Some Properties of String Field Algebra

岸本　功

(京大・基研)

based on

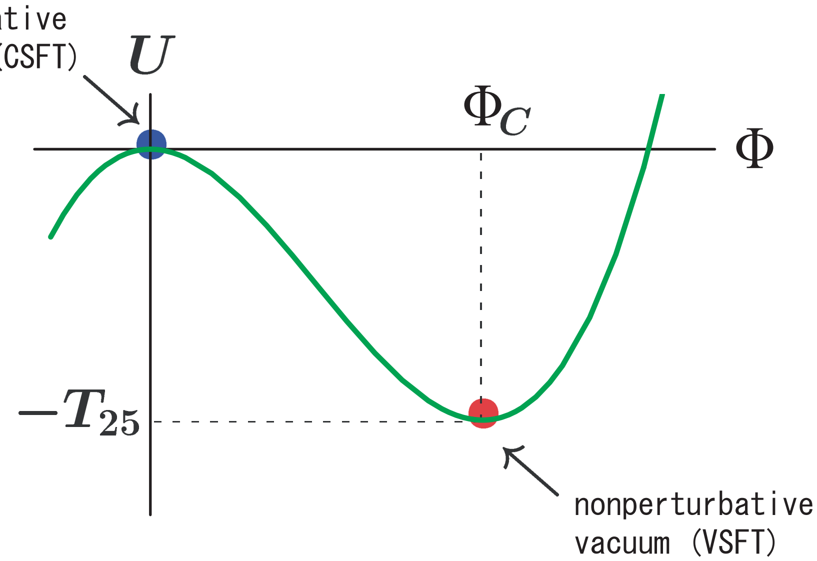
I. K., JHEP12(2001)007[hep-th/0110124]

I.K., K.Ohmori, hep-th/0112169

1. Introduction

Sen's conjecture (for bosonic open string field theory)

open string (D25-brane)



CSFT(cubic string field theory) Witten

$$S_{\mathrm{CSFT}} = -\frac{1}{g_o^2}\left(\frac{1}{2}\langle\Phi, Q_B\Phi\rangle + \frac{1}{3}\langle\Phi, \Phi * \Phi\rangle\right)$$

Sen's conjecture says there is a solution of CSFT Φ_c :
$Q_B\Phi_c + \Phi_c * \Phi_c = 0$ **and** $-S_{\mathrm{CSFT}}|_{\Phi_c}/V_{26} = T_{25}$.

VSFT(vacuum string field theory) Rastelli-Sen-Zwiebach (RSZ)

$$S_{\rm VSFT} = -\kappa_0 \left(\frac{1}{2} \langle \Phi, \mathcal{Q}\Phi \rangle + \frac{1}{3} \langle \Phi, \Phi * \Phi \rangle \right)$$

This describes the physics around nonperturbative vacuum (no D25-brane).
$\mathcal{Q}$ should satisfy the following conditions to define a gauge theory

$$\mathcal{Q}^2 = 0, \mathcal{Q}(A * B) = \mathcal{Q}A * B + (-1)^{|A|} A * \mathcal{Q}B, \langle \mathcal{Q}A, B \rangle = -(-1)^{|A|} \langle A, \mathcal{Q}B \rangle$$

and have vanishing cohomology and universality (no matter information).

These requirements are satisfied by

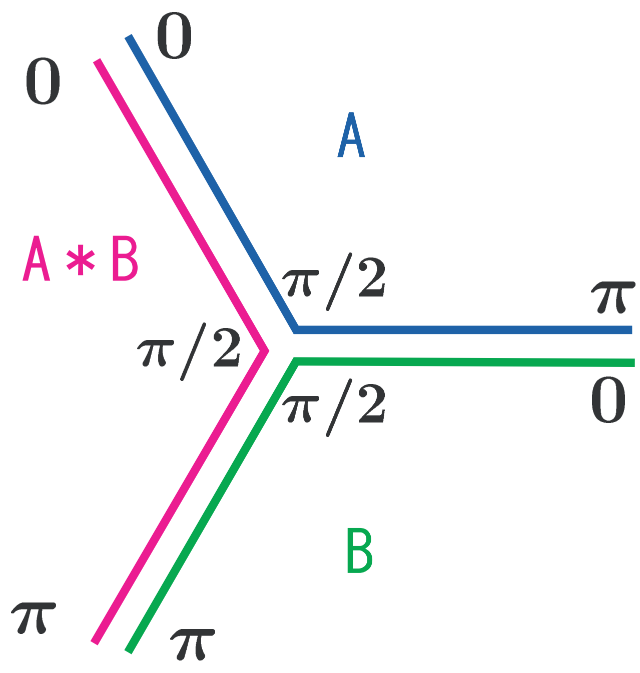
$$\mathcal{Q} = \Sigma_n f_n (c_n + (-1)^n c_{-n}),$$

where f_n is some coefficient. Later, its canonical choice was given by Gaiotto-Rastelli-Sen-Zwiebach (GRSZ):

$$\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i)) = c_0 - (c_2 + c_{-2}) + (c_4 + c_{-4}) + \cdots$$

To realize this scenario, it is necessary to have an analytic solution of CSFT or VSFT which relates them. We investigate Witten's $*$ product for this purpose.

The Witten's $*$ product represents string interaction. This is represented by operator formalism using oscillators or CFT technique.



In the context of VSFT, some techniques using oscillator representation have been developed in *matter part* **especially to construct projectors which satisfy** *reduced* **equation of motion of VSFT :**
$\Phi_M \star \Phi_M = \Phi_M$.
We extend them to *ghost part* **and solve** *full* **equation of motion of VSFT :** $\mathcal{Q}\Phi + \Phi \star \Phi = 0$.

Because $\mathcal{Q}$ is linear in c-ghost, one can take the ansatz $|\Phi_c\rangle = |\Phi_M\rangle|\Phi_G\rangle$ and the e.o.m is reduced to $\Phi_M \star \Phi_M = \Phi_M$ in matter part by assuming the existence of a solution Φ_G in ghost part which satisfies $\mathcal{Q}\Phi_G + \Phi_G \star \Phi_G = 0$. Many authors discussed D-brane solutions of VSFT with this strategy before.

In the context of purely CSFT, Horowitz et.al. discussed (formal) solutions. Using CFT technique, we reexamine them to construct a solution of CSFT which derives GRSZ's proposed VSFT action.

Plan of the talk

§1. **Introduction**

§2. **Oscillator Approach**

Neumann coefficient matrices,
reduced star product,
some formulas for wedge-like states,
application to VSFT,
subtlety of the identity state

§3. **CFT Approach**

Generalized Gluing and Resmoothing Theorem (GGRT),
some formulas for wedge states,
a derivation of VSFT from CSFT

§4. **Summary and Discussion**

2. Oscillator Approach

For two string fields A, B, which are represented by some oscillators on a particular Fock vacuum, we define the Witten's $\star$ product as

$$|A \star B\rangle_1 := {}_2\langle A|_3\langle B|1,2,3\rangle = \langle 2,4|A\rangle_4\langle 3,5|B\rangle_5|1,2,3\rangle,$$

where 3-string vertex $|1,2,3\rangle$ and reflector $\langle 1,2|$ are represented by

$$\begin{aligned}
|V_3\rangle &= |1,2,3\rangle = \tilde{\mu}_3 \int d^d p^{(1)} d^d p^{(2)} d^d p^{(3)} (2\pi)^d \delta^d(p^{(1)}+p^{(2)}+p^{(3)}) e^{E_3}|0,p\rangle,\\
E_3 &= -\frac{1}{2}\sum_{r,s=1}^{3}\sum_{n.m\geq 1} a_n^{(r)\dagger} V_{nm}^{rs} a_m^{(s)\dagger} - \sum_{r,s=1}^{3}\sum_{n\geq 1} p^{(r)} V_{0n}^{rs} a_n^{(r)\dagger} - \frac{1}{2}\sum_{r,s=1}^{3} p^{(r)} V_{00}^{rs} p^{(s)} - \sum_{r,s=1}^{3}\sum_{n\geq 1, m\geq 0} c_{-n}^{(r)} X_{nm}^{rs} b_{-m}^{(s)},\\
|0,p\rangle &= |0,p^{(1)}\rangle|0,p^{(2)}\rangle|0,p^{(3)}\rangle, \quad b_n^{(i)}|0,p^{(i)}\rangle = 0, \quad n\geq 1, \quad c_m^{(i)}|0,p^{(i)}\rangle = 0, \quad m \geq 0,\\
\langle V_2| &= \langle 1,2| = \int d^d p^{(1)} d^d p^{(2)} \langle 0,p| e^{E_2} \delta^d(p^{(1)}+p^{(2)}) \delta(c_0^{(1)}+c_0^{(2)})\\
E_2 &= -\sum_{n,m\geq 1} a_n^{(1)} C_{nm} a_m^{(2)} - \sum_{n,m\geq 1} (c_n^{(1)} C_{nm} b_m^{(2)} + c_n^{(2)} C_{nm} b_m^{(1)}), \quad \langle 0,p| = {}_1\langle 0,p^{(1)}|_2\langle 0,p^{(2)}|, \quad C_{nm} := (-1)^n \delta_{n,m}.
\end{aligned}$$

This 3-string vertex is a solution of the connection condition:

$$\left(X^{(r)}(\sigma) - X^{(r-1)}(\pi-\sigma)\right)|V_3\rangle = 0, \ \left(P^{(r)}(\sigma) + P^{(r-1)}(\pi-\sigma)\right)|V_3\rangle = 0, \quad 0 \leq \sigma \leq \frac{\pi}{2},$$
$$\left(c^{\pm(r)}(\sigma) + c^{\pm(r-1)}(\pi-\sigma)\right)|V_3\rangle = 0, \quad \left(b^{\pm(r)}(\sigma) - b^{\pm(r-1)}(\pi-\sigma)\right)|V_3\rangle = 0, \ r = 1,2,3.$$

We can prove useful relations among Neumann coefficients V^{rs}_{nm}, X^{rs}_{nm} :

For the matrices [Gross-Jevicki,Kosteleckey-Potting,RSZ]

$$M_0 := CV^{rr}, \quad M_\pm := CV^{rr\pm 1}, \quad \tilde{M}_0 := -CX^{rr}, \quad \tilde{M}_\pm := -CX^{rr\pm 1}$$

whose indices run from 1 to ∞, there are some relations

$$\begin{array}{cc} CM_0 = M_0 C,\ CM_+ = M_- C, & C\tilde{M}_0 = \tilde{M}_0 C,\ C\tilde{M}_+ = \tilde{M}_- C, \\ [M_0, M_\pm] = [M_+, M_-] = 0, & [\tilde{M}_0, \tilde{M}_\pm] = [\tilde{M}_+, \tilde{M}_-] = 0, \\ M_0 + M_+ + M_- = 1, & \tilde{M}_0 + \tilde{M}_+ + \tilde{M}_- = 1, \\ M_+ M_- = M_0^2 - M_0, & \tilde{M}_+ \tilde{M}_- = \tilde{M}_0^2 - \tilde{M}_0, \\ M_0^2 + M_+^2 + M_-^2 = 1, & \tilde{M}_0^2 + \tilde{M}_+^2 + \tilde{M}_-^2 = 1. \end{array}$$

Neumann coefficient matrices of ghost *nonzero* mode part satisfy the same relations as matter part.

For Neumann coefficients which have zero mode indices, using vector notation, we have found

$$CV^{rs}_{\ \ 0} = V^{sr}_{\ \ 0},\ \sum_{t=1}^{3} V^{ts}_{\ \ 0} = \sum_{t=1}^{3} V^{rt}_{\ \ 0} = 0, \qquad CX^{rs}_{\ \ 0} = X^{sr}_{\ \ 0},\ \sum_{t=1}^{3} X^{ts}_{\ \ 0} = \sum_{t=1}^{3} X^{rt}_{\ \ 0} = 0,$$

$$V^{21}_{\ \ 0} = \frac{3M_+ - 2}{1 + 3M_0} V^{11}_{\ \ 0},\ V^{31}_{\ \ 0} = \frac{3M_- - 2}{1 + 3M_0} V^{11}_{\ \ 0}, \qquad X^{21}_{\ \ 0} = -\frac{\tilde{M}_+}{1 - \tilde{M}_0} X^{11}_{\ \ 0},\ X^{31}_{\ \ 0} = -\frac{\tilde{M}_-}{1 - \tilde{M}_0} X^{11}_{\ \ 0}.$$

Matter part

We consider particular squeezed states : 'wedge-like' state [Furuuchi-Okuyama]

$$|n_\beta\rangle := e^{\beta a^\dagger}|n\rangle = \mu_n \exp\left(\beta a^\dagger - \frac{1}{2}a^\dagger C T_n a^\dagger\right)|0\rangle$$

where $|n\rangle$ is given by the state which is obtained by taking $\star$ product $n-1$ times with a particular squeezed states $|2\rangle$:

$$|n\rangle := (|2\rangle)_\star^{n-1}, \quad |2\rangle = \mu_2 e^{-\frac{1}{2}a^\dagger C T_2 a^\dagger}|0\rangle, \quad CT_2 = T_2 C, \; T_2^T = T_2, \quad [M_0, T_2] = 0, \quad T_2 \neq 1.$$

Here T_n, μ_n are given by

$$\begin{aligned} T_n &= \frac{T(1-T_2T)^{n-1} + (T_2-T)^{n-1}}{(1-T_2T)^{n-1} + T(T_2-T)^{n-1}}, \quad M_0T^2 - (M_0+1)T + M_0 = 0, \\ \mu_n &= \mu_2\left(\mu_2\mu_3^M \det{}^{-\frac{d}{2}}\left(\frac{1-T}{1-T+T^2}\right)\right)^{n-2} \det{}^{\frac{d}{2}}\left(\frac{1-T^2}{(1-T_2T)^{n-1}+T(T_2-T)^{n-1}}\right). \end{aligned}$$

We have $\star$ product formula between them [RSZ]:

$$|n_{\beta_1} \star m_{\beta_2}\rangle = \exp\left(-\mathcal{C}_{n_{\beta_1}, m_{\beta_2}}\right)\left|(n+m-1)_{\beta_1\rho_{1(n,m)}+\beta_2\rho_{2(n,m)}}\right\rangle,$$

where

$$\begin{aligned} \mathcal{C}_{n_{\beta_1}, m_{\beta_2}} &= \frac{1}{2}(\beta_1, \beta_2)\frac{C}{T_{n,m}}\begin{pmatrix} M_0(1-T_m) & M_- \\ M_+ & M_0(1-T_n)\end{pmatrix}\begin{pmatrix}\beta_1^T \\ \beta_2^T\end{pmatrix} = \mathcal{C}_{m_{\beta_2 C}, n_{\beta_1 C}}, \\ \rho_{1(n,m)} &= \frac{M_- + M_+T_m}{T_{n,m}}, \quad \rho_{2(n,m)} = \frac{M_+ + M_-T_n}{T_{n,m}}, \quad C\rho_{1(n,m)} = \rho_{2(m,n)}C, \quad T_{n,m} = 1 + M_0(T_nT_m - T_n - T_m). \end{aligned}$$

One can calculate $\star$ product between states of the form $a_k^\dagger \cdots a_l^\dagger|n\rangle$ by differentiating it with parameter β and setting $\beta = 0$ appropriately.

Ghost part

Noting similarity of relations among Neumann coefficients matrices for matter and ghost nonzero modes, we define *reduced* product (denoted as $\star^r$):

$$|A \star^r B\rangle := {}_2\langle A^r|_3\langle B^r|V_3^r\rangle_{123}, \quad \langle A^r| := \langle V_2^r|A\rangle,$$

where we restrict string fields $|A\rangle, |B\rangle$ such that they have no b_0, c_0 modes on the Fock vacuum $|+\rangle_G$. $(c_0|+\rangle_G = 0, b_0|+\rangle_G \neq 0)$

Here we introduced reduced reflector $\langle V_2^r|$ and reduced 3-string vertex $|V_3^r\rangle$ which contain no b_0, c_0 modes on the vacuum ${}_G\langle\tilde{+}|$, $|+\rangle_G$, i.e. they are related with usual reflector and 3-string vertex by

$${}_{12}\langle V_2| = {}_{12}\langle V_2^r|(c_0^{(1)} + c_0^{(2)}), \quad |V_3\rangle_{123} = e^{-\sum_{r,s=1}^3 c^{\dagger(r)} X^{rs}_{0} b_0^{(s)}} |V_3^r\rangle_{123}.$$

Under this $\star^r$ product in ghost part, one can obtain similar formulas to those of matter part as follows.

We define ghost squeezed state $|n_{\xi,\eta}\rangle$ with Grassmann odd parameters ξ, η which corresponds to $|n_\beta\rangle$ in matter part :

$$|n_{\xi,\eta}\rangle := e^{\xi b^\dagger + \eta c^\dagger}|n\rangle_G = \tilde{\mu}_n \exp\left(\xi b^\dagger + \eta c^\dagger + c^\dagger C\tilde{T}_n b^\dagger\right)|+\rangle_G.$$

Here we defined $|n\rangle_G$ as the state which is obtained by taking the $\star^r$ product $n-1$ times with a particular ghost squeezed state $|2\rangle_G$:

$$|n\rangle_G = (|2\rangle_G)^{n-1}_{\star^r}, \quad |2\rangle_G = \exp\left(c^\dagger C\tilde{T}_2 b^\dagger\right)|+\rangle_G, \quad C\tilde{T}_2 = \tilde{T}_2 C, \quad [\tilde{M}_0, \tilde{T}_2] = 0, \quad \tilde{T}_2 \neq 1,$$

and then we have obtained formulas for $\tilde{T}_n$, $\tilde{\mu}_n$,

$$\begin{aligned}\tilde{T}_n &= \frac{\tilde{T}(1-\tilde{T}_2\tilde{T})^{n-1} + (\tilde{T}_2-\tilde{T})^{n-1}}{(1-\tilde{T}_2\tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n-1}}, \quad \tilde{M}_0\tilde{T}^2 - (\tilde{M}_0+1)\tilde{T} + \tilde{M}_0 = 0,\\ \tilde{\mu}_n &= \tilde{\mu}_2\left(\tilde{\mu}_2\tilde{\mu}_3^r \det\left(\frac{1-\tilde{T}}{1-\tilde{T}+\tilde{T}^2}\right)\right)^{n-2} \det\left(\frac{(1-\tilde{T}_2\tilde{T})^{n-1} + \tilde{T}(\tilde{T}_2-\tilde{T})^{n-1}}{1-\tilde{T}^2}\right),\end{aligned}$$

by solving the same recurrence equation as that in matter part.

For these ghost squeezed states, we have the $\star^r$ product formula:

$$|n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle = \exp\left(-\mathcal{C}_{n_{\xi,\eta},m_{\xi',\eta'}}\right)\left|(n+m-1)_{\xi\tilde{\rho}_{1(n,m)}+\xi'\tilde{\rho}_{2(n,m)},\eta\tilde{\rho}^T_{1(n,m)}+\eta'\tilde{\rho}^T_{2(n,m)}}\right\rangle,$$

where

$$\mathcal{C}_{n_{\xi,\eta},m_{\xi',\eta'}} = (\xi,\xi')\frac{C}{\tilde{T}_{n,m}}\begin{pmatrix}\tilde{M}_0(1-\tilde{T}_m) & \tilde{M}_- \\ \tilde{M}_+ & \tilde{M}_0(1-\tilde{T}_n)\end{pmatrix}\begin{pmatrix}\eta^T \\ \eta'^T\end{pmatrix} = \mathcal{C}_{m_{\xi'C,\eta'C},n_{\xi C,\eta C}},$$

$$\tilde{\rho}_{1(n,m)} = \frac{\tilde{M}_- + \tilde{M}_+\tilde{T}_m}{\tilde{T}_{n,m}}, \quad \tilde{\rho}_{2(n,m)} = \frac{\tilde{M}_+ + \tilde{M}_-\tilde{T}_n}{\tilde{T}_{n,m}}, \quad C\tilde{\rho}_{1(n,m)} = \tilde{\rho}_{2(m,n)}C, \quad \tilde{T}_{n,m} = 1 + \tilde{M}_0(\tilde{T}_n\tilde{T}_m - \tilde{T}_n - \tilde{T}_m).$$

Using the $\star^r$ product, we get the $\star$ product formula between string fields $|\Phi\rangle = b_0|\phi\rangle$, $|\Psi\rangle = b_0|\psi\rangle$ in the Siegel gauge:

$$
\begin{aligned}
|\Phi \star \Psi\rangle &= |\phi \star^r \psi\rangle + b_0 \left({}_2\langle \phi^r|_3\langle\psi^r| \sum_{s=1}^{3} c^{(s)\dagger} X^{s1}_{\ 0} |V_3^r\rangle_{123} \right) \\
&= (1 + b_0 c^\dagger X^{11}_{\ 0})|\phi \star^r \psi\rangle + b_0 \sum_{s=2,3} {}_2\langle\phi^r|_3\langle\psi^r|c^{(s)\dagger} X^{s1}_{\ 0}|V_3^r\rangle_{123}.
\end{aligned}
$$

Especially, we have obtained $\star$ product formula between squeezed states in ghost part in the Siegel gauge:

$$
\begin{aligned}
&|(b_0 n_{\xi,\eta}) \star (b_0 m_{\xi',\eta'})\rangle \\
&= \left(1 + b_0 \left(c^\dagger X^{11}_{\ 0} + \left(\xi C + \frac{\partial}{\partial \eta}\tilde{T}_n\right) X^{21}_{\ 0} + \left(\xi' C + \frac{\partial}{\partial \eta'}\tilde{T}_m\right) X^{31}_{\ 0} \right)\right) |n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle \\
&= \left(1 + b_0 c^\dagger \frac{1 - \tilde{T}_n \tilde{T}_m}{\tilde{T}_{n,m}} X^{11}_{\ 0} - b_0(\xi \tilde{\rho}_{1(n,m)} + \xi' \tilde{\rho}_{2(n,m)}) \frac{1}{1 - \tilde{M}_0} X^{11}_{\ 0} \right) |n_{\xi,\eta} \star^r m_{\xi',\eta'}\rangle.
\end{aligned}
$$

We can obtain $\star$ product between the states of the form $b_0 b_k^\dagger \cdots c_l^\dagger |n\rangle_G$ by differentiating it with respect to parameters ξ, η and setting them zero appropriately.

Later, Okuyama further investigated and rearranged these algebra in the Siegel gauge elegantly. Especially, our $\star^r$ corresponds to his $\star_{b_0}$: $|\Phi \star_{b_0} \Psi\rangle = b_0|\phi \star^r \psi\rangle$.

Application to VSFT

Equation of motion of VSFT:

$$\mathcal{Q}|\Psi\rangle + |\Psi \star \Psi\rangle = 0, \quad \mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n(c_n + (-1)^n c_n^\dagger) = c_0 + f\cdot(c + Cc^\dagger).$$

To solve it we set the ansatz in the Siegel gauge :

$$|\Psi\rangle = b_0|P\rangle_M \left(\sum_{n=1}^{\infty} g_n|n\rangle_G\right), \quad |P\star P\rangle_M = |P\rangle_M.$$

As usual, the matter part is factorized and solved by a projector $|P\rangle_M$ which was well investigated earlier. [Gross-Taylor,RSZ,Kawano-Okuyama]

We have obtained some solutions by using previous formulas in ghost part:

1. **identity-like solution**

$$\mathcal{Q} = c_0, \quad |\Psi\rangle = -b_0|P\rangle_M|I^r\rangle_G.$$

2. **sliver-like solution**

$$\mathcal{Q} = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0}X^{11}_{\ \ 0}, \quad |\Psi\rangle = -b_0|P\rangle_M|\Xi^r\rangle_G.$$

   This solution was constructed by Hata-Kawano (HK). **(Our formula is simpler than HK's.)**

3. **another solution**

$$\mathcal{Q} = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0}X^{11}_{\ \ 0}, \quad |\Psi\rangle = -b_0|P\rangle_M(|I^r\rangle_G - |\Xi^r\rangle_G).$$

Here we denoted as

$$|n=1\rangle_G =: |I^r\rangle_G, \quad |n=\infty\rangle_G =: |\Xi^r\rangle_G,$$

which are analogies of identity and sliver states with respect to the $\star^r$ product:

$$|I^r \star^r A\rangle = |A \star^r I^r\rangle = |A\rangle, \quad |\Xi^r \star^r \Xi^r\rangle = |\Xi^r\rangle.$$

These $|I^r\rangle_G, |\Xi^r\rangle_G$ are *not* the ghost part of identity or sliver state which are defined as surface states.

Later, GRSZ proposed their canonical choice of the kinetic term of VSFT : $\mathcal{Q} = \frac{1}{2i}\left(c(i) - c(-i)\right)$, and observed that it would coincide with that of HK solution numerically, and then Okuyama proved

$$\frac{1}{2i}\left(c(i) - c(-i)\right) = c_0 - (c + c^\dagger)\frac{1}{1-\tilde{M}_0}X^{11}_{\ \ 0}$$

analytically.

GRSZ also observed numerically $|\Xi^r\rangle_G$ would coincide with their sliver state $|\Xi'\rangle_G$ with respect to the $*'$ product on twisted bc-ghost system, and then Okuda proved $|\Xi^r\rangle_G = |\Xi'\rangle_G$ analytically.

Subtlety of the identity state

The identity state $|\mathcal{I}\rangle$ is defined by

$$(X(\sigma)-X(\pi-\sigma))|\mathcal{I}\rangle=0,\quad 0\le\sigma\le\pi/2,$$

in matter part and corresponding connection condition in bc-ghost, but there is subtlety which comes from midpoint singularity especially in ghost part.
The identity state $|\mathcal{I}\rangle$ is expected to be the *identity* with respect to the $\star$ product at least naively.

The identity state $|\mathcal{I}\rangle$ in oscillator representation is given as [LPP]

$$\begin{aligned}
\langle\mathcal{I}| &= \mu_{1M}\langle 0|_G\langle\Omega|c_{-1}c_0c_1\\
&\quad\cdot\int_{\zeta_1\zeta_0\zeta_{-1}}\exp\Bigg(\frac{1}{2}\sum_{n,m\ge1}\alpha_nN_{nm}\alpha_m+\sum_{n\ge2,m\ge-1}c_n\tilde{N}_{nm}b_m-\sum_{i=\pm1,0,m\ge1}\zeta_iM_{im}b_m\Bigg),\\
N_{nm} &= \frac{1}{nm}\oint\frac{dz}{2\pi i}z^{-n}f'(z)\oint\frac{dw}{2\pi i}w^{-m}f'(w)\frac{1}{(f(z)-f(w))^2},\\
\tilde{N}_{nm} &= \oint\frac{dz}{2\pi i}z^{-n+1}(f'(z))^2\oint\frac{dw}{2\pi i}w^{-m-2}(f'(w))^{-1}\frac{-1}{f(z)-f(w)},\\
M_{im} &= \oint\frac{dz}{2\pi i}z^{-m-2}(f'(z))^{-1}(f(z))^{i+1}
\end{aligned}$$

where the map $f(z)$ is defined by $f(z)=\frac{2z}{1-z^2}$.

This formula gives the oscillator representation of the identity state $|\mathcal{I}\rangle$ which is the same as that in Gross-Jevicki(II):

$$|\mathcal{I}\rangle = \frac{1}{4i} b^{+}\left(\frac{\pi}{2}\right) b^{-}\left(\frac{\pi}{2}\right) |I\rangle_M |I^r\rangle_G = [b^\dagger]_{\mathcal{O}} \left(b_0 + 2[b^\dagger]_{\mathcal{E}}\right) |I\rangle_M |I^r\rangle_G,$$

$$[\ ]_{\mathcal{E}} := \sum_{n=1}^{\infty} (-1)^n [\ ]_{2n}, \quad [\ ]_{\mathcal{O}} := \sum_{n=0}^{\infty} (-1)^n [\ ]_{2n+1}.$$

By pure oscillator calculation, we can show the following equations :

$$\left(a_n - (-1)^n a_n^\dagger\right) |\mathcal{I}\rangle = 0, \quad \left(b_n - (-1)^n b_n^\dagger\right) |\mathcal{I}\rangle = 0,$$

$$\left(c_{2k} + c_{2k}^\dagger\right) |\mathcal{I}\rangle = (-1)^k 2 c_0 |\mathcal{I}\rangle, \quad \left(c_{2k+1} - c_{2k+1}^\dagger\right) |\mathcal{I}\rangle = (-1)^k (c_1 - c_{-1}) |\mathcal{I}\rangle,$$

$$Q_B |\mathcal{I}\rangle = -\frac{d-26}{2} \sum_{l=1}^{\infty} l c_{2l}^\dagger |\mathcal{I}\rangle + (1 - a_0) c_0 |\mathcal{I}\rangle = 0. \quad (d = 26, a_0 = 1)$$

Note $|\mathcal{I}\rangle$ is BRST invariant, but $(c_k + (-1)^k c_k^\dagger)|\mathcal{I}\rangle \neq 0$, i.e., there is anomaly for c-ghost in oscillator representation.

If we use the relations among Neumann coefficients *formally*, we have

$$
\begin{aligned}
{}_3\langle \mathcal{I}|1,2,3\rangle &= \mu_1\mu_3\,(\det(1-M_0))^{-\frac{d}{2}}\det(1-\tilde{M}_0)|1,2\rangle_M|1,2\rangle'_G\ (\neq |1,2\rangle),\\
|1,2\rangle_M &= \exp\left(-\sum_{n,m\geq 0} a_n^{\dagger(1)}C_{nm}a_m^{\dagger(2)}\right)|0\rangle_{M12},\\
|1,2\rangle'_G &= (1-2[(1-\tilde{M}_0)^{-1}X^{11}_{\ 0}]_{\mathcal{E}})\cdot\\
&\quad\cdot\Big([(1-\tilde{M}_0)^{-1}X^{21}_{\ 0}]_{\mathcal{O}}(b_0^{(1)}-b_0^{(2)})-[(1-\tilde{M}_0)^{-1}(\tilde{M}_+b^{\dagger(1)}+\tilde{M}_-b^{\dagger(2)})]_{\mathcal{O}}\Big)\cdot\\
&\quad\cdot\exp\left(\sum_{n,m\geq 1}(c_{-n}^{(1)}C_{nm}b_{-m}^{(2)}+c_{-n}^{(2)}C_{nm}b_{-m}^{(1)})\right)e^{\Delta E}|+\rangle_{G12},\\
\Delta E &= -(c^{\dagger(1)}-c^{\dagger(2)})\frac{1}{1-\tilde{M}_0}X^{11}_{\ 0}(b_0^{(1)}-b_0^{(2)}),
\end{aligned}
$$

and this shows the identity state in oscillator representation is *not* the identity with respect to the $\star$ product because ${}_3\langle\mathcal{I}|1,2,3\rangle = |1,2\rangle$ should be satisfied if $\mathcal{I}\star A = A\star\mathcal{I} = A,\ \forall A$.

This would be caused by c-ghost anomaly in oscillator representation. But the above calculation might be subtle because we treated $\infty\times\infty$ matrices as usual number here.

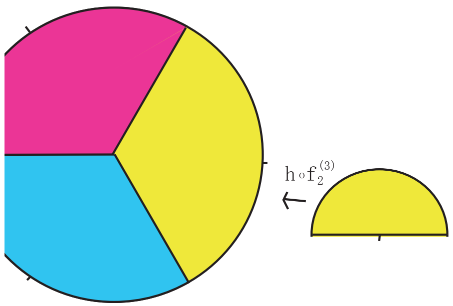
3. CFT Approach

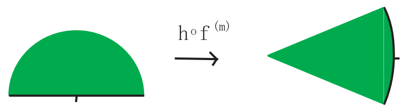
The Witten's $*$ product in CFT language which was developed by LeClair-Peskin-Preitschopf (LPP) is defined as:

$$\langle A, B * C\rangle = \left\langle f_1^{(3)} \circ A(0)\ f_2^{(3)} \circ B(0)\ f_3^{(3)} \circ C(0)\right\rangle_{\mathrm{UHP}},$$

where conformal maps are given by

$$f_1^{(3)}(z) = h^{-1}\left(e^{-\frac{2}{3}\pi i} h(z)^{\frac{2}{3}}\right), \quad f_2^{(3)}(z) = h^{-1}\left(h(z)^{\frac{2}{3}}\right), \quad f_3^{(3)}(z) = h^{-1}\left(e^{\frac{2}{3}\pi i} h(z)^{\frac{2}{3}}\right), \quad h(z) = \frac{1+iz}{1-iz}.$$





For wedge state $|m\rangle$ which is defined by

$$\langle m, \varphi\rangle = \left\langle f^{(m)} \circ \varphi(0)\right\rangle_{\mathrm{UHP}}, \quad f^{(m)}(z) = h^{-1}\left(h(z)^{\frac{2}{m}}\right),$$

we have the $*$ product between them [David]

$$\langle \varphi, m * n\rangle = \langle \varphi, m+n-1\rangle, \quad \forall \varphi.$$

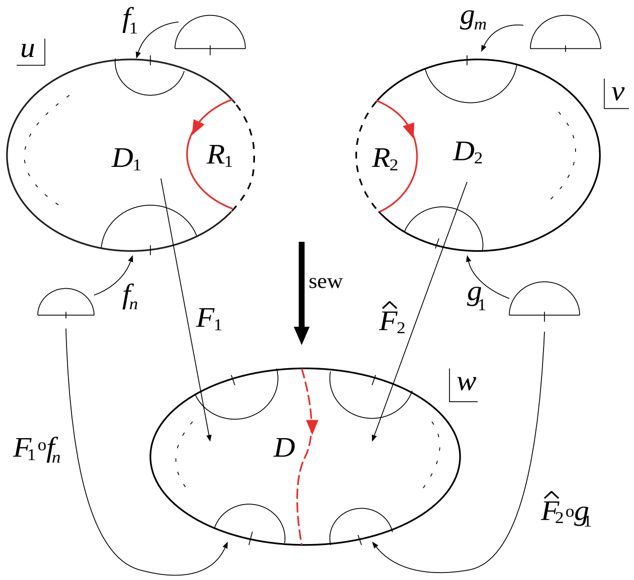
To prove this algebra we followed only the definition of wedge state and generalized gluing and resmoothing theorem (GGRT)[Schwarz-Sen]:

$$\sum_r \langle f_1\circ\Phi_{r_1}(0)\dots f_n\circ\Phi_{r_n}(0)\ f\circ\Phi_r(0)\rangle_{\mathcal{D}_1}\left\langle g_1\circ\Phi_{s_1}(0)\dots g_m\circ\Phi_{s_m}(0)\ g\circ\Phi^c_r(0)\right\rangle_{\mathcal{D}_2}$$
$$=\left\langle F_1\circ f_1\circ\Phi_{r_1}(0)\dots F_1\circ f_n\circ\Phi_{r_n}(0)\ \widehat{F}_2\circ g_1\circ\Phi_{s_1}(0)\dots\widehat{F}_2\circ g_m\circ\Phi_{s_m}(0)\right\rangle_{\mathcal{D}},\qquad F_1\circ f=\hat{F}_2\circ g\circ I.$$

and constructed resmoothing maps $F_1,\hat{F}_2$ concretely.

Our strategy for computation of the $*$ product including a wedge state $|m\rangle$ is as follows.
First insert complete set $\sum_r|\Phi_r\rangle\langle\Phi^c_r|$, and then apply GGRT:

$$\langle\varphi,A*(\mathcal{O}_Bm)\rangle=\sum_r\langle\varphi,A*\Phi_r\rangle\langle\Phi^c_r,\mathcal{O}_Bm\rangle$$
$$=\sum_r\left\langle f^{(3)}_1\circ\varphi\ f^{(3)}_2\circ A\ f^{(3)}_3\circ\Phi_r\right\rangle\left\langle f^{(m)}\circ I\circ\mathcal{O}_B\ f^{(m)}\circ\Phi^c_r\right\rangle$$
$$=\left\langle F_1\circ f^{(3)}_1\circ\varphi\ F_1\circ f^{(3)}_2\circ A\ \hat{F}_2\circ f^{(m)}\circ I\circ\mathcal{O}_B\right\rangle.$$



In this case, $F_1,\hat{F}_2$ are given by

$$F_1(z)=h^{-1}\left(e^{\frac{m+2}{m+1}\pi i}h(z)^{\frac{3}{m+1}}\right),\ \hat{F}_2(z)=h^{-1}\left(e^{\frac{m+2}{m+1}\pi i}h(z)^{\frac{m}{m+1}}\right),\quad F_1\circ f^{(3)}_3=\hat{F}_2\circ f^{(m)}\circ I.$$

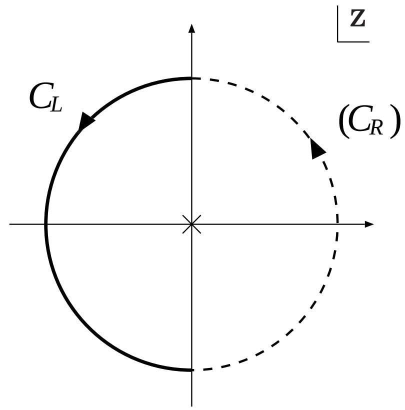
Using this technique, we proved some algebras about the identity state $|\mathcal{I}\rangle = |m=1\rangle$:

$$\langle\varphi, \mathcal{I} * \psi\rangle = \langle\varphi, \psi * \mathcal{I}\rangle = \langle\varphi, \psi\rangle, \quad \langle\varphi, \mathcal{I} * \mathcal{O}\mathcal{I}\rangle = \langle\varphi, \mathcal{O}\mathcal{I} * \mathcal{I}\rangle = \langle\varphi, \mathcal{O}\mathcal{I}\rangle$$

In this sense, we found the identity state $\mathcal{I}$ behaves like the identity with respect to the $*$ product in CFT language.

In the same way, we have checked 'partial integration formula'

$$\langle\varphi, (Q_R A) * B\rangle = -(-1)^{|A|}\langle\varphi, A * (Q_L B)\rangle,$$



even on the wedge state: $|A\rangle = \mathcal{O}_A|m\rangle$ or $|B\rangle = \mathcal{O}_B|m\rangle$.
Here we defined $Q_{L(R)}$ using the primary BRST current j_B as $Q_{L(R)} := \int_{C_{L(R)}} \frac{dz}{2\pi i} j_B(z)$.

From these results we have verified that

$$|\Phi_0\rangle = -Q_L|\mathcal{I}\rangle + \frac{a}{2}\mathcal{Q}^\epsilon|\mathcal{I}\rangle, \quad \left(\mathcal{Q}^\epsilon := \frac{1}{2i}\left(e^{-i\epsilon}c(ie^{i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon})\right)\right)$$

satisfies equation of motion of CSFT :

$$\langle\varphi, Q_B\Phi_0 + \Phi_0 * \Phi_0\rangle = 0, \quad \forall\varphi.$$

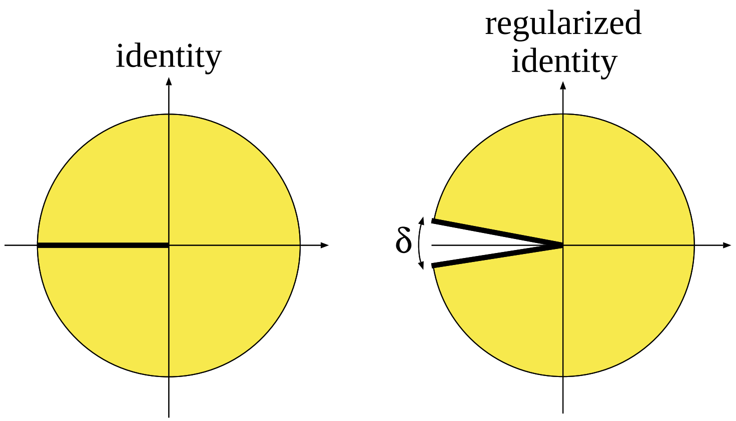
By expanding CSFT action around our solution Φ_0 :

$$S_{\mathrm{CSFT}}|_{\Phi_0+\Psi} = -\frac{1}{g_o^2}\left(\frac{a}{2}\langle\Psi, \mathcal{Q}_\epsilon\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right) + S_{\mathrm{CSFT}}|_{\Phi_0},$$

we have derived GRSZ's VSFT action which is regularized by ϵ in the kinetic term:

$$\mathcal{Q}_\epsilon = \frac{1}{4i}\left(e^{-i\epsilon}c(ie^{i\epsilon}) + e^{i\epsilon}c(ie^{-i\epsilon}) - e^{-i\epsilon}c(-ie^{-i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon})\right) \xrightarrow{\epsilon\to 0} \frac{1}{2i}(c(i) - c(-i)).$$

Naively one might think the value of the CSFT action at Φ_0 would be zero, but it may be possible to give a nonzero value for D25-brane tension.



In fact we have

$$\langle\mathcal{Q}^\epsilon\widetilde{\mathcal{I}}_\delta, Q_B\mathcal{Q}^\epsilon\widetilde{\mathcal{I}}_\delta\rangle$$
$$= -\delta^2\sin^2\epsilon\left[\frac{1}{2}\left\{\left(\tan\frac{\epsilon}{2}\right)^{\frac{2}{\delta}} + \left(\tan\frac{\epsilon}{2}\right)^{-\frac{2}{\delta}}\right\} + 3\right]V_{26},$$

where $\widetilde{\mathcal{I}}_\delta$ is regularized identity state which is necessary to apply GGRT. (At $\delta = 0$ this quantity would vanish if one uses equation of motion naively.)

Remark

The solution of the CSFT such as

$$\Psi_c = -Q_L\mathcal{I} + C_L(f)\mathcal{I}, \quad C_L(f) = \int_{C_L} d\sigma f(\sigma)(c(\sigma) + c(-\sigma)), \quad f(\pi - \sigma) = f(\sigma), f\left(\frac{\pi}{2}\right) = 0$$

was considered earlier by Horowitz et.al. in the context of purely cubic SFT, but they treated identity state rather formally (i.e., they treated $\mathcal{I}$ as a formal object which behaves like the identity).

If one uses the equations which were proved formally

$$Q_B\Psi_c + \Psi_c \star \Psi_c = 0, \quad Q_L\mathcal{I} \star Q_L\mathcal{I} = C_L(f)\mathcal{I} \star C_L(f)\mathcal{I} = 0,$$

the value of the action at this solution vanishes :

$$S|_{\Psi_c} \propto \langle \Psi_c, \Psi_c \star \Psi_c \rangle = 0.$$

Recently Takahashi-Tanimoto constructed a solution of CSFT of the form $-Q_L(f)\mathcal{I} + C_L(g)\mathcal{I}, \quad f \neq 1.$

4. Summary and Discussion

We examined Witten's $*$ product both in oscillator and in CFT language.

We constructed solutions of VSFT in oscillator representation and a solution of CSFT in CFT language. The latter one derives GRSZ's VSFT action from Witten's CSFT, but to confirm Sen's conjecture we should obtain D25-brane tension from potential height.

The identity state $\mathcal{I}$ is rather complicated in ghost part in oscillator representation, and *naive* computation (using relations among Neumann coefficient matrices *formally*) gives some unexpected results: for example $\mathcal{I} \star \mathcal{I} = 0$.
This subtlety would be caused not only by c-ghost anomaly but also by regarding $\infty \times \infty$ matrices as usual number. We might have to treat them more carefully using Neumann coefficient matrices spectroscopy [RSZ].
On the other hand, we proved some relations expected of the identity state using GGRT in CFT language. But the evaluation of the action including $\mathcal{I}$ is still rather subtle because an appropriate regularization is required.

Appendix

Gaussian integral formula:

matter part (momentum zero sector)

$$
\begin{aligned}
&\exp\left(\frac{1}{2}aMa+\lambda a\right)\exp\left(\frac{1}{2}a^\dagger N a^\dagger+\mu a^\dagger\right)|0\rangle\\
&=\frac{1}{\sqrt{\det(1-MN)}}\exp\left(\frac{1}{2}\lambda N(1-MN)^{-1}\lambda+\frac{1}{2}\mu M(1-NM)^{-1}\mu+\lambda(1-NM)^{-1}\mu\right)\\
&\cdot\exp\left((\lambda N+\mu)(1-MN)^{-1}a^\dagger+\frac{1}{2}a^\dagger N(1-MN)^{-1}a^\dagger\right)|0\rangle,\quad [a_m,a_n^\dagger]=\delta_{mn},\ a_n|0\rangle=0,n\geq 1.
\end{aligned}
$$

ghost part

$$
\begin{aligned}
&\exp(cAb+c_0\alpha b+c\mu+\nu b+c_0\gamma)\exp(c^\dagger Bb^\dagger+c^\dagger\beta b_0+c^\dagger\rho+\sigma b^\dagger+\delta b_0)|+\rangle=\det(1+BA)\det\Delta\cdot e^{E_1+E_0}|+\rangle,\\
&\Delta=1+\alpha(1+BA)^{-1}\beta,\\
&E_1=c^\dagger(1+BA)^{-1}Bb^\dagger+c^\dagger(1+BA)^{-1}(\rho-B\mu)+(\nu B+\sigma)(1+AB)^{-1}b^\dagger\\
&+\nu(1+BA)^{-1}(\rho-B\mu)-\sigma(1+AB)^{-1}(A\rho+\mu),\\
&E_0=-c^\dagger(1+BA)^{-1}\beta\Delta^{-1}(\alpha(1+BA)^{-1}Bb^\dagger-b_0)-c^\dagger(1+BA)^{-1}\beta\Delta^{-1}(\alpha(1+BA)^{-1}(\rho-B\mu)+\gamma)\\
&-((\nu-\sigma A)(1+BA)^{-1}\beta+\delta)\Delta^{-1}(\alpha(1+BA)^{-1}Bb^\dagger-b_0)\\
&-((\nu-\sigma A)(1+BA)^{-1}\beta+\delta)\Delta^{-1}(\alpha(1+BA)^{-1}(\rho-B\mu)+\gamma),\\
&\{c_n,b_m\}=\delta_{n+m,0},\ c_n|+\rangle=0,n\geq 0,\ b_n|+\rangle=0,n\geq 1,\qquad c_n^\dagger:=c_{-n},b_n^\dagger:=b_{-n},n\geq 1.
\end{aligned}
$$

More on Identity State

Oscillator language

- $$|\mathcal{I} \star \mathcal{I}\rangle = 0 \neq |\mathcal{I}\rangle \quad (?)$$

  by *naive* computation in ghost part

- $$\mathcal{Q}_\epsilon|\mathcal{I}\rangle = \left(1 + 2\sum_{n=1}^{\infty} \cos 2n\epsilon\right) c_0|\mathcal{I}\rangle \neq 0$$

  for $\epsilon \neq 0$

- $$\frac{1}{2i}\left(c(i) - c(-i)\right)|\mathcal{I}\rangle = (1 + 2\zeta(0))c_0|\mathcal{I}\rangle = 0$$

CFT language (LPP+GGRT)

- $$\langle \varphi, \mathcal{I} * \mathcal{I}\rangle = \langle \varphi, \mathcal{I}\rangle, \ \forall \varphi.$$

- $$\langle \varphi, \mathcal{Q}_\epsilon \mathcal{I}\rangle = 0,$$

  for $\epsilon \neq 0$.

- [GRSZ]

$$\begin{aligned}\left\langle \varphi, \frac{1}{2i}\left(c(i) - c(-i)\right)\mathcal{I}\right\rangle &= \left\langle \varphi, \lim_{\epsilon\to 0} \mathcal{Q}_\epsilon \mathcal{I}\right\rangle \\ &:= \lim_{\epsilon\to 0}\langle \varphi, \mathcal{Q}_\epsilon \mathcal{I}\rangle = 0.\end{aligned}$$

$\boxed{\text{Mystery on } c_0\mathcal{I}}$ [Rastelli-Zwiebach]

Formally,

$$c_0 A = c_0(\mathcal{I} * A) = (c_0\mathcal{I}) * A + \mathcal{I} * (c_0 A) = (c_0\mathcal{I}) * A + c_0 A,$$
$$\therefore\ (c_0\mathcal{I}) * A = 0, \quad \forall A.$$

If we take $A = \mathcal{I}$, $0 \neq c_0\mathcal{I} = (c_0\mathcal{I}) * \mathcal{I} = 0.$ (??) $\leftarrow$ inconsistent!

- Oscillator language (resolved ?)

$$_3\langle\mathcal{I}|c_0^{(3)}|V_3\rangle_{123} = 0 \ \Rightarrow\ |(c_0\mathcal{I}) \star A\rangle = 0,\ \forall|A\rangle \quad \therefore |(c_0\mathcal{I}) \star \mathcal{I}\rangle = 0.$$

  But, in this case

$$|(c_0\mathcal{I}) \star \mathcal{I}\rangle \neq |c_0\mathcal{I}\rangle(\neq 0) \quad \because\ _3\langle\mathcal{I}|V_3\rangle_{123} \neq |1,2\rangle.$$

  No inconsistency!

- CFT language (unresolved)

  c_0 might not be derivation on $\mathcal{I}$(?) or $c_0\mathcal{I}$ would be ill-defined.