On Algebraic Structure in Open-Closed String Field Theory

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⋆ A 'Definition' of SFT

(for a given string theory,) —

Action consisting of the kinetic term and the interaction terms such that the perturbative expansion reproduces the on-shell string scattering amplitudes of the corresponding string theory.

(Recall) String amplitudes are given
by integrating correlation functions over the moduli space of punctured Riemann surfaces

(by summing over all the equivalence classes of the 'shapes' of the string interaction described by the Riemann surfaces)

★ How to construct an SFT action:

∘ to decompose the moduli space of punctured Riemann surfaces into parts

so that each Feynman graph corresponds to the integration of the correlation functions over each part.

⇔ By this construction, the SFT action has a special algebraic structure

(in terms of BV-formalism).

. . . ., Zwiebach, ...

Problem How is such an algebraic structure useful ??

★ Main motivations to construct SFT

may be

(1) to discuss non-perturbative effect (classical equation, etc) of the SFT action.

(2) to calculate string amplitudes only by the Feynman rule of SFT.

- For string theory = bosonic CFT,

  Witten's cubic open SFT $\cdots$ (1) (2) ok

  (for quantum level, maybe ok?)

  The others, not easy both for (1) and (2)

- For string theory = topological string,

  For (1), only for marginal deformations.

  For (2), there are some useful ones:

  Chern Simons theory and its generalizations

  (=topological open SFT)

  BCOV (=a topological closed SFT)

BV-master equation implies:

tree SFT $\Longleftrightarrow$ Homotopy algebra

- tree open SFT

$\Longleftrightarrow A_\infty$-algebra (J. Stasheff'63)

(Gaberdiel-Zwiebach'95,

Zwiebach'97, Nakatsu'01, H.K'01, $\cdots$)

- tree closed SFT

$\Longleftrightarrow L_\infty$-algebra (Lada-Stasheff'92)

(Zwiebach'92, $\cdots$)

- tree open-closed SFT

$\Longleftrightarrow$ OCHA (H.K-Stasheff'04)

Open-Closed Homotopy Algebra

⋆ What we can do:

1a) A tree open SFT is homotopy equivalent to the corresponding tree string scattering amplitudes (H.K'01 for open case)

1b) Any tree SFT of a fixed string theory is related to each other by field redefinition.

(H.K'03 for open case)

(cf. Hata-Zwiebach'93)

∘ Tree closed case (L_∞) follows from A_∞ case.

∘ Tree open-closed case follows from a Thm. in H.K-Stasheff'04.

2) ∃classical open-closed correspondence (explained later)

The classical BV-master equation

Let $(x^1, \ldots, x^n, y^1, \ldots, y^n)$:

coordinates of $\mathbb{R}^{2n}$ with $\mathbb{Z}$-gradings such that

$$\deg(x^i) + \deg(y^i) = -1.$$

Rem. ∘ x^i: fields or ghosts; y^i: antifields.

∘ In SFT, the $\mathbb{Z}$-grading comes from the ghost number.

Then, $(\mathbb{R}^{2n}, \omega)$ forms a symplectic graded manifold, where

$$\omega(d/dx^i, d/dy^j) = -\omega(d/dy^j, d/dx^i) = \delta_{ij}.$$

The corresponding (odd) Poisson bracket

$$\{\ ,\ \} := \sum_{i=1}^{n} \left(\frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial y^i} - \frac{\overleftarrow{\partial}}{\partial y^i} \frac{\overrightarrow{\partial}}{\partial x^i} \right)$$

The Poisson bracket in $\mathbb{C}[[x,y]]$:

$$\{\ ,\ \} := \sum_{i=1}^{n} \left(\frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial y^i} - \frac{\overleftarrow{\partial}}{\partial y^i} \frac{\overrightarrow{\partial}}{\partial x^i} \right)$$

Let $\{\phi^1, \ldots, \phi^{2n}\} = \{x^1, \ldots, x^n, y^1, \ldots, y^n\}$. For a degree zero formal power series

$$S(\phi) := c_{ij}\phi^j\phi^i + c_{ijk}\phi^k\phi^j\phi^i + \cdots \in \mathbb{C}[[\phi]],$$

the **classical BV-master equation** is

$$\{S, S\} = 0.$$

By the graded Jacobi identity of $\{\ ,\ \}$, this is equivalent to that $\delta := \{*, S\}$ satisfies

$$\delta^2 = 0.$$

$(\mathbb{C}[[\phi]], \delta)$ **is a homotopy algebra** in general; imposing various properties on $\mathbb{C}[[\phi]]$ leads to various homotopy algebras.

‘**open case**’ Let $\mathbb{C}[[\phi]]$ be the space of cyclic formal power series, i.e.,

$$\cdots \phi^1 \phi^2 \cdots \neq \cdots \phi^2 \phi^1 \cdots ,$$
$$\phi^1 \phi^2 \phi^3 \phi^4 = \pm \phi^4 \phi^1 \phi^2 \phi^3 .$$

Then, $\delta := \{*, S\}$ is of the form:

$$\delta = \frac{\overleftarrow{\partial}}{\partial \phi^i} \sum_{k \geq 1} m^i_{i_1 \cdots i_k} \left(\phi^{i_k} \cdots \phi^{i_1} \right) .$$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{2n}\}$
$:= \quad \{d/dx^1, \ldots, d/dx^n, d/dy^1, \ldots, d/dy^n\}$, and

$$m_k(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) := \mathbf{e}_i m^i_{i_1 \cdots i_k} .$$

Then, $\delta^2 = 0$ turns out to be the A_∞-relation:

$$\sum_{\substack{k+l=n+1 \\ j=0,\cdots,k-1}} (-1)^{|\mathbf{e}_{i_1}|+\cdots+|\mathbf{e}_{i_j}|} m_k(\mathbf{e}_{i_1},\cdots,\mathbf{e}_{i_j},$$

$$m_l(\mathbf{e}_{i_{j+1}},\cdots,\mathbf{e}_{i_{j+l}}),\mathbf{e}_{i_{j+l+1}},\cdots,\mathbf{e}_{i_n}) = 0\ .$$

For the $\mathbb{Z}$-graded vector space $\mathcal{H}$ spanned by bases $\mathbf{e}_i$, $(\mathcal{H}, m_1, m_2, \dots)$ is an A_∞-algebra.

The A_∞-relation turns out to be

$[n=1]$ $\quad (m_1)^2 = 0, \cdots\cdots m_1 \leftrightarrow Q_B$

$[n=2]$ $\quad$ ' $[m_1, m_2] = m_1 m_2 + m_2 m_1 = 0$ ' $\cdots$

$\cdots$ Leibniz rule of m_1 w.r.t. the product m_2.

$[n=3]$ $\quad$ ' $[m_1, m_3] = m_2 m_2$ '

$m_2 m_2 = 0$ implies m_2 is associative.

$[m_1, m_3]$ is a coboundary term

(cf. m_3 is a chain homotopy)

... ...

$$[m_1, m_n] = -\sum_{k+l-1=n} m_k \circ m_l,$$

$$\leftrightarrow \partial(S_{n+1}) = -\sum_{k+l-1=n} \{S_{k+1}, S_{l+1}\}.$$

where $S = S_2 + S_3 + \cdots$,

$S_n := c_{i_1 \cdots i_n} \phi^{i_n} \cdots \phi^{i_1}$ and $\partial := \{S_2, *\}$.

Recall that the moduli space $\mathcal{M}_n$ of disk with n punctures on the boundary S^1
(configuration space of n-points on S^1
with three points being fixed at $0, 1, \infty$
to kill the automorphism $SL(2, \mathbb{R})$)
is isomorphic to $\mathbb{R}^{n-3}$.

Thus, in SFT, S_n is constructed from the integral of suitable correlation functions over a cell in $\mathbb{R}^{n-3}$.

S_2: kinetic term ($\leftrightarrow m_1$)

S_3: 3-vertex ($\leftrightarrow m_2$) ——— dim. $=$ 0,

S_4: 4-vertex ($\leftrightarrow m_3$) ——— dim. $=$ 1,

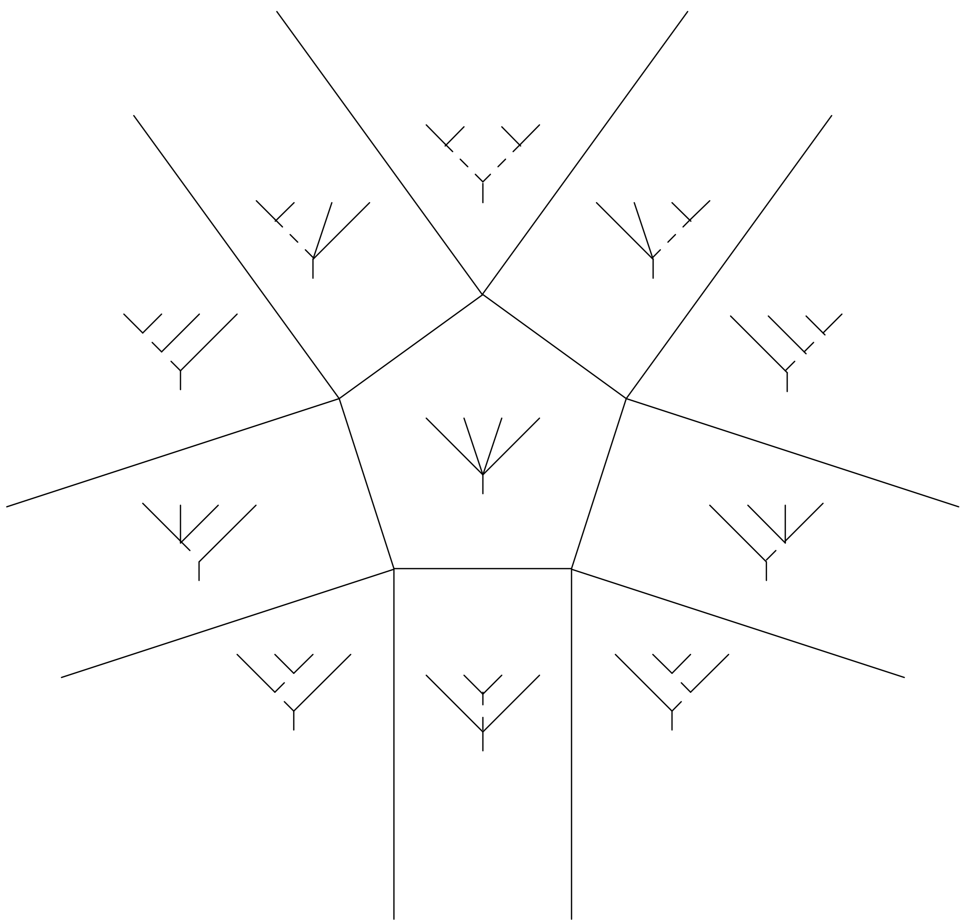
S_5: 5-vertex ($\leftrightarrow m_4$) ——— dim. $=$ 2,

…

(parameter of the length)

Propagator —————————— dim. $=$ 1.

Decomposition of moduli space $\mathcal{M}_5$:



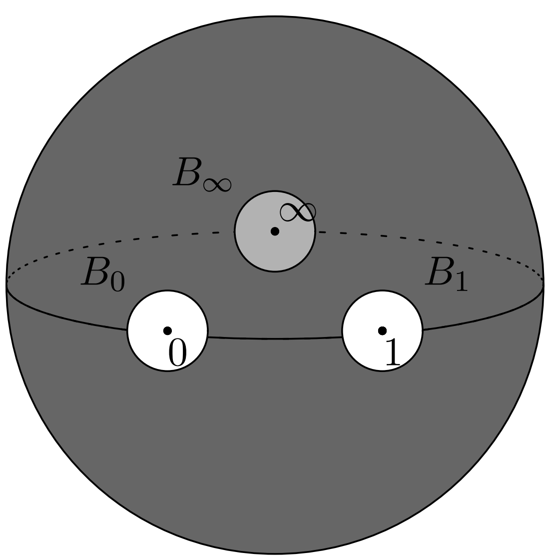
‘closed string case’

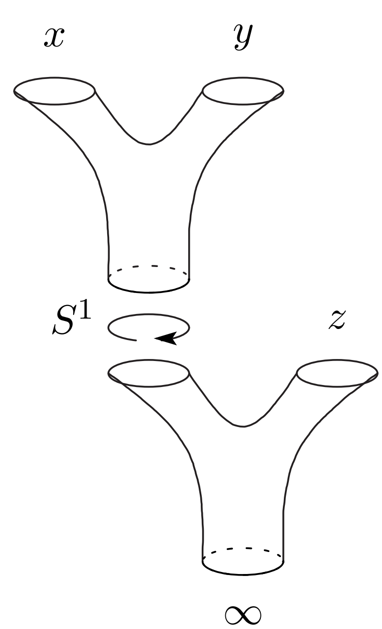
We may start from a differential graded commutative algebra $(\mathbb{C}[[\psi]], \delta)$.

The condition $\delta^2 = 0$ defines an L_∞-algebra.

(L indicates Lie $\leftrightarrow$ A indicates associative.)

Instead of the associativity up to homotopy, we obtain the Jacobi identity up to homotopy: the corresponding moduli space (a subspace of the moduli space of sphere with four punctures) are





open-closed case

Consider the graded algebra $\mathbb{C}[[\phi,\psi]] := \mathbb{C}[[\phi]]\otimes\mathbb{C}[[\psi]]$; where ψ's graded-commute with any others, and ϕ's are cyclic.

Let $S = S_S + S_D$ ($_S$=sphere, $_D$=disk),

where $S_S \in \mathbb{C}[[\psi]]$, $S_D \in \mathbb{C}[[\phi,\psi]]$.

"the action of tree open-closed SFT"

We set the OCHA structure by

$$\delta = \{*, S_S\}_c + \{*, S_D\}_o.$$

$\delta^2 = 0$ **defines an OCHA**.

By the Jacobi-Id, $\delta^2 = 0$ is equivalent to

$$0 = \{S_S, S_S\}_c, \qquad \{S_D, S_S\}_c + \{S_D, S_D\}_o = 0,$$

which are just the classical part of Zwiebach's BV-master equation for quantum open-closed SFT.

Homological Perturbation Theory (HPT) in homotopy algebras

⇔ Perturbation theory of SFTs

A version of (see H.K'03 and refs therein)

★ **HPT** for an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$: ————

Given a Hodge-Kodaira decomposition

$$dh + hd = \mathrm{Id}_{\mathcal{H}} - P, \qquad d := m_1,$$

$\exists$ an A_∞-structure $\mathfrak{m}'$ on the cohomology $H(\mathcal{H}) \simeq P\mathcal{H}$ and a homotopy equivalence $(H(\mathcal{H}), \mathfrak{m}') \to (\mathcal{H}, \mathfrak{m})$.

Note $h : \mathcal{H}^r \to \mathcal{H}^{r-1}$ is a propagator;
an explicit construction of $\mathfrak{m}'$ is given by the Feynman rule.

(1a) and (1b) follow from this fact $(+\alpha)$.

Classical open-closed correspondence

Given an open-closed SFT $S = S_S + S_D$.

∘ For the tree open string part,

$(\mathbb{C}[[\phi]], \delta_o, \{\ ,\ \}_o)$ forms a DGLA

(= differential graded Lie algebra, a special L_∞-algebra), where $\delta_o := \{*, S_D\}|_{\psi=0}$.

∘ For the tree closed string part,

$(\mathbb{C}[[\psi]], \delta_c)$ defines an L_∞-algebra.

What is the relation between them ?

There exists an L_∞-morphism from the L_∞-algebra of closed strings to the DGLA of open strings.

$\star$ Physically, the existence of an L_∞-morphism implies that, for a classical solution for the tree closed SFT S_S (= closed string condensation), the open part $(\mathbb{C}[[\phi]], \delta_o, \{\ ,\ \}_o)$ is deformed as a DGLA.

The deformation of $(\mathbb{C}[[\phi]], \delta_o)$ forms a weak A_∞-algebra. (The action is of the form $S(\phi) = S_0 + S_1(\phi) + S_2(\phi) + \cdots$.)

(cf. Zwiebach'97, H.K-Stasheff'04,'05)

$\star$ Mathematically, the DGLA $(\mathbb{C}[[\phi]], \delta_o, \{\ ,\ \}_o)$ controlls full deformation of the A_∞-structure δ_o as weak A_∞-algebras, where $(\mathbb{C}[[\phi]], \delta_o)$ is called the cyclic Hochschild complex. The bracket is related to the Gerstenhaber bracket on the Hochschild complex.

Examples and Applications of the classical OC correspondence

- Condensation of B-field

(Kawano-Takahashi'00, etc.)

For topological string case,

- Poisson sigma model and deformation quantization (Kontsevich'97, Cattaneo-Felder'99)

(see H.K-Stasheff'05)

$$(T_{poly}(M), [\ ,\ ]_{\text{Schouten}}) \xrightarrow{L_\infty} (D_{poly}(M), \delta, [\ ,\ ]_{\text{Gerstenhaber}}),$$

where $T_{poly}(M) := \oplus_{k\geq 0} \wedge^k TM$,
and $D_{poly}(M)$ is 'the differential Hochschild complex' = the space of multilinear maps $\{(C^\infty(M))^{\otimes k} \to C^\infty(M)\}_{k\geq 0}$ consisting of multi-differential operators.