Some Properties of String Field Algebra

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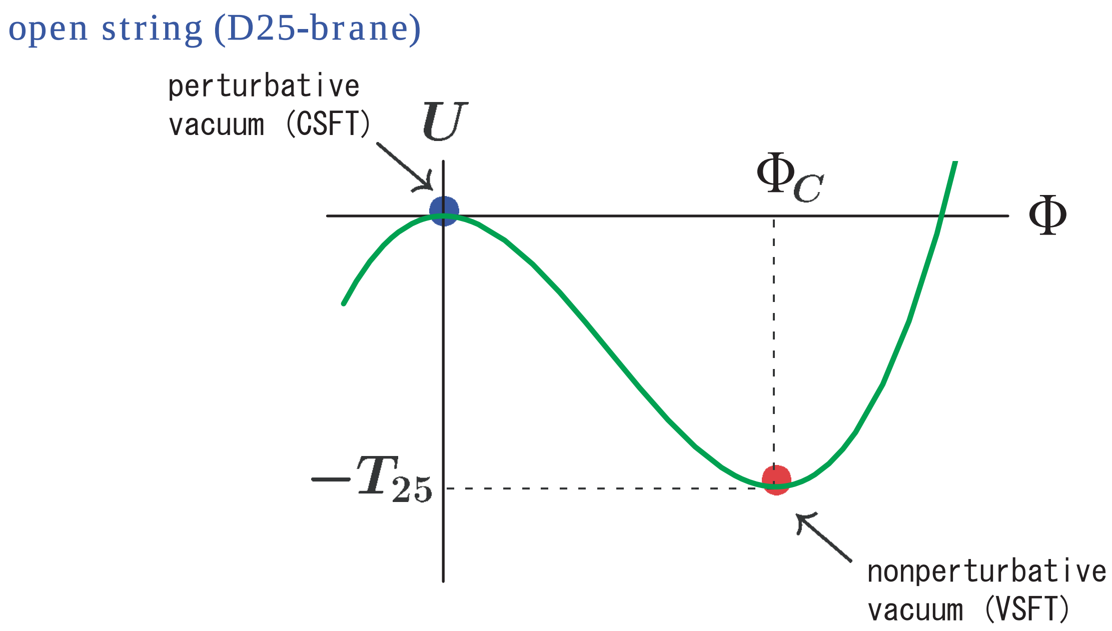
1. Introduction and Motivation

Sen's conjecture (for bosonic open string field theory)



CSFT(cubic string field theory) Witten

$$S_{\mathrm{CSFT}} = -\frac{1}{g_o^2}\left(\frac{1}{2}\langle\Phi, Q_B\Phi\rangle + \frac{1}{3}\langle\Phi, \Phi * \Phi\rangle\right)$$

Sen's conjecture says there is a solution of CSFT Φ_c : $Q_B\Phi_c + \Phi_c * \Phi_c = 0$ and $-S_{\mathrm{CSFT}}|_{\Phi_c}/V_{26} = T_{25}$.

VSFT(vacuum string field theory) Rastelli-Sen-Zwiebach

$$S_{\rm VSFT} = -\kappa_0 \left(\frac{1}{2}\langle \Phi, \mathcal{Q}\Phi \rangle + \frac{1}{3}\langle \Phi, \Phi * \Phi \rangle \right)$$

This describes the physics around nonperturbative vacuum (no D25-brane). $\mathcal{Q}$ should satisfy the following conditions to define a gauge theory

$$\mathcal{Q}^2 = 0, \mathcal{Q}(A * B) = \mathcal{Q}A * B + (-1)^{|A|} A * \mathcal{Q}B, \langle \mathcal{Q}A, B \rangle = -(-1)^{|A|} \langle A, \mathcal{Q}B \rangle$$

and have vanishing cohomology and universality (no matter information). This requirement is satisfied by

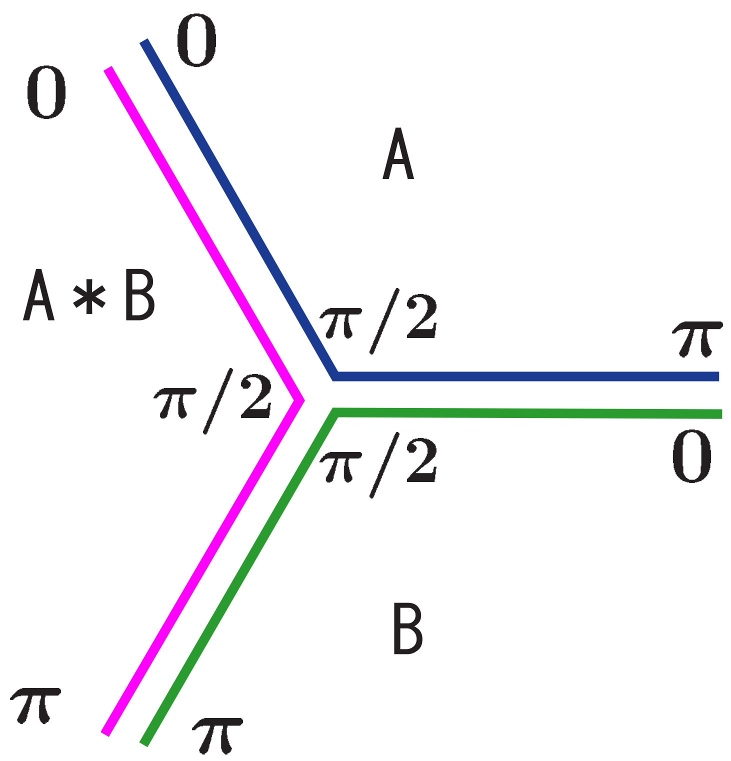
$$\mathcal{Q} = \Sigma_n f_n (c_n + (-1)^n c_{-n}),$$

where f_n is some coefficient. Later canonical choice is given by GRSZ:

$$\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i)) = c_0 - (c_2 + c_{-2}) + (c_4 + c_{-4}) + \cdots$$

To realize this scenario, it is necessary to have an analytic solution of CSFT or VSFT which relates them. We investigate Witten's $*$ product for this purpose.

Witten's $*$ product represents string interaction. This is represented by operator formalism using oscillators or CFT technique.



In the context of VSFT, some techniques using oscillator representation have been developed in *matter part* **especially to construct projection which satisfies reduced equation of motion of VSFT ($\Phi_M \star_M \Phi_M = \Phi_M$).**
We extend them to ghost part and solve *full* **equation of motion of VSFT($\mathcal{Q}\Phi + \Phi \star \Phi = 0$).**

In the context of purely CSFT, Horowitz et.al. discussed (formal) solutions.
We reexamine them to construct a solution of CSFT which derives conjectured VSFT action.

2. Algebraic Approach

For two string fields A, B, which are represented by some oscillators on a particular Fock vacuum, we define the Witten's $\star$ product as

$$|A \star B\rangle_1 := {}_2\langle A|_3\langle B|1,2,3\rangle = \langle 2,4|A\rangle_4\langle 3,5|B\rangle_5|1,2,3\rangle,$$

where 3-string vertex $|1,2,3\rangle$ and reflector $\langle 1,2|$ are represented by

$$\begin{aligned}
|V_3\rangle &= |1,2,3\rangle = \tilde{\mu}_3\int d^dp^{(1)}d^dp^{(2)}d^dp^{(3)}(2\pi)^d\delta^d(p^{(1)}+p^{(2)}+p^{(3)})e^{E_3}|0,p\rangle,\\
E_3 &= -\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n.m\geq 1}a_n^{(r)\dagger}V_{nm}^{rs}a_m^{(s)\dagger}-\sum_{r,s=1}^{3}\sum_{n\geq 1}p^{(r)}V_{0n}^{rs}a_n^{(r)\dagger}-\frac{1}{2}\sum_{r,s=1}^{3}p^{(r)}V_{00}^{rs}p^{(s)}-\sum_{r,s=1}^{3}\sum_{n\geq 1,m\geq 0}c_{-n}^{(r)}X_{nm}^{rs}b_{-m}^{(s)},\\
|0,p\rangle &= |0,p^{(1)}\rangle|0,p^{(2)}\rangle|0,p^{(3)}\rangle,\quad b_n^{(i)}|0,p^{(i)}\rangle=0,\quad n\geq 1,\quad c_m^{(i)}|0,p^{(i)}\rangle=0,\quad m\geq 0,\\
\langle V_2| &= \langle 1,2| = \int d^dp^{(1)}d^dp^{(2)}\langle 0,p|e^{E_2}\delta^d(p^{(1)}+p^{(2)})\delta(c_0^{(1)}+c_0^{(2)})\\
E_2 &= -\sum_{n,m\geq 1}a_n^{(1)}C_{nm}a_m^{(2)}-\sum_{n,m\geq 1}(c_n^{(1)}C_{nm}b_m^{(2)}+c_n^{(2)}C_{nm}b_m^{(1)}),\quad \langle 0,p| = {}_1\langle 0,p^{(1)}|_2\langle 0,p^{(2)}|,\quad C_{nm}:=(-1)^n\delta_{n,m}.
\end{aligned}$$

We can prove the useful relations among Neumann coefficients V_{nm}^{rs}, X_{nm}^{rs}:

$$\begin{aligned}
&M_0 := CV^{rr},\quad M_\pm := CV^{rr\pm1},\quad \tilde{M}_0 := -CX^{rr},\quad \tilde{M}_\pm := -CX^{rr\pm1}\quad \text{where these indices run from 1 to } \infty,\\
&CM_0 = M_0C,\ CM_+ = M_-C,\quad C\tilde{M}_0 = \tilde{M}_0C,\ C\tilde{M}_+ = \tilde{M}_-C,\\
&[M_0,M_\pm] = [M_+,M_-] = 0,\quad [\tilde{M}_0,\tilde{M}_\pm] = [\tilde{M}_+,\tilde{M}_-] = 0,\\
&M_0+M_++M_- = 1,\ \tilde{M}_0+\tilde{M}_++\tilde{M}_- = 1,\\
&M_+M_- = M_0^2-M_0,\quad \tilde{M}_+\tilde{M}_- = \tilde{M}_0^2-\tilde{M}_0,\quad M_0^2+M_+^2+M_-^2 = 1,\quad \tilde{M}_0^2+\tilde{M}_+^2+\tilde{M}_-^2 = 1,\cdots\\
&V^{21}{}_0 = \frac{3M_+-2}{1+3M_0}V^{11}{}_0,\quad V^{31}{}_0 = \frac{3M_--2}{1+3M_0}V^{11}{}_0,\quad X^{21}{}_0 = -\frac{\tilde{M}_+}{1-\tilde{M}_0}X^{11}{}_0,\quad X^{31}{}_0 = -\frac{\tilde{M}_-}{1-\tilde{M}_0}X^{11}{}_0,\cdots
\end{aligned}$$

Note Neumann coefficient matrices of ghost nonzero mode part satisfy the same relation as matter part.

We define *reduced* product (denoted as $\star^r$):

$$|A\star^r B\rangle := {}_2\langle A^r|_3\langle B^r|V_3^r\rangle_{123}, \quad \langle A^r| := \langle V_2^r|A\rangle,$$

where we restrict string fields $|A\rangle, |B\rangle$ such that they have no b_0, c_0 modes on the Fock vacuum $|+\rangle$. $(c_0|+\rangle = 0, b_0|+\rangle \neq 0)$ **Here we introduced reduced reflector $\langle V_2^r|$ and reduced 3-string vertex $|V_3^r\rangle$ which contain no b_0, c_0 modes on the vacuum ${}_G\langle\tilde{+}|,\ |+\rangle_G$, i.e. they are related with usual reflector and 3-string vertex by**

$${}_{12}\langle V_2| = {}_{12}\langle V_2^r|(c_0^{(1)}+c_0^{(2)}), \quad |V_3\rangle_{123} = \exp\left(-\sum_{r,s=1}^{3} c^{\dagger(r)} X^{rs}_{\ \ 0} b_0^{(s)}\right)|V_3^r\rangle_{123}.$$

Under the $\star^r$ product in ghost part, one can obtain similar formulas to those of matter part.

Using $\star^r$ product, we have $\star$ product formula between string fields in the Siegel gauge as

$$\begin{aligned}|\Phi\star\Psi\rangle &= |\phi\star^r\psi\rangle + b_0\left({}_2\langle\phi^r|_3\langle\psi^r|\sum_{s=1}^{3} c^{(s)\dagger}X^{s1}_{\ \ 0}|V_3^r\rangle_{123}\right)\\ &= (1+b_0c^\dagger X^{11}_{\ \ 0})|\phi\star^r\psi\rangle + b_0\sum_{s=2,3}{}_2\langle\phi^r|_3\langle\psi^r|c^{(s)\dagger}X^{s1}_{\ \ 0}|V_3^r\rangle_{123},\\ &|\Phi\rangle = b_0|\phi\rangle, \quad |\Psi\rangle = b_0|\psi\rangle.\end{aligned}$$

We have obtained $\star$ product formula between squeezed states in ghost part in the Siegel gauge:

$$
\begin{aligned}
&|(b_0 n_{\xi,\eta}) \star (b_0 m_{\xi',\eta'})\rangle \\
=&\ \left(1 + b_0\left(c^\dagger X^{11}_{\ 0} + \left(\xi C + \frac{\partial}{\partial\eta}\tilde{T}_n\right) X^{21}_{\ 0} + \left(\xi' C + \frac{\partial}{\partial\eta'}\tilde{T}_m\right) X^{31}_{\ 0}\right)\right) |n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle \\
=&\ \left(1 + b_0 c^\dagger \frac{1 - \tilde{T}_n\tilde{T}_m}{\tilde{T}_{n,m}} X^{11}_{\ 0} - b_0(\xi\tilde{\rho}_{1(n,m)} + \xi'\tilde{\rho}_{2(n,m)})\frac{1}{1-\tilde{M}_0} X^{11}_{\ 0}\right) |n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
&|n_{\xi,\eta}\rangle := e^{\xi b^\dagger + \eta c^\dagger}|n\rangle_G = \tilde{\mu}_n \exp\left(\xi b^\dagger + \eta c^\dagger + c^\dagger C \tilde{T}_n b^\dagger\right)|+\rangle_G, \\
&|n\rangle_G = (|2\rangle_G)^{n-1}_{\star^r}, \quad |2\rangle_G = \exp\left(c^\dagger C \tilde{T}_2 b^\dagger\right)|+\rangle_G. \\
&|n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle = \exp\left(-\mathcal{C}_{n_{\xi,\eta}, m_{\xi',\eta'}}\right)|(n+m-1)_{\xi\tilde{\rho}_{1(n,m)} + \xi'\tilde{\rho}_{2(n,m)},\, \eta\tilde{\rho}^T_{1(n,m)} + \eta'\tilde{\rho}^T_{2(n,m)}}\rangle,
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{T}_n = \frac{\tilde{T}(1-\tilde{T}_2\tilde{T})^{n-1} + (\tilde{T}_2 - \tilde{T})^{n-1}}{(1-\tilde{T}_2\tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2 - \tilde{T})^{n-1}}, \quad \tilde{\mu}_n = \tilde{\mu}_2\left(\tilde{\mu}_2\tilde{\mu}_3^r \det\left(\frac{1-\tilde{T}}{1-\tilde{T}+\tilde{T}^2}\right)\right)^{n-2} \det\left(\frac{(1-\tilde{T}_2\tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n-1}}{1-\tilde{T}^2}\right), \\
&\tilde{M}_0 = \frac{\tilde{T}}{1-\tilde{T}+\tilde{T}^2}, \quad \mathcal{C}_{n_{\xi,\eta}, m_{\xi',\eta'}} = (\xi,\xi')\frac{C}{\tilde{T}_{n,m}}\begin{pmatrix} \tilde{M}_0(1-\tilde{T}_m) & \tilde{M}_- \\ \tilde{M}_+ & \tilde{M}_0(1-\tilde{T}_n)\end{pmatrix}\begin{pmatrix}\eta^T \\ \eta'^T\end{pmatrix} = \mathcal{C}_{m_{\xi' C,\eta' C}, n_{\xi C,\eta C}}, \\
&\tilde{\rho}_{1(n,m)} = \frac{\tilde{M}_- + \tilde{M}_+\tilde{T}_m}{\tilde{T}_{n,m}}, \quad \tilde{\rho}_{2(n,m)} = \frac{\tilde{M}_+ + \tilde{M}_-\tilde{T}_n}{\tilde{T}_{n,m}}, \quad C\tilde{\rho}_{1(n,m)} = \tilde{\rho}_{2(m,n)}C, \\
&\tilde{T}_{n,m} = \frac{(1+\tilde{T})(1-\tilde{T})^2}{1-\tilde{T}+\tilde{T}^2}\frac{(1-\tilde{T}_2\tilde{T})^{n+m-2} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n+m-2}}{((1-\tilde{T}_2\tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n-1})((1-\tilde{T}_2\tilde{T})^{m-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{m-1})} = 1 + \tilde{M}_0(\tilde{T}_n\tilde{T}_m - \tilde{T}_n - \tilde{T}_m) = \tilde{T}_{m,n}.
\end{aligned}
$$

Later, Okuyama further investigated and rearranged these algebra in the Siegel gauge elegantly. Especially, our $\star^r$ corresponds to his $\star_{b_0}$: $|\Phi \star_{b_0} \Psi\rangle = b_0|\phi \star^r \psi\rangle$.

Application:

Equation of motion of VSFT:

$$\mathcal{Q}|\Psi\rangle + |\Psi \star \Psi\rangle = 0, \quad \mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n(c_n + (-1)^n c_n^\dagger) = c_0 + f \cdot (c + Cc^\dagger).$$

To solve it we put the ansatz

$$|\Psi\rangle = b_0|P\rangle_M \left(\sum_{n=1}^{\infty} g_n|n\rangle_G\right), \quad |P \star_M P\rangle_M = |P\rangle_M.$$

Then matter part is factorized and we have obtained some solutions by using previous formula in ghost part:

1. identity-like solution

$$\mathcal{Q} = c_0, \quad |\Psi\rangle = -b_0|P\rangle_M|I^r\rangle_G.$$

2. sliver-like solution

$$\mathcal{Q} = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0} X^{11}{}_0, \quad |\Psi\rangle = -b_0|P\rangle_M |\Xi^r\rangle_G.$$

This was constructed in Hata-Kawano (HK). (This formula is simpler than HK's.)

3. another solution

$$\mathcal{Q} = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0} X^{11}{}_0, \quad |\Psi\rangle = -b_0|P\rangle_M (|I^r\rangle_G - |\Xi^r\rangle_G).$$

where

$$|n = 1\rangle_G =: |I^r\rangle_G, \quad |n = \infty\rangle_G =: |\Xi^r\rangle_G,$$

which are analogies of identity and sliver states with respect to $\star^r$.

Later, Gaiotto, Rastelli, Sen and Zwiebach (GRSZ) proposed their canonical choice of kinetic term $\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i))$ for VSFT, and observed that this coincides with that of HK solution numerically, and Okuyama proved $\frac{1}{2i}(c(i) - c(-i)) = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0} X^{11}{}_0$ analytically.
GRSZ also observed $|\Xi^r\rangle_G$ would coincide with their sliver state with respect to $*'$ product on twisted bc-ghost system.

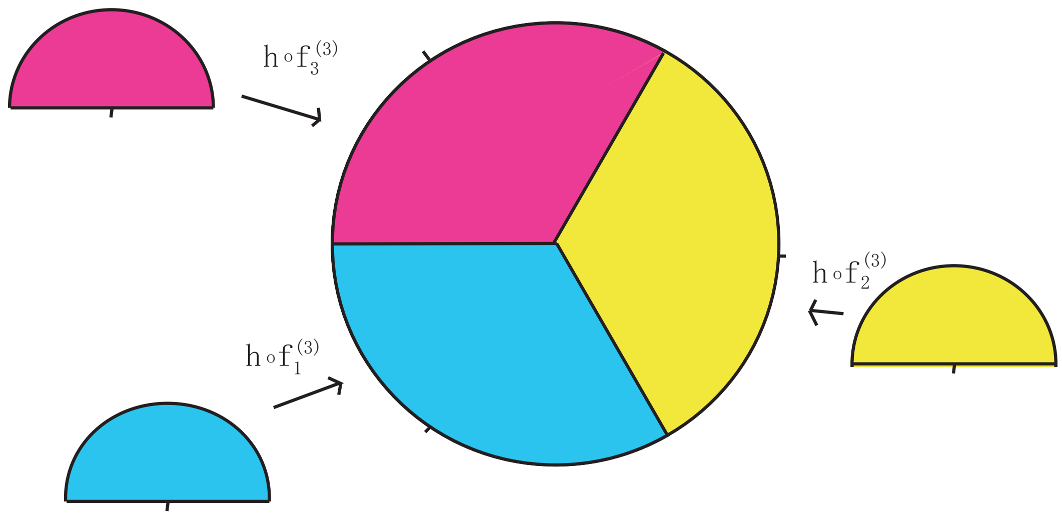
3. CFT Approach

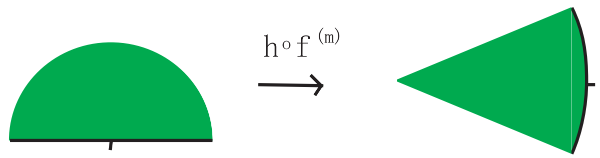
Witten's $*$ product in CFT language which was developed by LeClair-Peskin-Preitschopf (LPP):

$$\langle A, B * C\rangle = \left\langle f_1^{(3)} \circ A(0)\ f_2^{(3)} \circ B(0)\ f_3^{(3)} \circ C(0)\right\rangle_{\mathrm{UHP}},$$

where conformal maps are given by

$$f_1^{(3)}(z) = h^{-1}\left(e^{-\frac{2}{3}\pi i} h(z)^{\frac{2}{3}}\right), \quad f_2^{(3)}(z) = h^{-1}\left(h(z)^{\frac{2}{3}}\right), \quad f_3^{(3)}(z) = h^{-1}\left(e^{\frac{2}{3}\pi i} h(z)^{\frac{2}{3}}\right), \quad h(z) = \frac{1+iz}{1-iz}.$$





For wedge state $|m\rangle$ which is defined by

$$\langle m, \varphi \rangle = \left\langle f^{(m)} \circ \varphi(0) \right\rangle_{\mathrm{UHP}}, \quad f^{(m)}(z) = h^{-1}\left(h(z)^{\frac{2}{m}} \right),$$

we have the $*$ product between them [David]

$$\langle \varphi, m * n \rangle = \langle \varphi, m + n - 1 \rangle, \quad \forall \varphi.$$

For the proof of this algebra, we followed only the definition of wedge state and generalized gluing and resmoothing theorem (GGRT)[Schwarz-Sen]:

$$\sum_r \langle f_1 \circ \Phi_{r_1}(0) \ldots f_n \circ \Phi_{r_n}(0) \ f \circ \Phi_r(0) \rangle_{\mathcal{D}_1} \langle g_1 \circ \Phi_{s_1}(0) \ldots g_m \circ \Phi_{s_m}(0) \ g \circ \Phi_r^c(0) \rangle_{\mathcal{D}_2}$$

$$= \left\langle F_1 \circ f_1 \circ \Phi_{r_1}(0) \ldots F_1 \circ f_n \circ \Phi_{r_n}(0) \ \widehat{F}_2 \circ g_1 \circ \Phi_{s_1}(0) \ldots \widehat{F}_2 \circ g_m \circ \Phi_{s_m}(0) \right\rangle_{\mathcal{D}}, \quad F_1 \circ f = \hat{F}_2 \circ g \circ I,$$

and constructed resmoothing maps $F_1, \hat{F}_2$ concretely.

Using this technique, we proved some algebras about the identity state $|\mathcal{I}\rangle := |m = 1\rangle$:

$$\langle \varphi, \mathcal{I} * \psi \rangle = \langle \varphi, \psi * \mathcal{I} \rangle = \langle \varphi, \psi \rangle, \quad \langle \varphi, \mathcal{I} * \mathcal{O}\mathcal{I} \rangle = \langle \varphi, \mathcal{O}\mathcal{I} * \mathcal{I} \rangle = \langle \varphi, \mathcal{O}\mathcal{I} \rangle$$

In this sense, we found $\mathcal{I}$ behaves like the identity with respect to the $*$ product in this framework.

In the same way, we have checked 'partial integration formula'

$$\langle\varphi, (Q_R A) * B\rangle = -(-1)^{|A|}\langle\varphi, A * (Q_L B)\rangle,$$

even on the wedge state: $|A\rangle = \mathcal{O}_A|m\rangle$ or $|B\rangle = \mathcal{O}_B|m\rangle$.

Using these results we have verified that

$$|\Phi_0\rangle := -Q_L|\mathcal{I}\rangle + \frac{a}{2}\mathcal{Q}^{\epsilon}|\mathcal{I}\rangle,$$

$$Q_L := \int_{C_L}\frac{dz}{2\pi i}j_B(z), \quad \mathcal{Q}^{\epsilon} := \frac{1}{2i}\left(e^{-i\epsilon}c(ie^{i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon})\right)$$

satisfies equation of motion of CSFT :

$$\langle\varphi, Q_B\Phi_0 + \Phi_0 * \Phi_0\rangle = 0, \quad \forall\varphi.$$

By expanding CSFT action around our solution Φ_0, we have derived GRSZ's VSFT action which is regularized by ϵ in the kinetic term:

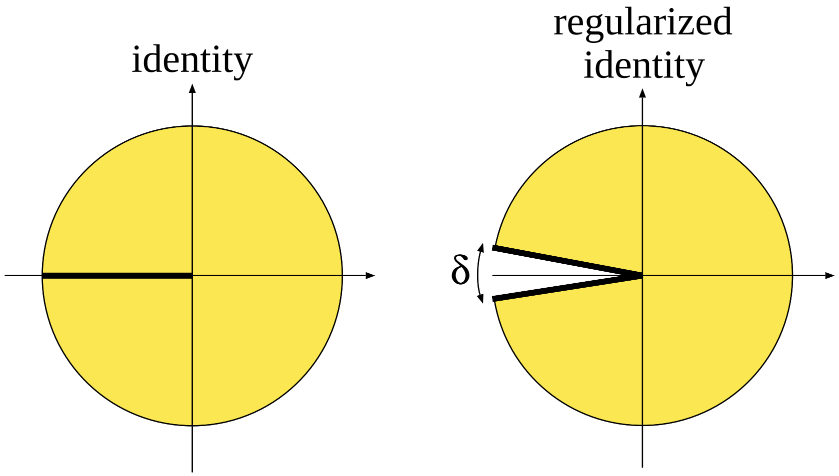
$$\mathcal{Q}_{\epsilon} = \frac{1}{4i}\left(e^{-i\epsilon}c(ie^{i\epsilon}) + e^{i\epsilon}c(ie^{-i\epsilon}) - e^{-i\epsilon}c(-ie^{-i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon})\right).$$

Naively one might think the value of the CSFT action at Φ_0 would be zero, but it may be possible to give a nonzero value for D25-brane tension.

In fact we have

$$\langle \mathcal{Q}^\epsilon \widetilde{\mathcal{I}}_\delta, Q_B \mathcal{Q}^\epsilon \widetilde{\mathcal{I}}_\delta \rangle = -\delta^2 \sin^2 \epsilon \left[\frac{1}{2} \left\{ \left(\tan \frac{\epsilon}{2} \right)^{\frac{2}{\delta}} + \left(\tan \frac{\epsilon}{2} \right)^{-\frac{2}{\delta}} \right\} + 3 \right] V_{26},$$

where $\widetilde{\mathcal{I}}_\delta$ is regularized identity state which is necessary to apply GGRT. (At $\delta = 0$ this quantity would vanish if one uses equation of motion naively.)



Solution of the form

$$\Psi = -Q_L \mathcal{I} + C_L(f) \mathcal{I},$$
$$C_L(f) = \int_{C_L} d\sigma f(\sigma)(c(\sigma) + c(-\sigma)),$$
$$f(\pi - \sigma) = f(\sigma), \quad f\left(\frac{\pi}{2}\right) = 0$$

was considered earlier by Horowitz et.al. in the context of purely cubic SFT, but they treated identity state rather formally.

Recently Takahashi-Tanimoto constructed a solution of CSFT of the form $-Q_L(f)\mathcal{I} + C_L(g)\mathcal{I}, \quad f \neq 1$.

4. Summary and Discussion

We examined Witten's $*$ product both in oscillator and in CFT language.

We constructed solutions of VSFT in oscillator representation and a solution of CSFT in CFT language. The latter one derives GRSZ's VSFT action from Witten's CSFT, but to confirm Sen's conjecture we should obtain D25-brane tension from potential height.

The identity state $\mathcal{I}$ is rather complicated in ghost part in oscillator representation, and naive computation (using relations among Neumann coefficient matrices formally) gives some unexpected results: for example $\mathcal{I} \star \mathcal{I} = 0$. This subtlety may come from treating $\infty \times \infty$ matirices as usual number and we should treat them more carefully using Neumann coefficient matrices spectroscopy [RSZ].
On the other hand, we proved some relations expected of the identity state using GGRT in CFT language. But the evaluation of the action including $\mathcal{I}$ is still rather subtle.