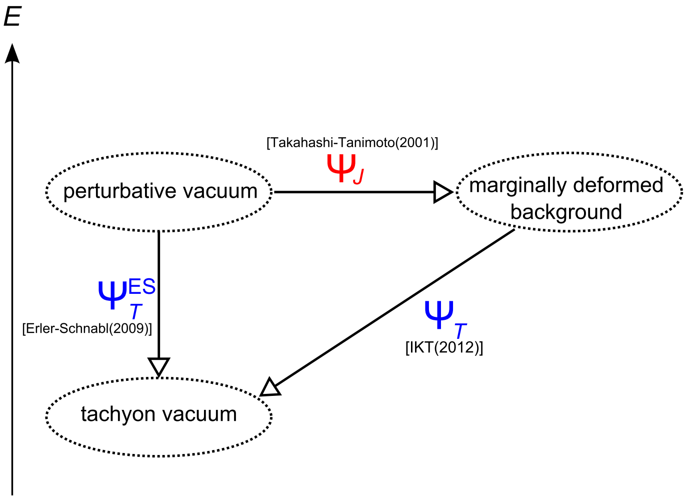



Figure: Identity/wedge-based solutions in bosonic SFT

⇓ extention to SSFT

$$e^{-\Phi_J}\hat{Q}e^{\Phi_J} = e^{-\Phi_T^{\mathrm{E}}}\hat{Q}e^{\Phi_T^{\mathrm{E}}} - e^{-\Phi_T}\hat{Q}_{\Phi_J}e^{\Phi_T} + \int_0^1 \hat{Q}_{\tilde{\Phi}_T(t)}\Lambda_t\,dt$$

Using this relation among identity-based marginal solution $\Phi_J$ and wedge-based tachyon vacuum solutions, $\Phi_T^{\mathrm{E}}$ and $\Phi_T$, the GIO for identity-based marginal solution can be evaluated:

$$\langle\Phi_J\rangle_{\mathcal{V}} = \langle\Phi_T^{\mathrm{E}}\rangle_{\mathcal{V}} - \langle\Phi_T\rangle_{\mathcal{V}}.$$

Actually, $\Phi_T^{\mathrm{E}}$ and $\log(e^{\Phi_J}e^{\Phi_T})$ are *gauge equivalent* and we have

$$S[\Phi_J;\hat{Q}] = S[\Phi_T^{\mathrm{E}};\hat{Q}] - S[\Phi_T;\hat{Q}_{\Phi_J}] = 1/(2\pi^2) - 1/(2\pi^2) = 0.$$

Namely, the energy for the identity-based marginal solution $\Phi_J$ is zero, which agrees with the previous result as a consequence of $\xi$ zeromode counting [KT(2005)].

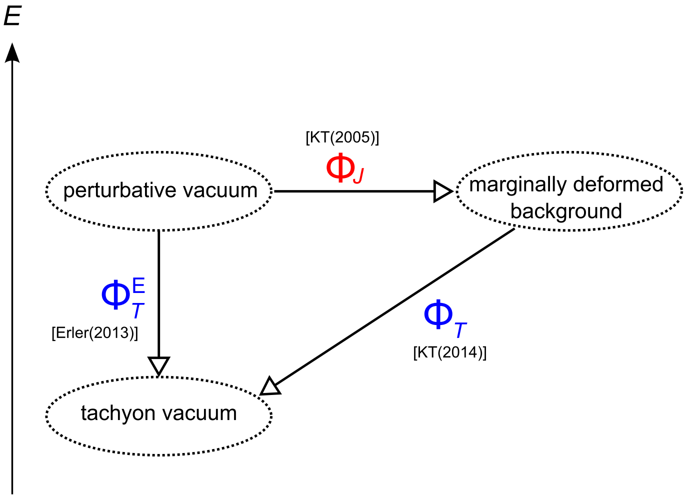



Figure: Identity/wedge-based solutions in Berkovits' WZW-like SSFT including GSO(−) sector

# Tachyon vacuum around the identity-based marginal solution

# Identity-based marginal solutions

Identity-based marginal solution in Berkovits' WZW-like SSFT:

$$\Phi_J = \tilde{V}_L^a(F_a)I,$$

$$\tilde{V}_L^a(f) \equiv \int_{C_{\rm L}} \frac{dz}{2\pi i} f(z) \frac{1}{\sqrt{2}} c\gamma^{-1}\psi^a(z), \quad \gamma^{-1}(z) = e^{-\phi}\xi(z).$$

$F_a(-1/z) = z^2 F_a(z)$, $C_{\rm L}$: a half unit circle: $|z| = 1$, ${\rm Re}\, z \geq 0$
$I$: the identity state, $\psi^a$: matter worldsheet fermion.

$\psi^a(y)\psi^b(z) \sim \frac{1}{y-z}\frac{1}{2}\Omega^{ab}$, $J^a(y)\psi^b(z) \sim \frac{1}{y-z} f^{ab}{}_c \psi^c(z)$,
$J^a(y)J^b(z) \sim \frac{1}{(y-z)^2}\frac{1}{2}\Omega^{ab} + \frac{1}{y-z} f^{ab}{}_c J^c(z)$,
$\Omega^{ab} = \Omega^{ba}$, $f^{ab}{}_c \Omega^{cd} + f^{ad}{}_c \Omega^{cb} = 0$, $f^{ab}{}_c = -f^{ba}{}_c$, $f^{ab}{}_d f^{cd}{}_e + f^{bc}{}_d f^{ad}{}_e + f^{ca}{}_d f^{bd}{}_e = 0$.

EOM in the NS sector is satisfied:

$$\eta_0(e^{-\Phi_J} Q_{\rm B} e^{\Phi_J}) = 0$$

By expanding the NS action $S[\Phi; Q_{\rm B}]$ of Berkovits' WZW-like SSFT around $\Phi_J$ as

$$e^{\Phi} = e^{\Phi_J} e^{\Phi'},$$

we have

$$S[\Phi; Q_{\rm B}] = S[\Phi_J; Q_{\rm B}] + S[\Phi'; Q_{\Phi_J}],$$

where $S[\Phi'; Q_{\Phi_J}]$ is given by a deformed BRST operator:

$$Q_{\Phi_J} = Q_{\rm B} - V^a(F_a) + \frac{1}{8}\Omega^{ab} C(F_a F_b).$$

$V^a(F_a)$ and $C(F_a F_b)$ are given by integrations along the whole unit circle:

$$V^a(f) \equiv \oint \frac{dz}{2\pi i}\frac{1}{\sqrt{2}} f(z)(cJ^a(z) + \gamma\psi^a(z)), \quad C(f) \equiv \oint \frac{dz}{2\pi i} f(z)c(z).$$

# Deformed algebra

A version of the extended $KBc$ algebra with

$Q_{\mathrm{B}} \to Q' \equiv Q_{\Phi_J}$

$L_n \to L'_n = \{Q', b_n\} = L_n - \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} F_{a,k} J^a_{n-k} + \frac{1}{8}\Omega^{ab} \sum_{k\in\mathbb{Z}} F_{a,n-k} F_{b,k}$

$F_{a,n} \equiv \oint \frac{d\sigma}{2\pi} e^{i(n+1)\sigma} F_a(e^{i\sigma}), \quad F_{a,n} = -(-1)^n F_{a,-n}$

Relations among string fields $K', B, c, \gamma$:

$$B^2 = 0,\ c^2 = 0,\ Bc + cB = 1, BK' = K'B,$$

$$K'c - cK' = Kc - cK \equiv \partial c,\ \gamma B + B\gamma = 0,\ c\gamma + \gamma c = 0,$$

$$K'\gamma - \gamma K' = K\gamma - \gamma K \equiv \partial\gamma,\ \hat{Q}'B = K',\ \hat{Q}'K' = 0,$$

$$\hat{Q}'c = cK'c - \gamma^2 = cKc - \gamma^2 = c\partial c - \gamma^2,\ \hat{Q}'\gamma = \hat{Q}\gamma = c\partial\gamma - \frac{1}{2}(\partial c)\gamma$$

where $\hat{Q}' \equiv Q'\sigma_3, \hat{Q} \equiv Q_{\mathrm{B}}\sigma_3$ and $\sigma_i$ are Pauli matrices (CP factor) for the GSO$(-)$ sector

$B = \frac{\pi}{2} B_1^L I \sigma_3, \quad c = \frac{2}{\pi}\hat{U}_1 \tilde{c}(0)|0\rangle\sigma_3, \quad \gamma = \sqrt{\frac{2}{\pi}}\hat{U}_1\tilde{\gamma}(0)|0\rangle\sigma_2 \quad K' = \frac{\pi}{2}K_1'^L I, \quad K = \frac{\pi}{2}K_1^L I$

For the string fields $\gamma^{-1}, \zeta, V$, we have

$$\gamma^{-1}\gamma = \gamma\gamma^{-1} = 1, \quad \gamma^{-1}B + B\gamma^{-1} = 0, \quad \gamma^{-1}c + c\gamma^{-1} = 0,$$
$$K'\gamma^{-1} - \gamma^{-1}K' = K\gamma^{-1} - \gamma^{-1}K \equiv \partial\gamma^{-1},$$
$$\hat{Q}'\gamma^{-1} = \hat{Q}\gamma^{-1} = c\partial\gamma^{-1} + \frac{1}{2}(\partial c)\gamma^{-1}, \qquad \hat{Q}'\zeta = \hat{Q}\zeta = cV + \gamma$$

where

$\gamma^{-1} = \sqrt{\frac{\pi}{2}}\hat{U}_1\tilde{\gamma}^{-1}(0)|0\rangle\sigma_2, \quad \zeta = \gamma^{-1}c = \sqrt{\frac{2}{\pi}}\hat{U}_1\tilde{\gamma}^{-1}\tilde{c}(0)|0\rangle i\sigma_1,$

$V = \frac{1}{2}\gamma^{-1}\partial c = \sqrt{\frac{\pi}{2}}\hat{U}_1\frac{1}{2}\tilde{\gamma}^{-1}\tilde{\partial}\tilde{c}(0)|0\rangle i\sigma_1.$

$K'$, $B$, $c$, $\gamma$, $\gamma^{-1}$, $\zeta$, $V$ and $\hat{Q}'$ have the same algebraic structure as that of the extended $KBc$ algebra with $\hat{Q}$ [Erler(2013)].

From the result in [Erler(2013)] and the above algebra, we can immediately construct a solution $\Phi_T$ in the theory with $\hat{Q}'$:

$$e^{\Phi_T} = 1 - c\frac{B}{1+K'} + q\left(\zeta + (\hat{Q}'\zeta)\frac{B}{1+K'}\right)$$

($q$ is a nonzero constant.) Actually, $\Phi_T$ satisfies

$$e^{-\Phi_T}\hat{Q}'e^{\Phi_T} = c - (\hat{Q}'c)\frac{B}{1+K'} = (c + \hat{Q}'(Bc))\frac{1}{1+K'}$$

which is in the *small* Hilbert space, and therefore the EOM in the NS sector holds:

$$\boldsymbol{\hat{\eta}(e^{-\Phi_T}\hat{Q}'e^{\Phi_T}) = 0}$$

($\hat{\eta} \equiv \eta_0\sigma_3$)

Expanding the action $S[\Phi';\hat{Q}']$ around the solution $\Phi_T$ as $e^{\Phi'}=e^{\Phi_T}e^{\Phi''}$, we have a new BRST operator $\hat{Q}'_{\Phi_T}$:

$$\hat{Q}'_{\Phi_T}\,\Xi=\hat{Q}'\,\Xi+(e^{-\Phi_T}\hat{Q}'e^{\Phi_T})\Xi-(-1)^{|\Xi|}\Xi(e^{-\Phi_T}\hat{Q}'e^{\Phi_T}).$$

Note that $e^{-\Phi_T}\hat{Q}'e^{\Phi_T}$ is the tachyon vacuum solution on the marginally deformed background in the modified cubic SSFT.

$\rightarrow$ We can find a homotopy operator $\hat{A}'$ for $\hat{Q}'_{\Phi_T}$:
$\hat{A}'\,\Xi=\frac{1}{2}(A'\,\Xi+(-1)^{|\Xi|}\Xi\,A')$, such as $\{\hat{Q}'_{\Phi_T},\hat{A}'\}=1,\ (\hat{A}')^2=0$, where $A'\equiv\dfrac{B}{1+K'}$ is a homotopy state in the *small* Hilbert space.

$\rightarrow$ No physical open string state around the solution $\Phi_T$
$\sim$ the tachyon vacuum solution in the marginally deformed background.

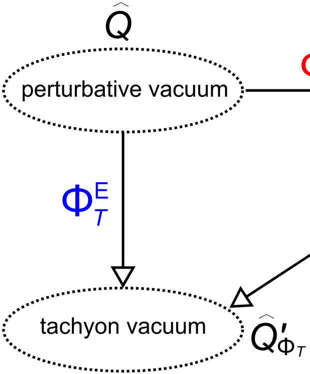



Figure: Identity/wedge-based solutions and BRST operators around them. $\hat{Q}'_{\Phi_T}$ has no cohomology in the small Hilbert space.

# Energy and gauge invariant overlaps

NS action $S[\Phi';\hat{Q}_{\Phi_J}]$ around $\Phi_J$:

$$S[\Phi';\hat{Q}_{\Phi_J}] = -\int_0^1 dt\,\mathrm{Tr}\left[\left(\hat{\eta}(g(t)^{-1}\partial_t g(t))\right)\left(g(t)^{-1}\hat{Q}_{\Phi_J} g(t)\right)\right].$$

$g(t)$: an interpolating string field s.t. $g(0)=1$ and $g(1)=e^{\Phi'}$.
$\mathrm{Tr}\,A \equiv \frac{1}{2}\mathrm{tr}\langle I|A\rangle$; $\mathrm{tr}$ : trace for the Chan-Paton factor for GSO($\pm$) sector

We take an interpolating string field as $g_T(t) = 1 + t(e^{\Phi_T}-1)$ and the integrand in the action for the solution: $S[\Phi_T;\hat{Q}_{\Phi_J}]$, can be manipulated in the same way as the Erler solution, with $Q_{\mathrm{B}} \to Q' \equiv Q_{\Phi_J}$, $L_n \to L'_n$. As a result, we have

$$\begin{aligned}&\mathrm{Tr}\left[\left(\hat{\eta}(g_T(t)^{-1}\partial_t g_T(t))\right)\left(g_T(t)^{-1}\hat{Q}_{\Phi_J} g_T(t)\right)\right]\\ &= \frac{-2q^2t^2(1-t)(2q^2t-1)}{(1-t+q^2t^2)^3}\int_0^1 d\theta(1-\theta)\mathcal{X}(\theta) + \frac{q^2t(1-t)}{(1-t+q^2t^2)^2}\int_0^1 d\theta\,\mathcal{X}(\theta)\end{aligned}$$

where $\quad \mathcal{X}(\theta) = \mathrm{Tr}\Big[B(\hat{\eta}(cV))e^{-\theta K'}cVe^{-(1-\theta)K'}\Big].$

We use the result for the modified cubic SSFT in the marginally deformed background [IKT(2012)]:

$$e^{-\alpha K'} = e^{-\alpha\frac{\pi}{2}\mathcal{C}}\hat{U}_{\alpha+1}\mathbf{T}\exp\left(\frac{\pi}{4}\int_{-\alpha}^{\alpha}du\int_{-\infty}^{\infty}dv f_a(v)\tilde{J}^a(iv+\frac{\pi}{4}u)\right)|0\rangle,$$

$$f_a(v)\equiv\frac{F_a(\tan(iv+\frac{\pi}{4}))}{2\pi\sqrt{2}\cos^2(it+\frac{\pi}{4})},\qquad \mathcal{C}\equiv\frac{\pi}{2}\int_{-\infty}^{\infty}dv\,\Omega^{ab}f_a(v)f_b(v),$$

where $\mathbf{T}$ is an ordering symbol with respect to the real part of the argument of $\tilde{J}^a$.

Finally, the trace can be evaluated as

$$\begin{aligned}&\mathrm{Tr}\Big[B(\hat{\eta}(cV))e^{-\theta K'}cVe^{-(1-\theta)K'}\Big]\\ &=-\frac{1}{4}\left\langle(\eta_0\gamma^{-1}(\frac{\pi}{2}))\gamma^{-1}(\frac{\pi}{2}(1-2\theta))\right\rangle_{\xi\eta\phi}\left\langle B_1^L c\partial c(\frac{\pi}{2})c\partial c(\frac{\pi}{2}(1-2\theta))\right\rangle_{bc}\\ &\qquad\times e^{-\frac{\pi}{2}\mathcal{C}}\left\langle\exp\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}du\int_{-\infty}^{\infty}dv f_a(v)J^a(2iv+u)\right)\right\rangle_{\mathrm{mat}}\end{aligned}$$

The last factor in the trace, which comes from the matter sector, is $1$:

$$e^{-\frac{\pi}{2}\mathcal{C}}\left\langle \exp\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du \int_{-\infty}^{\infty} dv\, f_a(v) J^a(2iv+u)\right)\right\rangle_{\rm mat} = 1$$

as proved in [IKT(2012)], and so the trace becomes the **same** result as the case of the Erler solution.

Consequently, the vacuum energy is unchanged from the case in the original background without marginal deformations, namely,

$$E = -S[\Phi_T; \hat{Q}_{\Phi_J}] = -\frac{1}{2\pi^2}$$

Next, we will evaluate the gauge invariant overlap (GIO) in Berkovits' WZW-like SSFT. We define the GIO $\langle\Phi\rangle_{\mathcal{V}}$ as

$$\langle\Phi\rangle_{\mathcal{V}} \equiv \mathrm{Tr}[\mathcal{V}(i)\Phi].$$

$\mathcal{V}(i)$ : a midpoint insertion of a primary closed string vertex operator with picture #: $-1$, ghost #: $2$, conformal dim.: $(0,0)$. $\mathcal{V}$ is BRST invariant in the small Hilbert space: $[Q_{\mathrm{B}},\mathcal{V}(i)]=0$, $[\eta_0,\mathcal{V}(i)]=0$ and $\forall\Lambda,\Xi$

$$\langle\hat{Q}\Lambda\rangle_{\mathcal{V}}=0,\qquad \langle\hat{\eta}\Lambda\rangle_{\mathcal{V}}=0,\qquad \langle\Lambda\Xi\rangle_{\mathcal{V}}=(-)^{|\Lambda||\Xi|}\langle\Xi\Lambda\rangle_{\mathcal{V}}.$$

Infinitesimal gauge transformation: $\delta_\Lambda e^\Phi=(\hat{Q}\Lambda_0)e^\Phi+e^\Phi\hat{\eta}\Lambda_1$ ($\Lambda_0$, $\Lambda_1$: gauge parameter string field with the picture # $0,1$.) Namely,

$$\delta_\Lambda\Phi=\frac{\mathrm{ad}_\Phi}{e^{\mathrm{ad}_\Phi}-1}\hat{Q}\Lambda_0+\frac{-\mathrm{ad}_\Phi}{e^{-\mathrm{ad}_\Phi}-1}\hat{\eta}\Lambda_1=\sum_{n=0}^{\infty}\frac{B_n}{n!}(\mathrm{ad}_\Phi)^n(\hat{Q}\Lambda_0+(-1)^n\hat{\eta}\Lambda_1)$$

$(\mathrm{ad}_B(A)\equiv[B,A]=BA-AB)$

Thanks to the above properties, the GIO is invariant under this gauge transformation: $\langle \delta_\Lambda \Phi \rangle_{\mathcal{V}} = 0$.

Inserting $1 = \{Q_{\rm B}, \xi Y(i)\}$, ($Y(z) = c\partial\xi e^{-2\phi}(z)$ the inverse picture changing operator) the GIO can be rewritten as

$$\langle \Phi \rangle_{\mathcal{V}} = \mathrm{Tr}[\mathcal{V}(i)\{Q_{\rm B}, \xi Y(i)\}\Phi] = \mathrm{Tr}[\xi Y \mathcal{V}(i)\sigma_3 \hat{Q}\Phi] = \mathrm{Tr}[\xi Y \mathcal{V}(i)\sigma_3 \hat{Q}'\Phi]$$
$$= \mathrm{Tr}[\xi Y \mathcal{V}(i)\sigma_3\, e^{-\Phi}\hat{Q}' e^{\Phi}].$$

GIO for the tachyon vacuum $\Phi_T$: $\langle \Phi_T \rangle_{\mathcal{V}} = \mathrm{Tr}\left[\xi Y \mathcal{V}(i)\sigma_3\, c\dfrac{1}{1+K'}\right]$

In a similar way to the calculation of the vacuum energy, we obtain an expression of the GIO in the marginally deformed background:

$$\langle \Phi_T \rangle_{\mathcal{V}} = \frac{e^{-\pi\mathcal{C}}}{\pi}\left\langle \xi Y \mathcal{V}(i\infty) c(\frac{\pi}{2}) \exp\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du\, \mathcal{J}(u)\right)\right\rangle_{C_\pi}$$
$$\mathcal{J}(u) = \int_{-\infty}^{\infty} dv\, f_a(v) J^a(iv+u)$$

# Evaluation of observables for identity-based marginal solutions

# Interpolation

Let us calculate two observables, the vacuum energy and the GIO, for the *identity-based* marginal solutions.

An interpolation: $\Phi_J(t) = t\Phi_J$ s.t. $\Phi_J(0) = 0, \ \Phi_J(1) = \Phi_J$.
$\Phi_J(t)$ : a replacement of weighting function: $F_a(z) \to tF_a(z)$ in $\Phi_J$.
$\Rightarrow$ EOM is satisfied: $\hat{\eta}(e^{-\Phi_J(t)}\hat{Q}e^{\Phi_J(t)}) = 0$

A new BRST operator $Q_{\Phi_J(t)}$ for the theory around $\Phi_J(t)$:

$$Q_{\Phi_J(t)} = Q_{\rm B} - tV^a(F_a) + \frac{t^2}{8}\Omega^{ab}C(F_aF_b)$$

Following the same procedure as before, we define a string field $K'(t) \equiv \hat{Q}_{\Phi_J(t)}B$ ($\hat{Q}_{\Phi_J(t)} \equiv Q_{\Phi_J(t)}\sigma_3$). Then, we can construct a tachyon vacuum solution $\Phi_T(t)$ as

$$e^{\Phi_T(t)} = 1 - c\frac{B}{1+K'(t)} + q\left(\zeta + (\hat{Q}_{\Phi_J(t)}\zeta)\frac{B}{1+K'(t)}\right)$$

It satisfies the EOM around the identity-based solution $\Phi_J(t)$:

$$\hat{\eta}(e^{-\Phi_T(t)}\hat{Q}_{\Phi_J(t)}e^{\Phi_T(t)}) = 0.$$

In particular, $\Phi_T(t)$ satisfies $\Phi_T(1) = \Phi_T$ and $\Phi_T(0) = \Phi_T^{\mathrm{E}}$ (the Erler solution (2013)) because $Q_{\Phi_J(1)} = Q_{\Phi_J}$ and $Q_{\Phi_J(0)} = Q_{\mathrm{B}}$.

Using the above string fields, we **define** a string field $\tilde{\Phi}_T(t)$ with the parameter $t$ as $e^{\tilde{\Phi}_T(t)} \equiv e^{\Phi_J(t)}e^{\Phi_T(t)}$ and then we find a relation:

$$e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)} = e^{-\Phi_J(t)}\hat{Q}e^{\Phi_J(t)} + e^{-\Phi_T(t)}\hat{Q}_{\Phi_J(t)}e^{\Phi_T(t)}.$$

Hence, $\tilde{\Phi}_T(t)$ satisfies the EOM of the *original* theory:

$$\hat{\eta}(e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)}) = 0.$$

Expanding around the solution $\tilde{\Phi}_T(t)$ in the theory with $\hat{Q}$, we have the theory with the deformed BRST operator $\hat{Q}_{\tilde{\Phi}_T(t)}$:

$$\hat{Q}_{\tilde{\Phi}_T(t)}\Xi = \hat{Q}\Xi + (e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)})\Xi - (-1)^{|\Xi|}\Xi(e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)})$$
$$= \hat{Q}_{\Phi_J(t)}\Xi + (e^{-\Phi_T(t)}\hat{Q}_{\Phi_J(t)}e^{\Phi_T(t)})\Xi - (-1)^{|\Xi|}\Xi(e^{-\Phi_T(t)}\hat{Q}_{\Phi_J(t)}e^{\Phi_T(t)}).$$

The last expression implies that $\hat{Q}_{\tilde{\Phi}_T(t)}$ is the same as the BRST operator $\hat{Q}'_{\Phi_T(t)}$ in the theory around $\Phi_T(t)$, which is a tachyon vacuum solution in the theory around $\Phi_J(t)$.

Following the previous results with appropriate replacement, we find that there exists a homotopy state: $A'(t) \equiv \frac{B}{1+K'(t)}$ such as $\hat{Q}_{\tilde{\Phi}_T(t)}A'(t) = 1$, which implies that

there is no cohomology for $\hat{Q}_{\tilde{\Phi}_T(t)}$ in the small Hilbert space.

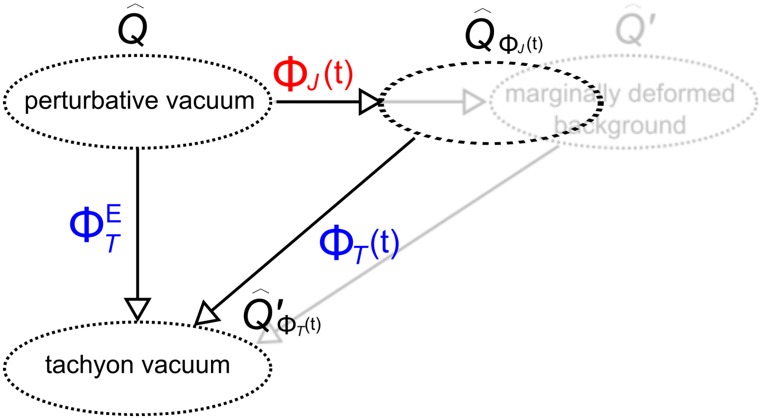



Figure: Interporating identity/wedge-based solutions and BRST operators around them. With $e^{\tilde{\Phi}_T(t)} \equiv e^{\Phi_J(t)} e^{\Phi_T(t)}$, a BRST operator around $\tilde{\Phi}_T(t)$ $\hat{Q}'_{\Phi_T(t)} = \hat{Q}_{\tilde{\Phi}_T(t)}$ has no cohomology in the small Hilbert space.

Differentiating an identity: $\hat{Q}(e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)}) + (e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)})^2 = 0$ with respect to $t$, we have

$$\hat{Q}_{\tilde{\Phi}_T(t)}\frac{d}{dt}(e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)}) = 0.$$

Therefore, there exists a state $\Lambda_t$ in the small Hilbert space such as

$$\frac{d}{dt}(e^{-\tilde{\Phi}_T(t)}\hat{Q}e^{\tilde{\Phi}_T(t)}) = \hat{Q}_{\tilde{\Phi}_T(t)}\Lambda_t.$$

Integrating the above, we have

$$e^{-\tilde{\Phi}_T(1)}\hat{Q}e^{\tilde{\Phi}_T(1)} = e^{-\Phi_T^{\mathrm{E}}}\hat{Q}e^{\Phi_T^{\mathrm{E}}} + \int_0^1 \hat{Q}_{\tilde{\Phi}_T(t)}\Lambda_t\, dt.$$

This relation implies that $\tilde{\Phi}_T(1) = \log(e^{\Phi_J}e^{\Phi_T})$ is *gauge equivalent* to the Erler solution $\Phi_T^{\mathrm{E}}$.

# On gauge equivalence relations

Here, we discuss some gauge equivalence relations in terms of the NS sector of Berkovits' WZW-like SSFT.

A gauge transformation of the superstring field $\Phi$ by group elements $h(t)$ and $g(t)$ with one parameter $t$ s.t. $g(0) = h(0) = 1$ is

$$e^{\Phi(t)} = h(t)\, e^{\Phi}\, g(t), \qquad Q_{\mathrm{B}} h(t) = \eta_0 g(t) = 0. \tag{1}$$

For the string fields, $\Phi$ and $\Phi(t)$, "one-form" string fields are defined by $\Psi \equiv e^{-\Phi}\, Q_{\mathrm{B}} e^{\Phi}$ and $\Psi(t) \equiv e^{-\Phi(t)}\, Q_{\mathrm{B}} e^{\Phi(t)}$. From (1), these turn out to be related by a transformation:

$$\Psi(t) = g(t)^{-1}\, Q_{\mathrm{B}}\, g(t) + g(t)^{-1} \Psi\, g(t), \qquad \eta_0 g(t) = 0. \tag{2}$$

It is the same form as gauge transformations in the modified cubic SSFT.

Conversely, given the relation (2), we find that the relation (1) holds for $h(t) = e^{\Phi(t)} g(t)^{-1} e^{-\Phi}$. In fact, from (2), we have $Q_{\mathrm{B}}(e^{\Phi(t)} g(t) e^{-\Phi}) = 0$.

Differentiating (2) with respect to $t$ and integrating it again, we find another relation between $\Psi$ and $\Psi(t)$:

$$\Psi(t) = \Psi + \int_0^t Q_{\Phi(t')}\Lambda(t')\,dt', \qquad \eta_0\Lambda(t) = 0, \tag{3}$$

where $\Lambda(t) = g(t)^{-1}\frac{d}{dt}g(t)$ and $Q_\phi$ is a modified BRST operator associated with $\psi \equiv e^{-\phi}Q_{\rm B}e^{\phi}$: $Q_\phi\lambda = Q_{\rm B}\lambda + \psi\lambda - (-1)^{|\lambda|}\psi\lambda$.

Conversely, supposing that the equations (3) for a given $\Lambda(t)$ hold, we find the relations $(2)$ hold for the group element $g(t)$ such as $g(0) = 1$:

$$g(t) = {\rm P}\exp\left(\int_0^t \Lambda(t')dt'\right),$$

where ${\rm P}\exp$ means a $t$-ordered exponent.

Consequently, the above relations (1), (2) and (3) are all equivalent.
(We can include internal Chan-Paton factors in these relations.)

# Energy and gauge invariant overlaps

From the gauge equivalence:

$$\log(e^{\Phi_J} e^{\Phi_T}) \sim \Phi_T^{\mathrm{E}}$$

we can analytically evaluate gauge invariants for the *identity-based* marginal solutions.

The value of the action: $S[\Phi_J;\hat{Q}] + S[\Phi_T;\hat{Q}_{\Phi_J}] = S[\Phi_T^{\mathrm{E}};\hat{Q}]$

$$\therefore\ E = -S[\Phi_J;\hat{Q}] = S[\Phi_T;\hat{Q}_{\Phi_J}] - S[\Phi_T^{\mathrm{E}};\hat{Q}] = 0$$

The result agrees with the previous one derived from $\xi$ zeromode counting:

$$S[\Phi_J, Q_{\mathrm{B}}] = -\int_0^1 dt\, \mathrm{Tr}[(\eta_0 \Phi_J)\,(e^{-t\Phi_J} Q_{\mathrm{B}} e^{t\Phi_J})] = 0$$

Evaluation of the GIO for the identity-based marginal solution $\langle \Phi_J \rangle_{\mathcal{V}}$:

Using an identity,

$$e^{-\tilde{\Phi}_T(1)} \hat{Q} e^{\tilde{\Phi}_T(1)} = e^{-\Phi_J} \hat{Q} e^{\Phi_J} + e^{-\Phi_T} \hat{Q}_{\Phi_J} e^{\Phi_T},$$

we have obtained

$$e^{-\Phi_J} \hat{Q} e^{\Phi_J} = e^{-\Phi_T^{\mathrm{E}}} \hat{Q} e^{\Phi_T^{\mathrm{E}}} - e^{-\Phi_T} \hat{Q}_{\Phi_J} e^{\Phi_T} + \int_0^1 \hat{Q}_{\tilde{\Phi}_T(t)} \Lambda_t \, dt.$$

It leads to a relation for the GIOs:

$$\begin{aligned} \langle \Phi_J \rangle_{\mathcal{V}} &= \langle \Phi_T^{\mathrm{E}} \rangle_{\mathcal{V}} - \langle \Phi_T \rangle_{\mathcal{V}} \\ &= \frac{1}{\pi} \left\langle \xi Y \mathcal{V}(i\infty) c(\frac{\pi}{2}) \left\{ 1 - e^{-\pi \mathcal{C}} \exp\Big( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du \, \mathcal{J}(u) \Big) \right\} \right\rangle_{C_\pi} . \end{aligned}$$

In the same way as the case of bosonic SFT [KT(2013)], it can be rewritten as a difference between two disk amplitudes with the boundary deformation by taking

$$F_a(z;s) = \frac{2\lambda_a s(1-s^2)}{\arctan \frac{2s}{1-s^2}} \frac{1+z^{-2}}{1-s^2(z^2+z^{-2})+s^4}$$

for the function $F_a(z)$ in $\Phi_J$, which satisfies

$$\int_{C_{\rm L}} \frac{dz}{2\pi i} F_a(z;s) = \frac{2\lambda_a}{\pi},$$

$$F_a(z;s) \to 4\lambda_a\{\delta(\theta)+\delta(\pi-\theta)\}, \qquad (s\to 1, \ \ z=e^{i\theta}).$$

Because the form of weighting function $F_a$ except the half-integration mode can be changed by a kind of gauge transformation [KT(2005)], we have the same value of the GIO for a fixed value of $\lambda_a$, which corresponds to a marginal deformation parameter).

Namely, for the limit $s \to 1$, the GIO is expressed as

$$\langle \Phi_J \rangle_{\mathcal{V}} = \frac{1}{\pi} \left\langle \xi Y \mathcal{V}(i\infty) c(\frac{\pi}{2}) \left\{ 1 - e^{-\pi \mathcal{C}} \exp\left( \frac{\sqrt{2}}{\pi} \lambda_a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du\, J^a(u) \right) \right\} \right\rangle_{C_\pi} .$$

In this expression, $\mathcal{C}$ becomes divergent for the function $\lim_{s\to 1} F_a(z;s)$ and then it cancels the contact term divergence due to singular OPE among the currents.

This expression of the GIO corresponds to the result in [Ellwood(2008)] for a *wedge-based* marginal solution.

# Conclusion

We have applied the method in [IKT(2012)] for cubic (S)SFT to the Erler solution $\Phi_T^{\mathrm{E}}$ for Berkovits' WZW-like SSFT.

- We have constructed a tachyon vacuum solution $\Phi_T$ around the identity-based marginal solution $\Phi_J$ [KT(2005)] in SSFT with an extended $KBc$ algebra in the marginally deformed background.
- Around $\Phi_T$, we have obtained a homotopy operator and evaluated vacuum energy and gauge invariant overlap (GIO) for it.
  The energy is the same value as that on the original background, but the GIO is deformed by the marginal operators.

Using the above, we have extended our computation for bosonic SFT [KT2013] to superstring: We have evaluated the energy and the GIO for the identity-based marginal solution $\Phi_J$ in the framework of Berkovits' WZW-like SSFT.

- The gauge equivalence relation between $\Phi_T^{\mathrm{E}}$ and $\log(e^{\Phi_J}e^{\Phi_T})$ is essential. It is derived from the vanishing cohomology in the small Hilbert space around the interpolating tachyon vacuum solutions.
- The energy for $\Phi_J$ vanishes and it is consistent with our previous result using $\xi$ zeromode counting.
- The GIO for $\Phi_J$ is expressed by a difference of those of the tachyon vacuum solutions on the undeformed and deformed backgrounds.

We hope that this approach to identity-based solutions will be useful to deeply understand bosonic and super SFT.

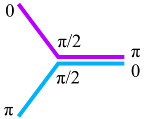
# Bosonic string field theory

The gauge invariant action for cubic bosonic open SFT

$$S[\Psi; Q_{\rm B}] = -\int \left( \frac{1}{2} \Psi * Q_{\rm B} \Psi + \frac{1}{3} \Psi * \Psi * \Psi \right)$$

string field: $|\Psi\rangle = t(x) c_1 |0\rangle + A_\mu(x) \alpha^\mu_{-1} c_1 |0\rangle + i B(x) c_0 |0\rangle + \cdots$

$Q_{\rm B}$: Kato-Ogawa's BRST operator




$*$ : interaction of open strings (star product)

$\int$ : contraction with the identity state $\langle I|$


The equation of motion

$$Q_{\rm B} \Psi + \Psi * \Psi = 0$$

# Gauge transformation and gauge invariant overlap

The action is invariant under the following transformation:

**Gauge transformation**

$$\Psi' = e^{-\Lambda} Q_{\mathrm{B}} e^{\Lambda} + e^{-\Lambda} \Psi e^{\Lambda}$$

$\Lambda$: gauge parameter string field, the symbol "$*$" is omitted.

Gauge invariants defined by an onshell closed string vertex $V$ at the string midpoint:

**GIO**

$$\langle \Psi \rangle_V \equiv \langle I | V(i) | \Psi \rangle$$

# The TT identity-based scalar solution

### Takahashi-Tanimoto (2002)

$$\Psi_0 = Q_L(e^h - 1)I - C_L((\partial h)^2 e^h)I$$

$Q_L(f) = \int_{C_{\rm left}} \frac{dz}{2\pi i} f(z) j_{\rm B}(z)$, $C_L(f) = \int_{C_{\rm left}} \frac{dz}{2\pi i} f(z) c(z)$ : half integration

A simple choice of a function with one parameter $a$:

$$h_a(z) = \log\left(1 + \frac{a}{2}(z + z^{-1})^2\right), \quad (a \geq -1/2)$$

### Interpretation of the solution

$$\Psi_0(a) = \begin{cases} a > -1/2 & : \text{trivial pure gauge solution} \\ a = -1/2 & : \text{tachyon vacuum solution} \end{cases}$$

There are several evidences for this expectation.

# $KBc$ algebra and the ES simple solution

$KBc$ algebra [Schnabl(2005), Okawa(2006)]

$$K = Q_\mathrm{B} B,\ \ Q_\mathrm{B} K = 0,\ \ Q_\mathrm{B} c = cKc,\ \ B^2 = 0,\ \ c^2 = 0,\ \ Bc + cB = 1$$

$$B = \tfrac{\pi}{2}(B_1)_L I,\quad c = \tfrac{1}{\pi}c(1)I,\quad (B_1)_L = \int_{C_\mathrm{left}} \tfrac{dz}{2\pi i}(1+z^2)b(z)$$

The simple tachyon vacuum solution [Erler-Schnabl(2009)]

$$\Psi_0(K,B,c) = \frac{1}{\sqrt{1+K}}(c + cKBc)\frac{1}{\sqrt{1+K}}$$

Gauge invariants for the ES simple solution can be evaluated:

$$-S[\Psi_0(K,B,c), Q_\mathrm{B}] = -\frac{1}{2\pi^2},\quad \langle \Psi_0(K,B,c)\rangle_V = \frac{1}{\pi}\left\langle V(i\infty)c(\frac{\pi}{2})\right\rangle$$

# The theory around the TT identity-based solution

Action expanding around the TT solution: $\Psi = \Psi_0(a) + \Phi$

$$S[\Psi; Q_{\rm B}] = S[\Psi_0(a); Q_{\rm B}] + S[\Phi; Q']$$

where $Q' = \oint \frac{dz}{2\pi i} \left( e^{h_a(z)} j_{\rm B}(z) - (\partial h_a(z))^2 \, e^{h_a(z)} c(z) \right)$

**The BRST operator $Q'$ around the solution $\Psi_0(a)$**

$a > -1/2$: $Q' = e^{\tilde{q}(h_a)} Q_{\rm B} e^{-\tilde{q}(h_a)}$, $\tilde{q}(h_a) = \oint \frac{dz}{2\pi i} h_a(z) \left( j_{\rm gh}(z) - \frac{3}{2z} \right)$

$a = -1/2$: $Q'$ has no cohomology [KT(2002), Inatomi-KT(2011)]

**EOM to the theory around $\Psi_0(a)$**

$$Q'\Phi + \Phi * \Phi = 0$$

# $K'Bc$ algebra and the ES-like solution $(a > -1/2)$

$K'Bc$ algebra for $\boldsymbol{a > -1/2}$

$$K' = Q'B,\ \ Q'K' = 0,\ \ Q'c = cK'c,\ \ B^2 = 0,\ \ c^2 = 0,\ \ Bc + cB = 1$$

The ES-like wedge-based solution to the EOM ◎

$$\Phi_0(K', B, c) = \frac{1}{\sqrt{1+K'}}(c + cK'Bc)\frac{1}{\sqrt{1+K'}}$$

In order to evaluate the gauge invariants, we find a relation:

$$\Phi_0(K', B, c) = e^{\tilde{q}(h_a)}\, U_f^{-1} \Psi_0(K, B, c)$$

where $U_f = \exp\big(\sum_n v_n (L_n - (-1)^n L_{-n})\big)$ is given by a conformal transformation determined by $h_a(z)$.

Using the above relation, we have

$$S[\Phi_0(K', B, c);\, Q'] = S[\Psi_0(K, B, c);\, U_f Q_{\rm B} U_f^{-1}]$$
$$= S[\Psi_0(K, B, c); Q_{\rm B}].$$

We also note

$$\langle I|V(i)\,\tilde{q}(h_a) = 0, \qquad \langle I|V(i)\,(L_n - (-1)^n L_{-n}) = 0.$$

Gauge invariants for the ES-like solution:

$$-S[\Phi_0(K', B, c), Q'] = -\frac{1}{2\pi^2}, \quad \langle \Phi_0(K', B, c)\rangle_V = \frac{1}{\pi}\left\langle V(i\infty)c(\frac{\pi}{2})\right\rangle$$

These are the **same** value for the ES simple solution for tachyon condensation in the original theory.

# $K'Bc$ algebra and the ES-like solution $(a=-1/2)$

**$K'Bc$ algebra for $a=-1/2$**

$$K' = Q'B, \quad Q'K' = 0, \quad Q'c = 0 \quad B^2 = 0, \quad c^2 = 0, \quad Bc + cB = 1$$

In particular, we have $K'c = cK'$ in this case.

The ES-like solution to the EOM ◎ is reduced to a $Q'$-closed form

$$\Phi_0(K', B, c) = \frac{1}{\sqrt{1+K'}}(c + cK'Bc)\frac{1}{\sqrt{1+K'}} = c$$

Actually, this should also be exact because $Q'$ has no cohomology.

Gauge invariants for the above solution become trivial:

$$-S[\Phi_0(K', B, c), Q'] = 0, \quad \langle \Phi_0(K', B, c) \rangle_V = 0$$

# An interpolating string field $\Psi_a$

$\Psi_a \equiv \Psi_0(a) + \Phi_0(K', B, c)$ satisfies EOM: $Q_{\rm B}\Psi_a + \Psi_a * \Psi_a = 0$.

Note: $\Psi_{a=0} = \Psi_0(K, B, c)$ (the ES simple solution)

Differentiating the EOM with respect to $a$, we have

$$Q_{\Psi_a}\frac{d}{da}\Psi_a = 0; \quad Q_{\Psi_a}A = Q'A + \Phi_0(K',B,c)*A - (-1)^{|A|}A*\Phi_0(K',B,c).$$

$Q_{\Psi_a}$ has a homotopy state, $\frac{B}{1+K'}$, and therefore it has no cohomology.

$$\Rightarrow \qquad \exists\Lambda_a \text{ s.t.} \qquad \frac{d}{da}\Psi_a = Q_{\Psi_a}\Lambda_a$$

Integrating this expression, we obtain a gauge equivalence relation.

# Gauge equivalence relation

$\Psi_a \sim \Psi_0(K, B, c)$

$$\Psi_0(a) + \Phi_0(K', B, c) = \Psi_0(K, B, c) + \int_0^a Q_{\Psi_a} \Lambda_a da$$

Or equivaletly, this can be rewritten as

$$\Psi_0(a) + \Phi_0(K', B, c) = g^{-1} Q_{\mathrm{B}} g + g^{-1}\, \Psi_0(K, B, c)\, g,$$

where $g = \mathrm{P}\exp\left(\int_0^a \Lambda_a da\right)$.

The above implies following relations among gauge invariants:

$$S[\Psi_0(a); Q_{\mathrm{B}}] + S[\Phi_0(K', B, c); Q'] = S[\Psi_0(K, B, c); Q_{\mathrm{B}}]$$

$$\langle \Psi_0(a) \rangle_V + \langle \Phi_0(K', B, c) \rangle_V = \langle \Psi_0(K, B, c) \rangle_V$$

# Gauge invariants for the identity-based solution

Then, we immediately obtain:

**Energy for the *identity-based* solution**

$$-S[\Psi_0(a);Q_{\rm B}]=\begin{cases}0 & (a>-1/2)\\[2ex] -\dfrac{1}{2\pi^2} & (a=-1/2)\end{cases}$$

**GIO for the *identity-based* solution**

$$\langle\Psi_0(a)\rangle_V=\begin{cases}0 & (a>-1/2)\\[2ex] \dfrac{1}{\pi}\left\langle V(i\infty)c(\dfrac{\pi}{2})\right\rangle & (a=-1/2)\end{cases}$$

These results strongly support our previous expectation ♢.

# Concluding remarks

We have extended the method in [KT2013] for marginal identity-based solutions to the TT identity-based scalar solution $\Psi_0(a)$ in bosonic SFT.
The result is consistent with our previous interpretation:

$$\Psi_0(a) = \begin{cases} a > -1/2 & : \text{trivial pure gauge solution} \\ a = -1/2 & : \text{tachyon vacuum solution} \end{cases}$$

In particular, $\Psi_0(a = -1/2)$ becomes nontrivial and corresponds to the tachyon vacuum in bosonic SFT, which is also supported by [Ishibashi(2014)] with other methods.