

# Some Properties of String Field Algebra

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based on

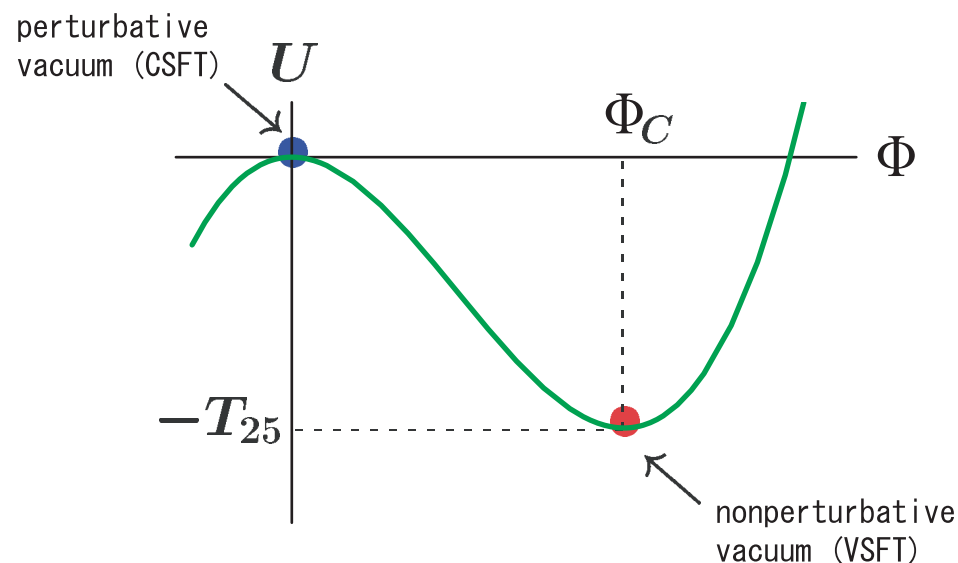
I. K., JHEP12(2001)007[hep-th/0110124]

I.K., K.Ohmori, hep-th/0112169

# 1. Introduction

## Sen's conjecture (for bosonic open string field theory)

open string (D25-brane)



### CSFT (cubic string field theory) Witten

$$S_{\text{CSFT}} = -\frac{1}{g_o^2} \left( \frac{1}{2} \langle \Phi, Q_B \Phi \rangle + \frac{1}{3} \langle \Phi, \Phi * \Phi \rangle \right)$$

Sen's conjecture says there is a solution of CSFT  $\Phi_c$  :

$$Q_B \Phi_c + \Phi_c * \Phi_c = 0 \text{ and } -S_{\text{CSFT}}|_{\Phi_c} / V_{26} = T_{25}.$$

## VSFT (vacuum string field theory) Rastelli-Sen-Zwiebach (RSZ)

$$S_{\text{VSFT}} = -\kappa_0 \left( \frac{1}{2} \langle \Phi, \mathcal{Q}\Phi \rangle + \frac{1}{3} \langle \Phi, \Phi * \Phi \rangle \right)$$

This describes the physics around nonperturbative vacuum (no D25-brane).

$\mathcal{Q}$  should satisfy the following conditions to define a gauge theory

$$\mathcal{Q}^2 = 0, \mathcal{Q}(A * B) = \mathcal{Q}A * B + (-1)^{|A|} A * \mathcal{Q}B, \langle \mathcal{Q}A, B \rangle = -(-1)^{|A|} \langle A, \mathcal{Q}B \rangle$$

and have vanishing cohomology and universality (no matter information).

These requirements are satisfied by

$$\mathcal{Q} = \sum_n f_n (c_n + (-1)^n c_{-n}),$$

where  $f_n$  is some coefficient. Later, its canonical choice was given by Gaiotto-Rastelli-Sen-Zwiebach (GRSZ):

$$\mathcal{Q} = \frac{1}{2i} (c(i) - c(-i)) = c_0 - (c_2 + c_{-2}) + (c_4 + c_{-4}) + \dots$$

To realize this scenario, it is necessary to have an analytic solution of CSFT or VSFT which relates them. We investigate Witten's  $*$  product for this purpose.

The Witten's  $*$  product represents string interaction. This is represented by operator formalism using oscillators or CFT technique.

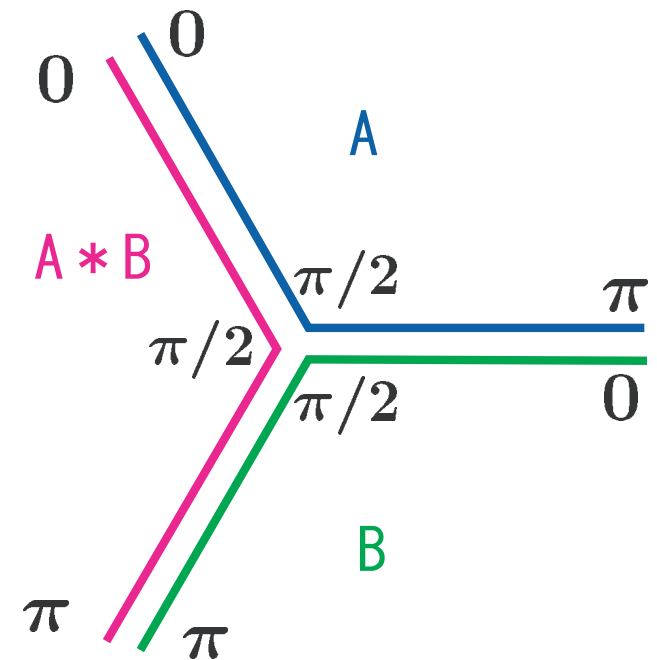
In the context of VSFT, some techniques using **oscillator representation** have been developed in *matter part* especially to construct projectors which satisfy *reduced* equation of motion of VSFT :

$$\Phi_M \star \Phi_M = \Phi_M.$$

We extend them to *ghost part* and solve *full* equation of motion of VSFT :  $Q\Phi + \Phi \star \Phi = 0$ .

Because  $Q$  is linear in  $c$ -ghost, one can take the ansatz  $|\Phi_c\rangle = |\Phi_M\rangle|\Phi_G\rangle$  and the e.o.m is reduced to  $\Phi_M \star \Phi_M = \Phi_M$  in matter part by assuming the existence of a solution  $\Phi_G$  in ghost part which satisfies  $Q\Phi_G + \Phi_G \star \Phi_G = 0$ . Many authors discussed D-brane solutions of VSFT with this strategy before.

In the context of purely CSFT, Horowitz et.al. discussed (formal) solutions. Using **CFT technique**, we reexamine them to construct a solution of CSFT which derives GRSZ's proposed VSFT action.



§1. Introduction

§2. Oscillator Approach

Neumann coefficient matrices,  
reduced star product,  
some formulas for wedge-like states,  
application to VSFT,  
subtlety of the identity state

§3. CFT Approach

Generalized Gluing and Resmoothing Theorem (GGRT),  
some formulas for wedge states,  
a derivation of VSFT from CSFT

§4. Summary and Discussion

## 2. Oscillator Approach

For two string fields  $A, B$ , which are represented by some oscillators on a particular Fock vacuum, we define the Witten's  $\star$  product as

$$|A \star B\rangle_1 := {}_2\langle A|_3\langle B|1, 2, 3\rangle = \langle 2, 4|A\rangle_4\langle 3, 5|B\rangle_5|1, 2, 3\rangle,$$

where 3-string vertex  $|1, 2, 3\rangle$  and reflector  $\langle 1, 2|$  are represented by

$$|V_3\rangle = |1, 2, 3\rangle = \tilde{\mu}_3 \int d^d p^{(1)} d^d p^{(2)} d^d p^{(3)} (2\pi)^d \delta^d(p^{(1)} + p^{(2)} + p^{(3)}) e^{E_3} |0, p\rangle,$$

$$E_3 = -\frac{1}{2} \sum_{r,s=1}^3 \sum_{n,m \geq 1} a_n^{(r)\dagger} V_{nm}^{rs} a_m^{(s)\dagger} - \sum_{r,s=1}^3 \sum_{n \geq 1} p^{(r)} V_{0n}^{rs} a_n^{(r)\dagger} - \frac{1}{2} \sum_{r,s=1}^3 p^{(r)} V_{00}^{rs} p^{(s)} - \sum_{r,s=1}^3 \sum_{n \geq 1, m \geq 0} c_{-n}^{(r)} X_{nm}^{rs} b_{-m}^{(s)},$$

$$|0, p\rangle = |0, p^{(1)}\rangle |0, p^{(2)}\rangle |0, p^{(3)}\rangle, \quad b_n^{(i)} |0, p^{(i)}\rangle = 0, \quad n \geq 1, \quad c_m^{(i)} |0, p^{(i)}\rangle = 0, \quad m \geq 0,$$

$$\langle V_2| = \langle 1, 2| = \int d^d p^{(1)} d^d p^{(2)} \langle 0, p| e^{E_2} \delta^d(p^{(1)} + p^{(2)}) \delta(c_0^{(1)} + c_0^{(2)})$$

$$E_2 = - \sum_{n,m \geq 1} a_n^{(1)} C_{nm} a_m^{(2)} - \sum_{n,m \geq 1} (c_n^{(1)} C_{nm} b_m^{(2)} + c_n^{(2)} C_{nm} b_m^{(1)}), \quad \langle 0, p| = {}_1\langle 0, p^{(1)}|_2\langle 0, p^{(2)}|, \quad C_{nm} := (-1)^n \delta_{n,m}.$$

This 3-string vertex is a solution of the connection condition:

$$\begin{aligned} (X^{(r)}(\sigma) - X^{(r-1)}(\pi - \sigma)) |V_3\rangle &= 0, \quad (P^{(r)}(\sigma) + P^{(r-1)}(\pi - \sigma)) |V_3\rangle = 0, \quad 0 \leq \sigma \leq \frac{\pi}{2}, \\ (c^{\pm(r)}(\sigma) + c^{\pm(r-1)}(\pi - \sigma)) |V_3\rangle &= 0, \quad (b^{\pm(r)}(\sigma) - b^{\pm(r-1)}(\pi - \sigma)) |V_3\rangle = 0, \quad r = 1, 2, 3. \end{aligned}$$

We can prove useful relations among Neumann coefficients  $V_{nm}^{rs}, X_{nm}^{rs}$  :

For the matrices [Gross-Jevicki,Kosteleckey-Potting,RSZ]

$$M_0 := CV^{rr}, \quad M_{\pm} := CV^{rr\pm 1}, \quad \tilde{M}_0 := -CX^{rr}, \quad \tilde{M}_{\pm} := -CX^{rr\pm 1}$$

whose indices run from 1 to  $\infty$ , there are some relations

$$\begin{aligned} CM_0 &= M_0C, \quad CM_+ = M_-C, \quad C\tilde{M}_0 = \tilde{M}_0C, \quad C\tilde{M}_+ = \tilde{M}_-C, \\ [M_0, M_{\pm}] &= [M_+, M_-] = 0, \quad [\tilde{M}_0, \tilde{M}_{\pm}] = [\tilde{M}_+, \tilde{M}_-] = 0, \\ M_0 + M_+ + M_- &= 1, \quad \tilde{M}_0 + \tilde{M}_+ + \tilde{M}_- = 1, \\ M_+M_- &= M_0^2 - M_0, \quad \tilde{M}_+\tilde{M}_- = \tilde{M}_0^2 - \tilde{M}_0, \\ M_0^2 + M_+^2 + M_-^2 &= 1, \quad \tilde{M}_0^2 + \tilde{M}_+^2 + \tilde{M}_-^2 = 1. \end{aligned}$$

Neumann coefficient matrices of ghost *nonzero* mode part satisfy the same relations as matter part.

For Neumann coefficients which have zero mode indices, using vector notation, we have found

$$\begin{aligned} CV_0^{rs} &= V_0^{sr}, \quad \sum_{t=1}^3 V_0^{ts} = \sum_{t=1}^3 V_0^{rt} = 0, & CX_0^{rs} &= X_0^{sr}, \quad \sum_{t=1}^3 X_0^{ts} = \sum_{t=1}^3 X_0^{rt} = 0, \\ V_0^{21} &= \frac{3M_+ - 2}{1 + 3M_0} V_0^{11}, \quad V_0^{31} = \frac{3M_- - 2}{1 + 3M_0} V_0^{11}, & X_0^{21} &= -\frac{\tilde{M}_+}{1 - \tilde{M}_0} X_0^{11}, \quad X_0^{31} = -\frac{\tilde{M}_-}{1 - \tilde{M}_0} X_0^{11}. \end{aligned}$$

We consider particular squeezed states : ‘wedge-like’ state [Furuuchi-Okuyama]

$$|n_\beta\rangle := e^{\beta a^\dagger} |n\rangle = \mu_n \exp\left(\beta a^\dagger - \frac{1}{2} a^\dagger C T_n a^\dagger\right) |0\rangle$$

where  $|n\rangle$  is given by the state which is obtained by taking  $\star$  product  $n - 1$  times with a particular squeezed states  $|2\rangle$ :

$$|n\rangle := (|2\rangle)_\star^{n-1}, \quad |2\rangle = \mu_2 e^{-\frac{1}{2} a^\dagger C T_2 a^\dagger} |0\rangle, \quad C T_2 = T_2 C, \quad T_2^T = T_2, \quad [M_0, T_2] = 0, \quad T_2 \neq 1.$$

Here  $T_n, \mu_n$  are given by

$$T_n = \frac{T(1 - T_2 T)^{n-1} + (T_2 - T)^{n-1}}{(1 - T_2 T)^{n-1} + T(T_2 - T)^{n-1}}, \quad M_0 T^2 - (M_0 + 1)T + M_0 = 0,$$

$$\mu_n = \mu_2 \left( \mu_2 \mu_3^M \det^{-\frac{d}{2}} \left( \frac{1 - T}{1 - T + T^2} \right) \right)^{n-2} \det^{\frac{d}{2}} \left( \frac{1 - T^2}{(1 - T_2 T)^{n-1} + T(T_2 - T)^{n-1}} \right),$$

We have  $\star$  product formula between them [RSZ]:

$$|n_{\beta_1} \star m_{\beta_2}\rangle = \exp\left(-\mathcal{C}_{n_{\beta_1}, m_{\beta_2}}\right) \left| (n + m - 1)_{\beta_1 \rho_{1(n,m)} + \beta_2 \rho_{2(n,m)}} \right\rangle,$$

where

$$\mathcal{C}_{n_{\beta_1}, m_{\beta_2}} = \frac{1}{2} (\beta_1, \beta_2) \frac{C}{T_{n,m}} \begin{pmatrix} M_0(1 - T_m) & M_- \\ M_+ & M_0(1 - T_n) \end{pmatrix} \begin{pmatrix} \beta_1^T \\ \beta_2^T \end{pmatrix} = \mathcal{C}_{m_{\beta_2} C, n_{\beta_1} C},$$

$$\rho_{1(n,m)} = \frac{M_- + M_+ T_m}{T_{n,m}}, \quad \rho_{2(n,m)} = \frac{M_+ + M_- T_n}{T_{n,m}}, \quad C \rho_{1(n,m)} = \rho_{2(m,n)} C, \quad T_{n,m} = 1 + M_0(T_n T_m - T_n - T_m).$$

One can calculate  $\star$  product between states of the form  $a_k^\dagger \cdots a_l^\dagger |n\rangle$  by differentiating it with parameter  $\beta$  and setting  $\beta = 0$  appropriately.



Noting similarity of relations among Neumann coefficients matrices for matter and ghost nonzero modes, we define *reduced* product (denoted as  $\star^r$ ):

$$|A \star^r B\rangle := {}_2\langle A^r | {}_3\langle B^r | V_3^r \rangle_{123}, \quad \langle A^r | := \langle V_2^r | A \rangle,$$

where we restrict string fields  $|A\rangle, |B\rangle$  such that they have no  $b_0, c_0$  modes on the Fock vacuum  $|+\rangle_G$ . ( $c_0|+\rangle_G = 0, b_0|+\rangle_G \neq 0$ )

Here we introduced reduced reflector  $\langle V_2^r |$  and reduced 3-string vertex  $|V_3^r\rangle$  which contain no  $b_0, c_0$  modes on the vacuum  ${}_G\langle \tilde{+} |, |+\rangle_G$ , i.e. they are related with usual reflector and 3-string vertex by

$${}_{12}\langle V_2 | = {}_{12}\langle V_2^r | (c_0^{(1)} + c_0^{(2)}), \quad |V_3\rangle_{123} = e^{-\sum_{r,s=1}^3 c^{\dagger(r)} X^{rs} b_0^{(s)}} |V_3^r\rangle_{123}.$$

Under this  $\star^r$  product in ghost part, one can obtain similar formulas to those of matter part as follows.

We define ghost squeezed state  $|n_{\xi,\eta}\rangle$  with Grassmann odd parameters  $\xi, \eta$  which corresponds to  $|n_{\beta}\rangle$  in matter part :

$$|n_{\xi,\eta}\rangle := e^{\xi b^{\dagger} + \eta c^{\dagger}} |n\rangle_G = \tilde{\mu}_n \exp\left(\xi b^{\dagger} + \eta c^{\dagger} + c^{\dagger} C \tilde{T}_n b^{\dagger}\right) |+\rangle_G.$$

Here we defined  $|n\rangle_G$  as the state which is obtained by taking the  $\star^r$  product  $n - 1$  times with a particular ghost squeezed state  $|2\rangle_G$  :

$$|n\rangle_G = (|2\rangle_G)_{\star^r}^{n-1}, \quad |2\rangle_G = \exp\left(c^\dagger C \tilde{T}_2 b^\dagger\right) |+\rangle_G, \quad C \tilde{T}_2 = \tilde{T}_2 C, \quad [\tilde{M}_0, \tilde{T}_2] = 0, \quad \tilde{T}_2 \neq 1,$$

and then we have obtained formulas for  $\tilde{T}_n, \tilde{\mu}_n$ ,

$$\tilde{T}_n = \frac{\tilde{T}(1 - \tilde{T}_2 \tilde{T})^{n-1} + (\tilde{T}_2 - \tilde{T})^{n-1}}{(1 - \tilde{T}_2 \tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2 - \tilde{T})^{n-1}}, \quad \tilde{M}_0 \tilde{T}^2 - (\tilde{M}_0 + 1) \tilde{T} + \tilde{M}_0 = 0,$$

$$\tilde{\mu}_n = \tilde{\mu}_2 \left( \tilde{\mu}_2 \tilde{\mu}_3^r \det \left( \frac{1 - \tilde{T}}{1 - \tilde{T} + \tilde{T}^2} \right) \right)^{n-2} \det \left( \frac{(1 - \tilde{T}_2 \tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2 - \tilde{T})^{n-1}}{1 - \tilde{T}^2} \right),$$

by solving the same recurrence equation as that in matter part.

For these ghost squeezed states, we have the  $\star^r$  product formula:

$$|n_{\xi, \eta} \star^r m_{\xi', \eta'}\rangle = \exp\left(-\mathcal{C}_{n_{\xi, \eta}, m_{\xi', \eta'}}\right) \left| (n + m - 1)_{\xi \tilde{\rho}_{1(n, m)} + \xi' \tilde{\rho}_{2(n, m)}, \eta \tilde{\rho}_{1(n, m)}^T + \eta' \tilde{\rho}_{2(n, m)}^T} \right\rangle,$$

where

$$\mathcal{C}_{n_{\xi, \eta}, m_{\xi', \eta'}} = (\xi, \xi') \frac{C}{\tilde{T}_{n, m}} \begin{pmatrix} \tilde{M}_0(1 - \tilde{T}_m) & \tilde{M}_- \\ \tilde{M}_+ & \tilde{M}_0(1 - \tilde{T}_n) \end{pmatrix} \begin{pmatrix} \eta^T \\ \eta'^T \end{pmatrix} = \mathcal{C}_{m_{\xi', \eta'}, n_{\xi, \eta}},$$

$$\tilde{\rho}_{1(n, m)} = \frac{\tilde{M}_- + \tilde{M}_+ \tilde{T}_m}{\tilde{T}_{n, m}}, \quad \tilde{\rho}_{2(n, m)} = \frac{\tilde{M}_+ + \tilde{M}_- \tilde{T}_n}{\tilde{T}_{n, m}}, \quad C \tilde{\rho}_{1(n, m)} = \tilde{\rho}_{2(m, n)} C, \quad \tilde{T}_{n, m} = 1 + \tilde{M}_0(\tilde{T}_n \tilde{T}_m - \tilde{T}_n - \tilde{T}_m).$$

Using the  $\star^r$  product, we get the  $\star$  product formula between string fields  $|\Phi\rangle = b_0|\phi\rangle$ ,  $|\Psi\rangle = b_0|\psi\rangle$  in the **Siegel gauge**:

$$\begin{aligned} |\Phi \star \Psi\rangle &= |\phi \star^r \psi\rangle + b_0 \left( {}_2\langle\phi^r|_3\langle\psi^r| \sum_{s=1}^3 c^{(s)\dagger} X_{0}^{s1} |V_3^r\rangle_{123} \right) \\ &= (1 + b_0 c^\dagger X_{0}^{11}) |\phi \star^r \psi\rangle + b_0 \sum_{s=2,3} {}_2\langle\phi^r|_3\langle\psi^r| c^{(s)\dagger} X_{0}^{s1} |V_3^r\rangle_{123}. \end{aligned}$$

Especially, we have obtained  $\star$  product formula between squeezed states in ghost part in the Siegel gauge:

$$\begin{aligned} &|(b_0 n_{\xi,\eta}) \star (b_0 m_{\xi',\eta'})\rangle \\ &= \left( 1 + b_0 \left( c^\dagger X_{0}^{11} + \left( \xi C + \frac{\partial}{\partial \eta} \tilde{T}_n \right) X_{0}^{21} + \left( \xi' C + \frac{\partial}{\partial \eta'} \tilde{T}_m \right) X_{0}^{31} \right) \right) |n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle \\ &= \left( 1 + b_0 c^\dagger \frac{1 - \tilde{T}_n \tilde{T}_m}{\tilde{T}_{n,m}} X_{0}^{11} - b_0 (\xi \tilde{\rho}_{1(n,m)} + \xi' \tilde{\rho}_{2(n,m)}) \frac{1}{1 - \tilde{M}_0} X_{0}^{11} \right) |n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle. \end{aligned}$$

We can obtain  $\star$  product between the states of the form  $b_0 b_k^\dagger \cdots c_l^\dagger |n\rangle_G$  by differentiating it with respect to parameters  $\xi, \eta$  and setting them zero appropriately.

Later, Okuyama further investigated and rearranged these algebra in the **Siegel gauge** elegantly. Especially, our  $\star^r$  corresponds to his  $\star_{b_0}$ :  $|\Phi \star_{b_0} \Psi\rangle = b_0 |\phi \star^r \psi\rangle$ .

Equation of motion of VSFT:

$$\mathcal{Q}|\Psi\rangle + |\Psi \star \Psi\rangle = 0, \quad \mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n(c_n + (-1)^n c_n^\dagger) = c_0 + f \cdot (c + Cc^\dagger).$$

To solve it we set the ansatz in the Siegel gauge :

$$|\Psi\rangle = b_0|P\rangle_M \left( \sum_{n=1}^{\infty} g_n|n\rangle_G \right), \quad |P \star P\rangle_M = |P\rangle_M.$$

As usual, the matter part is factorized and solved by a projector  $|P\rangle_M$  which was well investigated earlier.[Gross-Taylor,RSZ,Kawano-Okuyama]

We have obtained some solutions by using previous formulas in ghost part:

1. identity-like solution

$$\mathcal{Q} = c_0, \quad |\Psi\rangle = -b_0|P\rangle_M|I^r\rangle_G.$$

2. sliver-like solution

$$\mathcal{Q} = c_0 - (c + c^\dagger) \frac{1}{1 - \tilde{M}_0} X_{0}^{11}, \quad |\Psi\rangle = -b_0|P\rangle_M|\Xi^r\rangle_G.$$

This solution was constructed by Hata-Kawano (HK). (Our formula is simpler than HK's.)

3. another solution

$$\mathcal{Q} = c_0 - (c + c^\dagger) \frac{1}{1 - \tilde{M}_0} X_{0}^{11}, \quad |\Psi\rangle = -b_0|P\rangle_M(|I^r\rangle_G - |\Xi^r\rangle_G).$$

Here we denoted as

$$|n = 1\rangle_G =: |I^r\rangle_G, \quad |n = \infty\rangle_G =: |\Xi^r\rangle_G,$$

which are analogies of identity and sliver states with respect to the  $\star^r$  product:

$$|I^r \star^r A\rangle = |A \star^r I^r\rangle = |A\rangle, \quad |\Xi^r \star^r \Xi^r\rangle = |\Xi^r\rangle.$$

These  $|I^r\rangle_G, |\Xi^r\rangle_G$  are *not* the ghost part of identity or sliver state which are defined as surface states.

Later, GRSZ proposed their canonical choice of the kinetic term of VSFT :  $Q = \frac{1}{2i} (c(i) - c(-i))$ , and observed that it would coincide with that of HK solution numerically, and then Okuyama proved


$$\frac{1}{2i} (c(i) - c(-i)) = c_0 - (c + c^\dagger) \frac{1}{1 - \tilde{M}_0} X_{0}^{11}$$

analytically.

GRSZ also observed numerically  $|\Xi^r\rangle_G$  would coincide with their sliver state  $|\Xi'\rangle_G$  with respect to the  $\star'$  product on twisted  $bc$ -ghost system.

## Subtlety of the identity state

The identity state  $|\mathcal{I}\rangle$  is defined by

$$(X(\sigma) - X(\pi - \sigma))|\mathcal{I}\rangle = 0, \quad 0 \leq \sigma \leq \pi/2,$$


in matter part and corresponding connection condition in  $bc$ -ghost, but there is subtlety which comes from midpoint singularity especially in ghost part.

The identity state  $|\mathcal{I}\rangle$  is expected to be the *identity* with respect to the  $\star$  product at least naively.

The identity state  $|\mathcal{I}\rangle$  in oscillator representation is given as [LPP]

$$\begin{aligned} \langle \mathcal{I} | &= \mu_{1M} \langle 0 |_G \langle \Omega | c_{-1} c_0 c_1 \\ &\cdot \int_{\zeta_1 \zeta_0 \zeta_{-1}} \exp \left( \frac{1}{2} \sum_{n,m \geq 1} \alpha_n N_{nm} \alpha_m + \sum_{n \geq 2, m \geq -1} c_n \tilde{N}_{nm} b_m - \sum_{i=\pm 1, 0, m \geq 1} \zeta_i M_{im} b_m \right), \\ N_{nm} &= \frac{1}{nm} \oint \frac{dz}{2\pi i} z^{-n} f'(z) \oint \frac{dw}{2\pi i} w^{-m} f'(w) \frac{1}{(f(z) - f(w))^2}, \\ \tilde{N}_{nm} &= \oint \frac{dz}{2\pi i} z^{-n+1} (f'(z))^2 \oint \frac{dw}{2\pi i} w^{-m-2} (f'(w))^{-1} \frac{-1}{f(z) - f(w)}, \\ M_{im} &= \oint \frac{dz}{2\pi i} z^{-m-2} (f'(z))^{-1} (f(z))^{i+1} \end{aligned}$$

where the map  $f(z)$  is defined by  $f(z) = \frac{2z}{1-z^2}$ .

This formula gives **the oscillator representation of the identity state**  $|\mathcal{I}\rangle$  which is the same as that in Gross-Jevicki(II):

$$|\mathcal{I}\rangle = \frac{1}{4i} b^+ \left(\frac{\pi}{2}\right) b^- \left(\frac{\pi}{2}\right) |I\rangle_M |I^r\rangle_G = [b^\dagger]_{\mathcal{O}} (b_0 + 2[b^\dagger]_{\mathcal{E}}) |I\rangle_M |I^r\rangle_G,$$

$$[\ ]_{\mathcal{E}} := \sum_{n=1}^{\infty} (-1)^n [\ ]_{2n}, \quad [\ ]_{\mathcal{O}} := \sum_{n=0}^{\infty} (-1)^n [\ ]_{2n+1}.$$

By pure oscillator calculation, we can show the following equations :

$$(a_n - (-1)^n a_n^\dagger) |\mathcal{I}\rangle = 0, \quad (b_n - (-1)^n b_n^\dagger) |\mathcal{I}\rangle = 0,$$

$$(c_{2k} + c_{2k}^\dagger) |\mathcal{I}\rangle = (-1)^k 2c_0 |\mathcal{I}\rangle, \quad (c_{2k+1} - c_{2k+1}^\dagger) |\mathcal{I}\rangle = (-1)^k (c_1 - c_{-1}) |\mathcal{I}\rangle,$$

$$Q_B |\mathcal{I}\rangle = -\frac{d-26}{2} \sum_{l=1}^{\infty} l c_{2l}^\dagger |\mathcal{I}\rangle + (1 - a_0) c_0 |\mathcal{I}\rangle = 0. \quad (d = 26, a_0 = 1)$$

Note  $|\mathcal{I}\rangle$  is BRST invariant, but  $(c_k + (-1)^k c_k^\dagger) |\mathcal{I}\rangle \neq 0$ , i.e., there is anomaly for  $c$ -ghost in oscillator representation.

If we use the relations among Neumann coefficients *formally*, we have

$$\begin{aligned}
{}_3\langle \mathcal{I} | 1, 2, 3 \rangle &= \mu_1 \mu_3 (\det(1 - M_0))^{-\frac{d}{2}} \det(1 - \tilde{M}_0) |1, 2\rangle_M |1, 2\rangle'_G (\neq |1, 2\rangle), \\
|1, 2\rangle_M &= \exp \left( - \sum_{n, m \geq 0} a_n^{\dagger(1)} C_{nm} a_m^{\dagger(2)} \right) |0\rangle_{M12}, \\
|1, 2\rangle'_G &= (1 - 2[(1 - \tilde{M}_0)^{-1} X_0^{11}] \varepsilon) \cdot \\
&\quad \cdot \left( [(1 - \tilde{M}_0)^{-1} X_0^{21}] \circ (b_0^{(1)} - b_0^{(2)}) - [(1 - \tilde{M}_0)^{-1} (\tilde{M}_+ b^{\dagger(1)} + \tilde{M}_- b^{\dagger(2)})] \circ \right) \cdot \\
&\quad \cdot \exp \left( \sum_{n, m \geq 1} (c_{-n}^{(1)} C_{nm} b_{-m}^{(2)} + c_{-n}^{(2)} C_{nm} b_{-m}^{(1)}) \right) e^{\Delta E} |+\rangle_{G12}, \\
\Delta E &= -(c^{\dagger(1)} - c^{\dagger(2)}) \frac{1}{1 - \tilde{M}_0} X_0^{11} (b_0^{(1)} - b_0^{(2)}),
\end{aligned}$$

and this shows the identity state **in oscillator representation** is *not* the identity with respect to the  $\star$  product because  ${}_3\langle \mathcal{I} | 1, 2, 3 \rangle = |1, 2\rangle$  should be satisfied if  $\mathcal{I} \star A = A \star \mathcal{I} = A$ ,  $\forall A$ .

This would be caused by *c*-ghost anomaly in oscillator representation. But the above calculation might be subtle because we treated  $\infty \times \infty$  matrices as usual number here.



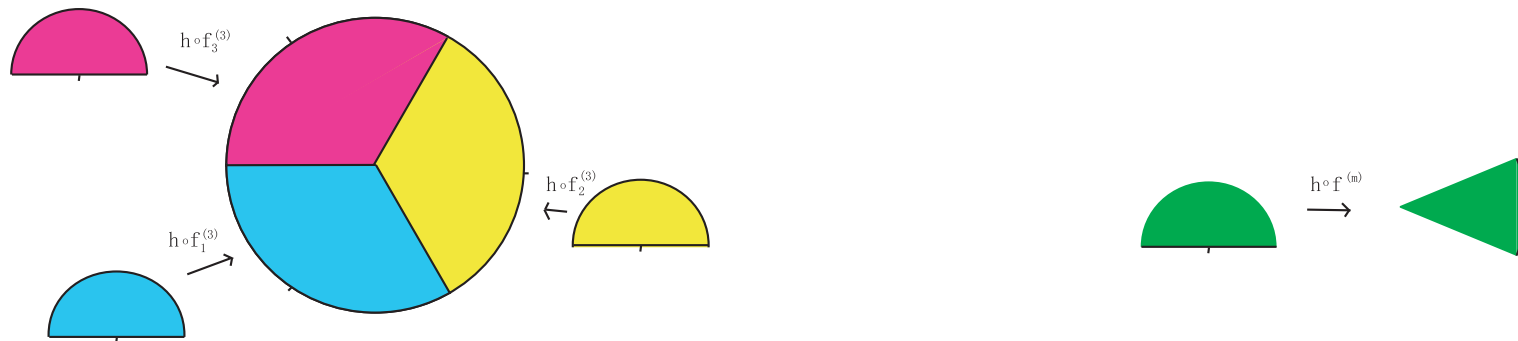
### 3. CFT Approach

The Witten's  $*$  product in CFT language which was developed by LeClair-Peskin-Preitschopf (LPP) is defined as:

$$\langle A, B * C \rangle = \left\langle f_1^{(3)} \circ A(0) f_2^{(3)} \circ B(0) f_3^{(3)} \circ C(0) \right\rangle_{\text{UHP}},$$

where conformal maps are given by

$$f_1^{(3)}(z) = h^{-1} \left( e^{-\frac{2}{3}\pi i} h(z)^{\frac{2}{3}} \right), \quad f_2^{(3)}(z) = h^{-1} \left( h(z)^{\frac{2}{3}} \right), \quad f_3^{(3)}(z) = h^{-1} \left( e^{\frac{2}{3}\pi i} h(z)^{\frac{2}{3}} \right), \quad h(z) = \frac{1 + iz}{1 - iz}.$$



For wedge state  $|m\rangle$  which is defined by

$$\langle m, \varphi \rangle = \left\langle f^{(m)} \circ \varphi(0) \right\rangle_{\text{UHP}}, \quad f^{(m)}(z) = h^{-1} \left( h(z)^{\frac{2}{m}} \right),$$

we have the  $*$  product between them [David]

$$\langle \varphi, m * n \rangle = \langle \varphi, m + n - 1 \rangle, \quad \forall \varphi.$$

To prove this algebra we followed only the definition of wedge state and generalized gluing and resmoothing theorem (GGRT)<sup>[Schwarz-Sen]</sup>:

$$\begin{aligned} & \sum_r \langle f_1 \circ \Phi_{r_1}(0) \dots f_n \circ \Phi_{r_n}(0) f \circ \Phi_r(0) \rangle_{\mathcal{D}_1} \langle g_1 \circ \Phi_{s_1}(0) \dots g_m \circ \Phi_{s_m}(0) g \circ \Phi_r^c(0) \rangle_{\mathcal{D}_2} \\ &= \left\langle F_1 \circ f_1 \circ \Phi_{r_1}(0) \dots F_1 \circ f_n \circ \Phi_{r_n}(0) \hat{F}_2 \circ g_1 \circ \Phi_{s_1}(0) \dots \hat{F}_2 \circ g_m \circ \Phi_{s_m}(0) \right\rangle_{\mathcal{D}}, \quad F_1 \circ f = \hat{F}_2 \circ g \circ I. \end{aligned}$$

and constructed resmoothing maps  $F_1, \hat{F}_2$  concretely.

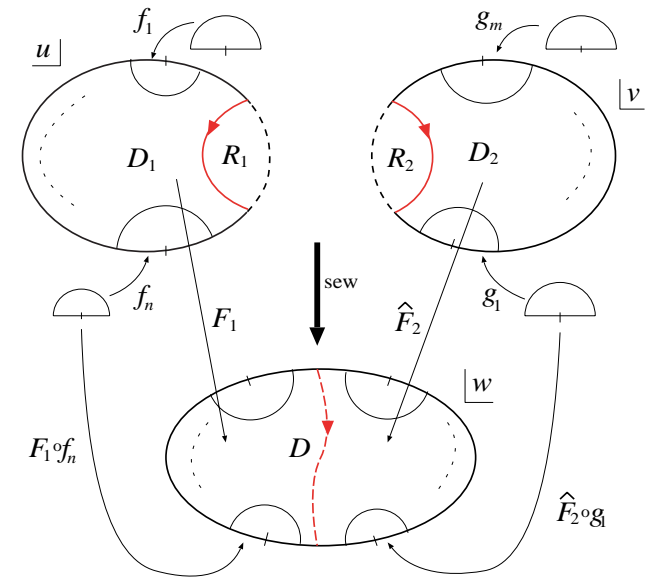
Our strategy for computation of the  $*$  product including a wedge state  $|m\rangle$  is as follows.

First insert complete set  $\sum_r |\Phi_r\rangle \langle \Phi_r^c|$ , and then apply GGRT:

$$\begin{aligned} \langle \varphi, A * (\mathcal{O}_B m) \rangle &= \sum_r \langle \varphi, A * \Phi_r \rangle \langle \Phi_r^c, \mathcal{O}_B m \rangle \\ &= \sum_r \left\langle f_1^{(3)} \circ \varphi f_2^{(3)} \circ A f_3^{(3)} \circ \Phi_r \right\rangle \left\langle f^{(m)} \circ I \circ \mathcal{O}_B f^{(m)} \circ \Phi_r^c \right\rangle \\ &= \left\langle F_1 \circ f_1^{(3)} \circ \varphi F_1 \circ f_2^{(3)} \circ A \hat{F}_2 \circ f^{(m)} \circ I \circ \mathcal{O}_B \right\rangle. \end{aligned}$$

In this case,  $F_1, \hat{F}_2$  are given by

$$F_1(z) = h^{-1} \left( e^{\frac{m+2}{m+1}\pi i} h(z)^{\frac{3}{m+1}} \right), \quad \hat{F}_2(z) = h^{-1} \left( e^{\frac{m+2}{m+1}\pi i} h(z)^{\frac{m}{m+1}} \right), \quad F_1 \circ f_3^{(3)} = \hat{F}_2 \circ f^{(m)} \circ I.$$



Using this technique, we proved some algebras about the identity state

$|\mathcal{I}\rangle = |m = 1\rangle$  :

$$\langle \varphi, \mathcal{I} * \psi \rangle = \langle \varphi, \psi * \mathcal{I} \rangle = \langle \varphi, \psi \rangle, \quad \langle \varphi, \mathcal{I} * \mathcal{O}\mathcal{I} \rangle = \langle \varphi, \mathcal{O}\mathcal{I} * \mathcal{I} \rangle = \langle \varphi, \mathcal{O}\mathcal{I} \rangle$$

In this sense, we found the identity state  $\mathcal{I}$  behaves like the identity with respect to the  $*$  product **in CFT language**.

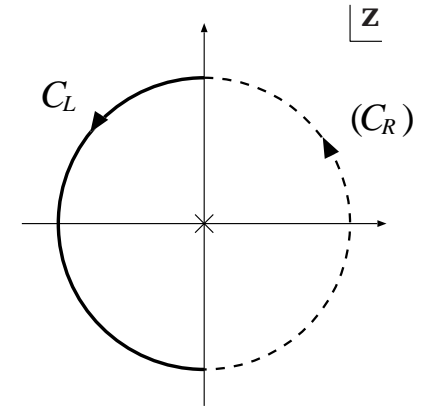
In the same way, we have checked ‘partial integration formula’

$$\langle \varphi, (Q_R A) * B \rangle = -(-1)^{|A|} \langle \varphi, A * (Q_L B) \rangle,$$

even on the wedge state:  $|A\rangle = \mathcal{O}_A |m\rangle$  or  $|B\rangle = \mathcal{O}_B |m\rangle$ .

Here we defined  $Q_{L(R)}$  using the primary BRST current  $j_B$  as

$$Q_{L(R)} := \int_{C_{L(R)}} \frac{dz}{2\pi i} j_B(z).$$



From these results we have verified that

$$|\Phi_0\rangle = -Q_L |\mathcal{I}\rangle + \frac{a}{2} \mathcal{Q}^\epsilon |\mathcal{I}\rangle, \quad \left( \mathcal{Q}^\epsilon := \frac{1}{2i} (e^{-i\epsilon} c(i e^{i\epsilon}) - e^{i\epsilon} c(-i e^{-i\epsilon})) \right)$$

satisfies equation of motion of CSFT :

$$\langle \varphi, Q_B \Phi_0 + \Phi_0 * \Phi_0 \rangle = 0, \quad \forall \varphi.$$

By expanding CSFT action around our solution  $\Phi_0$  :

$$S_{\text{CSFT}}|_{\Phi_0+\Psi} = -\frac{1}{g_o^2} \left( \frac{a}{2} \langle \Psi, \mathcal{Q}_\epsilon \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi * \Psi \rangle \right) + S_{\text{CSFT}}|_{\Phi_0},$$

we have derived GRSZ's VSFT action which is regularized by  $\epsilon$  in the kinetic term:

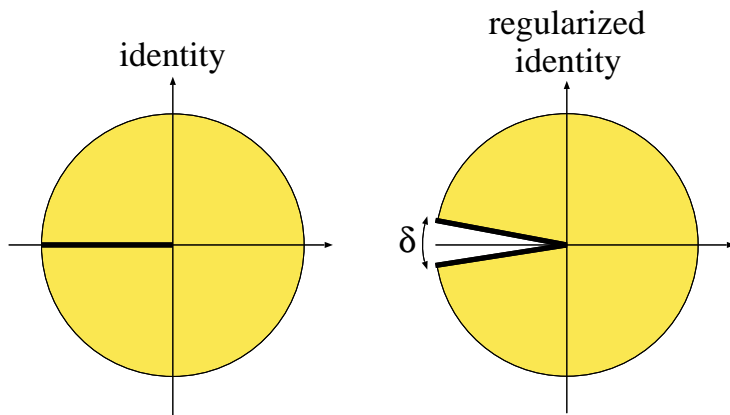
$$\mathcal{Q}_\epsilon = \frac{1}{4i} (e^{-i\epsilon} c(i e^{i\epsilon}) + e^{i\epsilon} c(i e^{-i\epsilon}) - e^{-i\epsilon} c(-i e^{-i\epsilon}) - e^{i\epsilon} c(-i e^{i\epsilon})) \xrightarrow{\epsilon \rightarrow 0} \frac{1}{2i} (c(i) - c(-i)).$$

Naively one might think the value of the CSFT action at  $\Phi_0$  would be zero, but it may be possible to give a nonzero value for D25-brane tension.

In fact we have

$$\begin{aligned} & \langle \mathcal{Q}^\epsilon \tilde{\mathcal{I}}_\delta, Q_B \mathcal{Q}^\epsilon \tilde{\mathcal{I}}_\delta \rangle \\ &= -\delta^2 \sin^2 \epsilon \left[ \frac{1}{2} \left\{ \left( \tan \frac{\epsilon}{2} \right)^{\frac{2}{\delta}} + \left( \tan \frac{\epsilon}{2} \right)^{-\frac{2}{\delta}} \right\} + 3 \right] V_{26}, \end{aligned}$$

where  $\tilde{\mathcal{I}}_\delta$  is regularized identity state which is necessary to apply GGRT. (At  $\delta = 0$  this quantity would vanish if one uses equation of motion naively.)



The solution of the CSFT such as

$$\Psi_c = -Q_L \mathcal{I} + C_L(f) \mathcal{I}, \quad C_L(f) = \int_{C_L} d\sigma f(\sigma)(c(\sigma) + c(-\sigma)), \quad f(\pi - \sigma) = f(\sigma), \quad f\left(\frac{\pi}{2}\right) = 0$$

was considered earlier by Horowitz et.al. in the context of purely cubic SFT, but they treated identity state rather formally (i.e., they treated  $\mathcal{I}$  as a formal object which behaves like the identity).

If one uses the equations which were proved formally

$$Q_B \Psi_c + \Psi_c \star \Psi_c = 0, \quad Q_L \mathcal{I} \star Q_L \mathcal{I} = C_L(f) \mathcal{I} \star C_L(f) \mathcal{I} = 0,$$

the value of the action at this solution vanishes :

$$S|_{\Psi_c} \propto \langle \Psi_c, \Psi_c \star \Psi_c \rangle = 0.$$

Recently Takahashi-Tanimoto constructed a solution of CSFT of the form  $-Q_L(f) \mathcal{I} + C_L(g) \mathcal{I}$ ,  $f \neq 1$ .

## 4. Summary and Discussion

We examined Witten's  $*$  product both in oscillator and in CFT language.

We constructed solutions of VSFT in oscillator representation and a solution of CSFT in CFT language. The latter one derives GRSZ's VSFT action from Witten's CSFT, but to confirm Sen's conjecture we should obtain D25-brane tension from potential height.

The identity state  $\mathcal{I}$  is rather complicated in ghost part **in oscillator representation**, and *naive* computation (using relations among Neumann coefficient matrices *formally*) gives some unexpected results: for example  $\mathcal{I} \star \mathcal{I} = 0$ .

This subtlety would be caused not only by  $c$ -ghost anomaly but also by regarding  $\infty \times \infty$  matrices as usual number. We might have to treat them more carefully using Neumann coefficient matrices spectroscopy [RSZ].

On the other hand, we proved some relations expected of the identity state **using GGRT in CFT language**. But the evaluation of the action including  $\mathcal{I}$  is still rather subtle because an appropriate regularization is required.

## Appendix

### Gaussian integral formula:

matter part (momentum zero sector)

$$\begin{aligned}
& \exp\left(\frac{1}{2}aMa + \lambda a\right) \exp\left(\frac{1}{2}a^\dagger Na^\dagger + \mu a^\dagger\right) |0\rangle \\
&= \frac{1}{\sqrt{\det(1 - MN)}} \exp\left(\frac{1}{2}\lambda N(1 - MN)^{-1}\lambda + \frac{1}{2}\mu M(1 - NM)^{-1}\mu + \lambda(1 - NM)^{-1}\mu\right) \\
&\cdot \exp\left((\lambda N + \mu)(1 - MN)^{-1}a^\dagger + \frac{1}{2}a^\dagger N(1 - MN)^{-1}a^\dagger\right) |0\rangle, \quad [a_m, a_n^\dagger] = \delta_{mn}, \quad a_n|0\rangle = 0, \quad n \geq 1.
\end{aligned}$$

ghost part

$$\begin{aligned}
& \exp(cAb + c_0\alpha b + c\mu + \nu b + c_0\gamma) \exp(c^\dagger Bb^\dagger + c^\dagger\beta b_0 + c^\dagger\rho + \sigma b^\dagger + \delta b_0)|+\rangle = \det(1 + BA) \det \Delta \cdot e^{E_1 + E_0}|+\rangle, \\
& \Delta = 1 + \alpha(1 + BA)^{-1}\beta, \\
& E_1 = c^\dagger(1 + BA)^{-1}Bb^\dagger + c^\dagger(1 + BA)^{-1}(\rho - B\mu) + (\nu B + \sigma)(1 + AB)^{-1}b^\dagger \\
& + \nu(1 + BA)^{-1}(\rho - B\mu) - \sigma(1 + AB)^{-1}(A\rho + \mu), \\
& E_0 = -c^\dagger(1 + BA)^{-1}\beta\Delta^{-1}(\alpha(1 + BA)^{-1}Bb^\dagger - b_0) - c^\dagger(1 + BA)^{-1}\beta\Delta^{-1}(\alpha(1 + BA)^{-1}(\rho - B\mu) + \gamma) \\
& - ((\nu - \sigma A)(1 + BA)^{-1}\beta + \delta)\Delta^{-1}(\alpha(1 + BA)^{-1}Bb^\dagger - b_0) \\
& - ((\nu - \sigma A)(1 + BA)^{-1}\beta + \delta)\Delta^{-1}(\alpha(1 + BA)^{-1}(\rho - B\mu) + \gamma), \\
& \{c_n, b_m\} = \delta_{n+m,0}, \quad c_n|+\rangle = 0, \quad n \geq 0, \quad b_n|+\rangle = 0, \quad n \geq 1, \quad c_n^\dagger := c_{-n}, \quad b_n^\dagger := b_{-n}, \quad n \geq 1.
\end{aligned}$$