Idempotency Equation and Boundary States in Closed String Field Theory

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[KMW1] I.K., Y. Matsuo, E. Watanabe, PRD68 (2003) 126006

[KMW2] I.K., Y. Matsuo, E. Watanabe, PTP111 (2004) 433

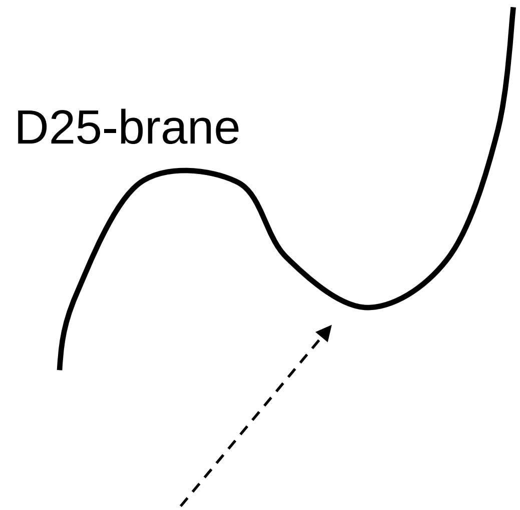
[KM] I.K., Y. Matsuo, PLB590(2004)303

Introduction

- Sen's conjecture:

Witten's open SFT

∃ tachyon vacuum



- Vacuum String Field Theory (VSFT)

[Rastelli-Sen-Zwiebach(2000)]

D-brane

~ Projector with respect to Witten's $*$ product.

(Sliver, Butterfly,…)

D-brane ~ Boundary state $\leftarrow$ closed string

Closed SFT description is more natural (!?)

$$S = \frac{1}{2}\Psi \cdot Q\Psi + \frac{1}{3}\Psi \cdot \Psi * \Psi$$

$$|\Xi\rangle * |\Xi\rangle = |\Xi\rangle$$

↓

$$S = \frac{1}{2}\Phi \cdot Q\Phi + \frac{1}{3}\Phi \cdot \Phi * \Phi \ (+\cdots)$$

HIKKO cubic CSFT (Nonpolynomial CSFT)

$$|B\rangle * |B\rangle = |B\rangle \ (?)$$

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Star product in closed SFT

$*$ product is defined by 3-string vertex:

$$|\Phi_1 * \Phi_2\rangle_3 = {}_1\langle\Phi_1|_2\langle\Phi_2|V(1,2,3)\rangle$$

- HIKKO (Hata-Itoh-Kugo-Kunitomo-Ogawa) type

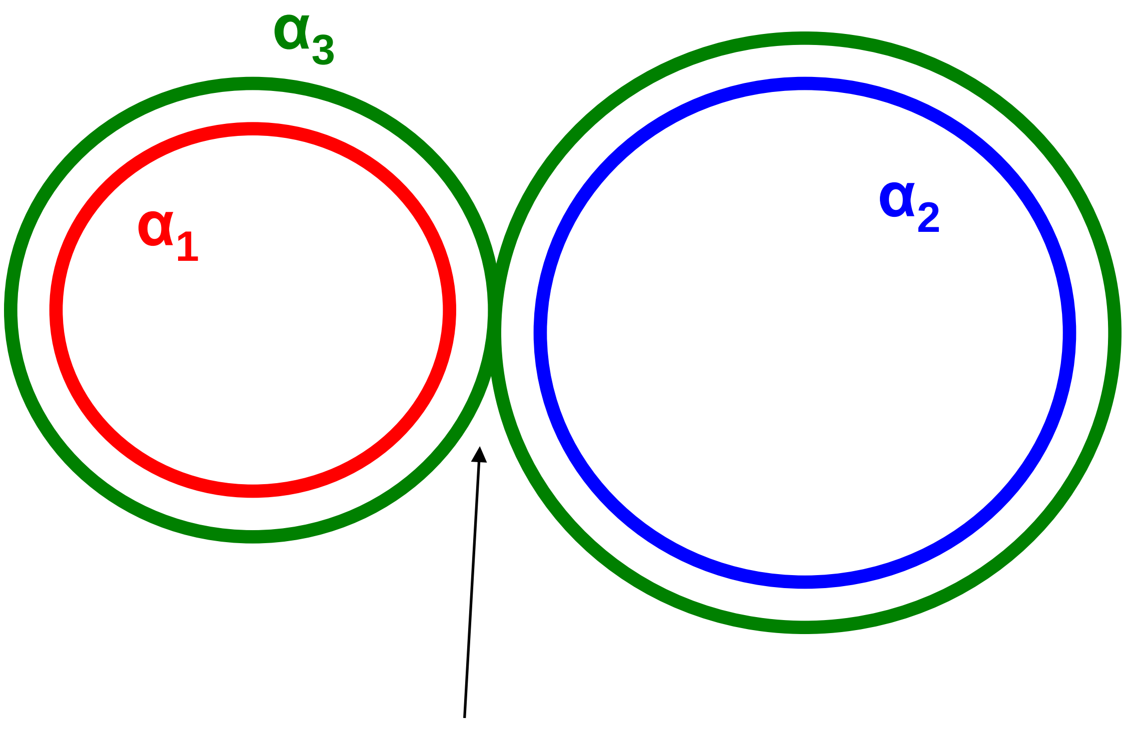
$$(X^{(3)} - \Theta_1 X^{(1)} - \Theta_2 X^{(2)})|V_0(1,2,3)\rangle = 0$$

and ghost sector (to be compatible with BRST invariance)

with projection:

$$|V(1,2,3)\rangle = \wp_1\wp_2\wp_3|V_0(1,2,3)\rangle, \quad \wp_r := \oint\frac{d\theta}{2\pi}e^{i\theta(L_0^{(r)}-\tilde{L}_0^{(r)})}$$

Overlapping condition for 3 closed strings



Interaction point

- <u>Explicit</u> representation of the 3-string vertex:

  solution to overlapping condition [HIKKO]

$$|V(1,2,3)\rangle = \int \delta(1,2,3)[\mu(1,2,3)]^2 \wp_1\wp_2\wp_3 \frac{\alpha_1\alpha_2}{\alpha_3}\Pi_c\,\delta\Big(\sum_{r=1}^3 \alpha_r^{-1}\pi_c^{0(r)}\Big)$$

$$\times\prod_{r=1}^3\Big[1+2^{-\frac{1}{2}}w_I^{(r)}\bar{c}_0^{(r)}\Big]e^{F(1,2,3)}|p_1,\alpha_1\rangle_1|p_2,\alpha_2\rangle_2|p_3,\alpha_3\rangle_3$$

$$F(1,2,3) = \sum_{r,s=1}^3\sum_{m,n\geq 1}\tilde{N}^{rs}_{mn}\Big[\frac{1}{2}a_m^{(r)\dagger}a_n^{(s)\dagger} + \sqrt{m}\alpha_r c_{-m}^{(r)}(\sqrt{n}\alpha_s)^{-1}b_{-n}^{(s)}$$

$$+\frac{1}{2}\tilde{a}_m^{(r)\dagger}\tilde{a}_n^{(s)\dagger} + \sqrt{m}\alpha_r\tilde{c}_{-m}^{(r)}(\sqrt{n}\alpha_s)^{-1}\tilde{b}_{-n}^{(s)}\Big]$$

$$+\frac{1}{2}\sum_{r=1}^3\sum_{n\geq 1}\tilde{N}_n^r(a_n^{(r)\dagger}+\tilde{a}_n^{(r)\dagger})\mathrm{P} - \frac{\tau_0}{4\alpha_1\alpha_2\alpha_3}\mathrm{P}^2$$

(Gaussian !)

$\tilde{N}^{rs}_{mn}, \quad \tilde{N}^{r}_{n}$: Neumann coefficients of light-cone type

$$\tilde{N}^{rs}_{mn} = \frac{mn\alpha_1\alpha_2\alpha_3}{\alpha_r n + \alpha_s m}\tilde{N}^r_m \tilde{N}^s_n,$$

$$\tilde{N}^r_m = \frac{\sqrt{m}}{\alpha_r m!}\frac{\Gamma(-m\alpha_{r+1}/\alpha_r)}{\Gamma(1+m\alpha_{r-1}/\alpha_r)}e^{\frac{m\tau_0}{\alpha_r}}, \qquad \tau_0 = \sum_{r=1}^{3}\alpha_r \log|\alpha_r|$$

We can prove various relations. [Mandelstam, Green-Schwarz,…]

$$\sum_{t=1}^{3}\sum_{p=1}^{\infty}\tilde{N}^{rt}_{mp}\tilde{N}^{ts}_{pn} = \delta_{r,s}\delta_{m,n}, \quad \sum_{t=1}^{3}\sum_{p=1}^{\infty}\tilde{N}^{rt}_{mp}\tilde{N}^{t}_{p} = -\tilde{N}^r_m,$$

$$\sum_{t=1}^{3}\sum_{p=1}^{\infty}\tilde{N}^{t}_{p}\tilde{N}^{t}_{p} = \frac{2\tau_0}{\alpha_1\alpha_2\alpha_3}, \quad \cdots$$

[Yoneya(1987)]

Star product of boundary state

The boundary state for Dp-brane with constant flux:

$$
\begin{aligned}
|B(x^\perp)\rangle &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\mathcal{O}\tilde{\alpha}_{-n}+\sum_{n=1}^{\infty}(c_{-n}\tilde{b}_{-n}+\tilde{c}_{-n}b_{-n})\right) \\
&\quad \times c_0^+ c_1 \tilde{c}_1 |p^\parallel = 0, x^\perp\rangle \otimes |0\rangle_{gh}, \\
\mathcal{O}^\mu{}_\nu &= \left[(1+F)^{-1}(1-F)\right]^\mu{}_\nu, \quad \mu,\nu = 0,1,\cdots,p, \\
\mathcal{O}^i{}_j &= -\delta^i_j, \qquad\qquad i,j = p+1,\cdots,d-1.
\end{aligned}
$$

↓

We *define* the string field $\Phi_B(x^\perp,\alpha)$:

$$
|\Phi_B(x^\perp,\alpha)\rangle = c_0^- b_0^+ |B(x^\perp)\rangle \otimes |\alpha\rangle
$$

$|\Phi_B(x^\perp, \alpha)\rangle$ and $|V(1,2,3)\rangle$ are "Gaussian." $\mathcal{O}$ is orthogonal.
Using *Yoneya formula* for Neumann matrices, we have obtained

$$|\Phi_B(x^\perp, \alpha_1)\rangle * |\Phi_B(y^\perp, \alpha_2)\rangle = \delta(x^\perp - y^\perp)\mathcal{C}c_0^+|\Phi_B(x^\perp, \alpha_1 + \alpha_2)\rangle$$

"idempotency equation"

$\mathcal{C}$ is given by

$$\mathcal{C} \;=\; [\mu(1,2,3)]^2[\det(1 - (\tilde{N}^{33})^2)]^{-\frac{d-2}{2}}$$

where $\mu(1,2,3) = e^{-\tau_0 \sum_{r=1}^{3} \alpha_r^{-1}}$.

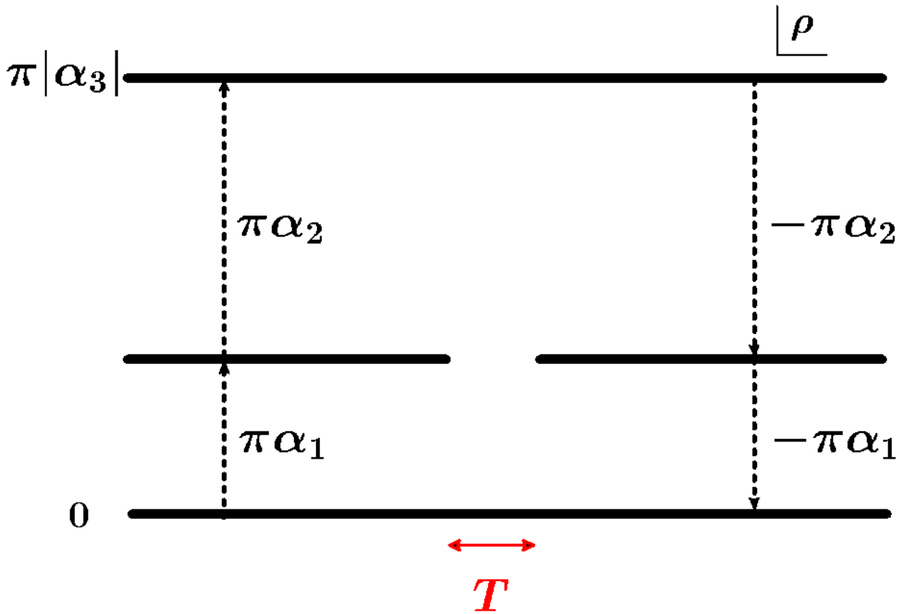
$\mathcal{C}$ is divergent because $\tilde{N}^{33}_{mn}$ is $\infty \times \infty$ matrix.

However, by *regularizing* with parameter T:

$$\tilde{N}^{33}_{mn} \to \tilde{N}^{33}_{mn} e^{-(m+n)\frac{T}{|\alpha_3|}}$$

$\mathcal{C}$ can be simplified drastically for $d = 26$.

We use *Cremmer-Gervais identity* to evaluate the regularized $\mathcal{C}$.



The result is $\mathcal{C} = 2^5\, T^{-3}|\alpha_1\alpha_2\alpha_3|$ for $T \to +0$.

On the other hand, we have computed $\mathcal{C}$ numerically by truncating the size of $\tilde{N}^{33}_{mn}$ to L. We have observed $\mathcal{C} \sim L^3|(\alpha_1/\alpha_3)(\alpha_2/\alpha_3)|$, therefore, $T \sim |\alpha_3|/L$.

Idempotency equation

$$|\Phi(\alpha_1)\rangle * |\Phi(\alpha_2)\rangle = K^3 \hat{\alpha}^2 c_0^+ |\Phi(\alpha_1+\alpha_2)\rangle$$

where $c_0^+ = \frac{1}{2}(c_0 + \tilde{c}_0)$,
$K(\sim T^{-1} \to \infty)$: constant and $\alpha_1\alpha_2 > 0$

$\hat{\alpha}^2 c_0^+$ is a "pure ghost" BRST operator which is nilpotent, partial integrable and derivation with respect to $*$ product.

The boundary state which corresponds to Dp-brane is a solution to this equation *in the following sense*.

- Boundary state as an “idempotent” :

$$|\Phi_f(\alpha)\rangle \;=\; \int d^{d-p-1}x^{\perp}\, f(x^{\perp})\, |\Phi_B(x^{\perp},\alpha)\rangle/\alpha$$

$f(x^{\perp})$ is a solution to $f(x^{\perp})^2 = f(x^{\perp})$.

Namely, “commutative soliton” $f(x^{\perp}) = \begin{cases} 1 & (x^{\perp} \in \Sigma) \\ 0 & (\text{otherwise}) \end{cases}$

for some subset Σ of R^{d-p-1}.

↓

$$|\Phi_f(\alpha_1)\rangle * |\Phi_f(\alpha_2)\rangle = K^3 \hat{\alpha}^2 c_0^{\dagger} |\Phi_f(\alpha_1+\alpha_2)\rangle$$

- “Non-associative” product for coefficient functions in “non-commutative” background.

KT operator: [Kawano-Takahashi (1999)]

$$V_{\theta,\sigma_c} = \exp\left(-\frac{i}{4}\oint d\sigma \oint d\sigma' P_i(\sigma)\theta^{ij}\epsilon(\sigma-\sigma')P_j(\sigma')\right)$$
$$:= \exp\left(-\frac{i}{4}\int_{\sigma_c}^{2\pi+\sigma_c} d\sigma \int_{\sigma_c}^{2\pi+\sigma_c} d\sigma' P_i(\sigma)\theta^{ij}\epsilon(\sigma,\sigma')P_j(\sigma')\right)$$

Note: there is an identity cf. [Murakami-Nakatsu(2002)]

$$\hat{V}_{\theta,\sigma_c}|p\rangle\rangle_D \;=\; V_p(\sigma_c)|B(F_{ij}=-(\theta^{-1})_{ij})\rangle$$

$V_p(\sigma_c)$: Tachyon vertex at σ_c,
$|p\rangle\rangle_D$: Dirichlet type Ishibashi state,
$|B(F)\rangle$: Neumann boundary state ($p=0$).

In the Seiberg-Witten limit: $\alpha' \sim \epsilon^{1/2}$, $g_{ij} \sim \epsilon$,

$$\hat{V}_{\theta,\sigma_c}|p_1\rangle\rangle_{D,\alpha_1} * \hat{V}_{\theta,\sigma_c}|p_2\rangle\rangle_{D,\alpha_2}$$

$$\sim \det{}^{-\frac{d}{2}}(1-(\tilde{N}^{33})^2)\oint\frac{d\sigma_1}{2\pi}\oint\frac{d\sigma_2}{2\pi}e^{i\Theta_{12}}\hat{V}_{\theta,\sigma_c}|p_1+p_2\rangle\rangle_{D,\alpha_1+\alpha_2}$$

$$_{\alpha_1+\alpha_2}\langle x|\int dy f_{\alpha_1}(y)\hat{V}_{\theta,\sigma_c}|B(y)\rangle_{\alpha_1} * \int dy' g_{\alpha_2}(y')\hat{V}_{\theta,\sigma_c}|B(y')\rangle_{\alpha_2}$$

$$\sim [\det{}^{-\frac{d}{2}}(1-(\tilde{N}^{33})^2)\,2\pi\delta(0)]\, f_{\alpha_1}(x)\diamond_\beta g_{\alpha_2}(x)$$

where

$$f_{\alpha_1}(x)\diamond_\beta g_{\alpha_2}(x)$$

$$= f_{\alpha_1}(x)\frac{\sin(-\beta\lambda)\sin((1+\beta)\lambda)}{(-\beta)(1+\beta)\lambda^2}g_{\alpha_2}(x) \qquad \left(\beta=\frac{-\alpha_1}{\alpha_1+\alpha_2},\quad \lambda=\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x^i}\theta^{ij}\frac{\overrightarrow{\partial}}{\partial x^j}\right)$$

$$= f_{\alpha_1}(x)\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}}{(2k+1)!}\sum_{l=0}^{k}\frac{(1+2\beta)^{2l}}{k+1}g_{\alpha_2}(x).$$

In particular, one of the two α-parameter becomes zero, this induced product becomes *Strachan product* :

$$f(x) \diamond_\beta g(x)|_{\beta=0 \text{ or } -1} = f(x)\frac{\sin\lambda}{\lambda}g(x)$$

which is also called the generalized star product: $*_2$.

By taking the Laplace transformation with an ansatz: $f_\alpha(x) = \alpha^{\delta-1}f(x)$ the idempotency equation is reduced to

$$f(x)\frac{\sin\lambda}{\lambda}f(x) = f(x)$$

Projector eq. with respect to the Strachan product which is commutative and non-associative.

Feature of the HIKKO $*$ product

Fluctuations

Infinitesimal deformation of "idempotency equation"
around $\Phi_B(x^\perp, \alpha)$:

cf. [Hata-Kawano(2001)]

$$\delta\Phi_B(x^\perp, \alpha_1) * \Phi_B(y^\perp, \alpha_2) + \Phi_B(x^\perp, \alpha_1) * \delta\Phi_B(y^\perp, \alpha_2)$$
$$= \delta^{d-p-1}(x^\perp - y^\perp)\mathcal{C}c_0^+ \delta\Phi_B(x^\perp, \alpha_1 + \alpha_2) \,.$$

Ansatz: $\delta\Phi_B(x^\perp, \alpha) = \oint \frac{d\sigma}{2\pi} V(\sigma)\Phi_B(x^\perp, \alpha)$

By *straightforward computation in oscillator language*, we found scalar and vector type "solutions":

$$V_S(\sigma) =: e^{ik_\mu X^\mu(\sigma)} :, \qquad k_\mu G^{\mu\nu} k_\nu = \alpha'^{-1},$$

$$V_V(\sigma) =: \zeta_\nu \partial_\sigma X^\nu e^{ik_\mu X^\nu(\sigma)} :, \quad k_\mu G^{\mu\nu} k_\nu = 0,$$

$$(G^{\mu\nu} = [(1+F)^{-1}\eta(1-F)^{-1}]^{\mu\nu} : \text{ open string metric}).$$

In computation of tachyon mass using Neumann coefficients, we enconunter

$$k_\mu G^{\mu\nu} k_\nu \left(\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{m=1}^{\infty} \frac{1}{m} \right)$$

at least naively. $\to$ *regularization*

By truncating the level of string r as is proportional to $|\alpha_r|$, we obtain on-shell condition uniquely:

$$(-\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu} + (1+\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu} = 1 \text{ for } V_S$$

where $\beta = \alpha_1/\alpha_3$

$\to$ *open* string tachyon: $k_\mu G^{\mu\nu} k_\nu = \alpha'^{-1}$.

For vector type fluctuation $\delta_V\Phi_B$, we compute

$$
\begin{aligned}
& |\delta_V\Phi_B(\alpha_1)\rangle * |\Phi_B(\alpha_2)\rangle + |\Phi_B(\alpha_1)\rangle * |\delta_V\Phi_B(\alpha_2)\rangle \\
= \; & ((-\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu + 1} + (1+\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu + 1}) \mathcal{C} c_0^+ |\delta_V\Phi_B(\alpha_1+\alpha_2)\rangle \\
& + ((-\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu} - (1+\beta)^{\alpha' k_\mu G^{\mu\nu} k_\nu}) \\
& \quad \times \left[-i\zeta_\mu G^{\mu\nu} k_\nu \sum_{p=1}^{\infty} \frac{\sin^2 p\pi\beta}{\pi p} \mathcal{C} c_0^+ |\delta_S\Phi_B(\alpha_1+\alpha_2)\rangle + \cdots \right].
\end{aligned}
$$

We obtain massless condition $k_\mu G^{\mu\nu} k_\nu = 0$.

However, the transversality condition is subtle because $((-\beta)^0 - (1+\beta)^0) \sum_{p=1}^{\infty} \frac{\sin^2 \pi p\beta}{\pi p} \sim 0 \times \infty$.

On the other hand, using LPP formulation for the HIKKO closed SFT, the equation for the fluctuation is reduced to

$$\wp\left(\oint \frac{d\sigma_1}{2\pi}\Sigma_1[V(\sigma_1)] + \oint \frac{d\sigma_2}{2\pi}\Sigma_2[V(\sigma_2)] + \oint \frac{d\sigma_3}{2\pi}V(\sigma_3)\right)|B(x^\perp)\rangle = 0.$$

cf.[Okawa(2002)]

A *sufficient* condition for this solution : primary with weight 1

$$\Sigma_r[V(\sigma_r)]\,|B(x^\perp)\rangle = \frac{d}{d\sigma_r}\Sigma_r(\sigma_r)\,V(\Sigma_r(\sigma_r))\,|B(x^\perp)\rangle\,.$$

→ *open* string spectrum!

However, Σ_r is a particular mapping.

Is this a *necessary* condition?

By modifying the vector type fluctuation [Murakami-Nakatsu(2002)] :

$$V_S(\sigma) = :e^{ik_\mu X^\mu(\sigma)}:, \quad V_V(\sigma) = :\zeta_\mu \partial_\sigma X^\mu(\sigma) e^{ik_\nu X^\nu(\sigma)}:,$$
$$\hat{V}_V(\sigma) \equiv V_V(\sigma) - (\zeta_\mu \theta^{\mu\nu} k_\nu / 4\pi) V_S(\sigma),$$

where $\theta \equiv \pi(\mathcal{O} - \mathcal{O}^T)/2 = -2\pi(1+F)^{-1}F(1-F)^{-1}$,

we obtain the finite transformation

$$(d\sigma)^\Delta V_S(\sigma)|B(x^\perp)\rangle\rangle = (d\lambda)^\Delta V_S(\lambda)|B(x^\perp)\rangle\rangle,$$
$$(d\sigma)^{\Delta+1}\hat{V}_V(\sigma)|B(x^\perp)\rangle\rangle = (d\lambda)^{\Delta+1}\left[\hat{V}_V(\lambda)|B(x^\perp)\rangle\rangle - \Xi \frac{\partial_\lambda^2 \sigma}{\partial_\lambda \sigma} V_S(\lambda)|B(x^\perp)\rangle\rangle\right],$$

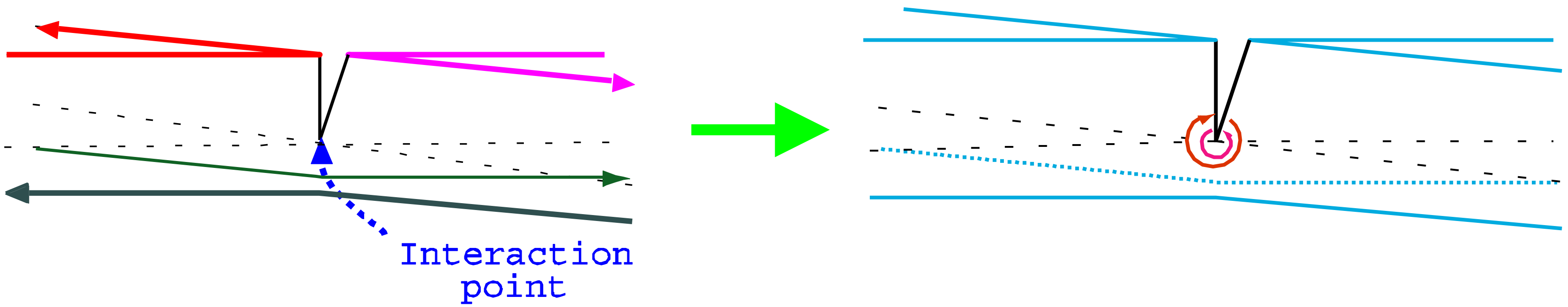
where

$$\Delta \equiv \alpha' k_\mu G^{\mu\nu} k_\nu, \qquad \Xi \equiv -i\zeta_\mu G^{\mu\nu} k_\nu / 2.$$

Σ_1, Σ_2 are linear mappings

$\rightarrow \quad \Delta = 1$ for V_S and $\Delta = 0$ for $\hat{V}_V$.

We should note the *singularity* at the interaction point for $\hat{V}_V$.



Around the interaction point for $\Delta = 0$

$$d\sigma \hat{V}_V(\sigma)|B(x^\perp)\rangle$$
$$= dz\left[\hat{V}_V(z)|B(x^\perp)\rangle - \Xi\left((z-z_0)^{-1} + \mathcal{O}((z-z_0)^0)\right) V_S(z)|B(x^\perp)\rangle\right]$$

$\rightarrow$

$$\wp\left(\oint \frac{d\sigma_1}{2\pi}\Sigma_1[V(\sigma_1)] + \oint \frac{d\sigma_2}{2\pi}\Sigma_2[V(\sigma_2)] + \oint \frac{d\sigma_3}{2\pi}V(\sigma_3)\right)|B(x^\perp)\rangle$$
$$= i\wp\,\Xi\, V_S(z_0)|B(x^\perp)\rangle = i\,\Xi \oint \frac{d\sigma}{2\pi} V_S(\sigma)|B(x^\perp)\rangle$$

$\rightarrow$ the transversality condition $2i\Xi = \zeta_\mu G^{\mu\nu} k_\nu = 0$ is imposed.

Correct open string spectrum!

Cardy states and idempotents

- On the flat (R^d) background, we have $*$ product formula for *Ishibashi states* :

$$|p_1^\perp\rangle\rangle_{\alpha_1} * |p_2^\perp\rangle\rangle_{\alpha_2} = \mathcal{C} c_0^+ |p_1^\perp + p_2^\perp\rangle\rangle_{\alpha_1+\alpha_2}.$$

$|p^\perp\rangle\rangle$ satisfies $(L_n - \tilde{L}_{-n})|p^\perp\rangle\rangle = 0$, but is *not* an idempotent. Its *Fourier transform* $|B(x^\perp)\rangle\rangle$ which is a Cardy state gives an idempotent.

Conjecture

Cardy states ～ idempotents in closed SFT

even on nontrivial backgrounds.

Cardy states $|B\rangle$:

1. $(L_n - \tilde{L}_{-n})|B\rangle = 0.$
2. $\langle B|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|B'\rangle = \sum_i N^i_{BB'}\chi_i(q),$

   $N^i_{BB'}$:nonnegative integer.

$\Updownarrow$

Closed SFT:

1. $(L_n - \tilde{L}_{-n})|B\rangle = 0, \quad (L_n - \tilde{L}_{-n})|B'\rangle = 0,$

   $\rightarrow \quad (L_n - \tilde{L}_{-n})|B\rangle * |B'\rangle = 0.$
2. idempotency: $|B\rangle * |B'\rangle = \delta_{B,B'}\,\mathcal{C}\,|B\rangle.$

- Orbifold（M/Γ）

twisted sector: $X(\sigma + 2\pi) = gX(\sigma) \quad (g \in \Gamma)$

$(g\text{-twisted}) * (g'\text{-twisted}) \sim (gg'\text{-twisted})$

$\rightarrow$ $*$ product of Ishibashi states should be

$$|g\rangle\rangle_{\alpha_1} * |g'\rangle\rangle_{\alpha_2} \sim |gg'\rangle\rangle_{\alpha_1+\alpha_2}$$

$\updownarrow$

Group ring $C^{[\Gamma]}$: $\sum_{g\in\Gamma} \lambda_g e_g \in C^{[\Gamma]}, \ \lambda_g \in C$

$$e_g \star e_{g'} = e_{gg'}$$

Γ:nonabelian $\quad e_g \;\to\; e_i = \sum_{g \in \mathcal{C}_i} e_g \quad$ ($\mathcal{C}_i$: conjugacy class).

Formula: $\quad e_i \star e_j = \mathcal{N}_{ij}^{\;k} e_k$

$$\mathcal{N}_{ij}^{\;k} = \frac{1}{|\Gamma|} \sum_{\alpha:\text{irreps.}} \frac{|\mathcal{C}_i||\mathcal{C}_j| \zeta_i^{(\alpha)} \zeta_j^{(\alpha)} \zeta_k^{(\alpha)*}}{\zeta_1^{(\alpha)}} . \quad (\zeta_i^{(\alpha)} : \text{character})$$

idempotents: $P^{(\alpha)} = \frac{\zeta_1^{(\alpha)}}{|\Gamma|} \sum_{i:\text{class}} \zeta_i^{(\alpha)} e_i, \quad P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha,\beta} P^{(\beta)}.$

$\Updownarrow$

Cardy states: $|\alpha\rangle = \frac{1}{\sqrt{|\Gamma|}} \sum_{i:\text{class}} \zeta_i^{(\alpha)} \sqrt{\sigma_i} |i\rangle\rangle, \quad |i\rangle\rangle := \sum_{g \in \mathcal{C}_i} |g\rangle\rangle$,

[cf. Billo et al.(2001)]

$$\sigma_i = \sigma(e, g), g \in \mathcal{C}_i, \quad \chi_h^g(q) = \mathrm{Tr}_{\mathcal{H}_h}(g q^{L_0 - \frac{c}{24}}) = \sigma(h, g) \chi_g^{h^{-1}}(\tilde{q})$$

$\to \; |\alpha\rangle$: idempotents in closed SFT (?)

- Fusion ring of RCFT

$$e_i \star e_j = N_{ij}^{\ k} e_k, \quad N_{ij}^{\ k} = \sum_l \frac{S_{il} S_{jl} S^*_{kl}}{S_{1l}}$$

[Verlinde(1988)]

idempotents: $P^{(\alpha)} = S^*_{1\alpha} \sum_{i:\text{primary}} S_{i\alpha} e_i, \quad P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha,\beta} P^{(\beta)}.$

[T.Kawai (1989)]

↕

Cardy states: $|\alpha\rangle = \sum_{i:\text{primary}} \frac{S_{\alpha i}}{\sqrt{S_{1i}}} |i\rangle\rangle$

Suppose $|i\rangle\rangle_{\alpha_1} * |j\rangle\rangle_{\alpha_2} \sim N_{ij}^{\ k} |k\rangle\rangle_{\alpha_1+\alpha_2}$,

then Cardy states $|\alpha\rangle \sim$ idempotents in closed SFT

$T^D, T^D/Z_2$ compactification

Explicit formulation of closed SFT on $T^D, T^D/Z_2$ is known. [HIKKO(1987), Itoh-Kunitomo(1988)]

3-string vertex is modified: cf. [Maeno-Takano(1989)]

$$(-1)^{p_2 w_2 - p_1 w_3} |V_0(1_u, 2_u, 3_u)\rangle,$$

$$(-1)^{p_1 n_3^f} \delta([n_3^f - n_2^f + w_1]) |V_0(1_u, 2_t, 3_t)\rangle$$

- cocycle factor $\leftarrow$ Jacobi identity,
- matter zero mode part.
- untwisted-twisted-twisted : different Neumann coefficients $\tilde{T}^{rs}_{n_r n_s}$,
- $\mathbf{Z}_2$ projection

We can compute $*$ product of Ishibashi states directly.

Ishibashi states:

$$|\iota(\mathcal{O},p,w)\rangle\rangle_u = e^{-\sum_{n=1}^{\infty}\frac{1}{n}\alpha^i_{-n}G_{ij}\mathcal{O}^j{}_k\tilde{\alpha}^k_{-n}}|p,w\rangle,$$
$$|\iota(\mathcal{O},n^f)\rangle\rangle_t = e^{-\sum_{r=1/2}^{\infty}\frac{1}{r}\alpha^i_{-r}G_{ij}\mathcal{O}^j{}_k\tilde{\alpha}^k_{-r}}|n^f\rangle,$$

$\mathcal{O}^T G\mathcal{O} = G$; p_i, w^j:integers such as $p_i = -F_{ij}w^j$,
$F = -(G+B-(G-B)\mathcal{O})(1+\mathcal{O})^{-1}$; $(n^f)^i = 0,1$: fixed point.

∗ products of these states are not diagonal.
→ We consider following linear combinations:

Dirichlet type ($\mathcal{O} = -1$)

$$|n^f\rangle_u := (\det(2G_{ij}))^{-\frac{1}{4}}\sum_{p_i}(-1)^{p\,n^f}|\iota(-1,p,0)\rangle\rangle_u,$$
$$|n^f\rangle_t := |\iota(-1,n^f)\rangle\rangle_t.$$

Neumann type ($\mathcal{O} \neq -1$)

$$|m^f,F\rangle_u := \left(\det(2G_O^{-1})\right)^{-\frac{1}{4}}\sum_{w}(-1)^{w\,m^f+wF_uw}|\iota(\mathcal{O},-Fw,w)\rangle\rangle_u,$$
$$|m^f,F\rangle_t := 2^{-\frac{D}{2}}\sum_{n^f\in\{0,1\}^D}(-1)^{m^f n^f+n^f F_u n^f}|\iota(\mathcal{O},n^f)\rangle\rangle_t,$$

where $(m^f)^i = 1,0$, $G_O^{-1} = (G+B+F)^{-1}G(G-B-F)^{-1}$.

∗ product (Dirichlet type)

$$|n_1^f, x^\perp, \alpha_1\rangle_u * |n_2^f, y^\perp, \alpha_2\rangle_u$$
$$= \left(\det(2G_{ij})\right)^{-\frac{1}{4}} (2\pi)^D \delta^D(0) \delta^D_{n_1,n_2} \delta^{d-p-1}(x^\perp - y^\perp)$$
$$\times\, \mu_u^2 \det{}^{-\frac{d+D-2}{2}}(1-(\tilde{N}^{33})^2) c_0^+ |n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_u,$$
$$|n_1^f, x^\perp, \alpha_1\rangle_u * |n_2^f, y^\perp, \alpha_2\rangle_t$$
$$= \left(\det(2G_{ij})\right)^{-\frac{1}{4}} (2\pi)^D \delta^D(0) \delta^D_{n_1,n_2} \delta^{d-p-1}(x^\perp - y^\perp)$$
$$\times\, \mu_t^2 \det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_t 3_t})^2) \det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2) c_0^+ |n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_t,$$
$$|n_1^f, x^\perp, \alpha_1\rangle_t * |n_2^f, y^\perp, \alpha_2\rangle_t$$
$$= \left(\det(2G_{ij})\right)^{\frac{1}{4}} \delta^D_{n_1,n_2} \delta^{d-p-1}(x^\perp - y^\perp)$$
$$\times\, \mu_t^2 \det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_u 3_u})^2) \det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2) c_0^+ |n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_u.$$

$$\mathcal{C} := \mu_u^2 \det{}^{-\frac{d+D-2}{2}}(1-(\tilde{N}^{33})^2) \quad (\sim |\alpha_1\alpha_2\alpha_3| T^{-3})$$
$$= \mu_t^2 \det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_t 3_t})^2) \det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2),$$

follows from *Cremmer-Gervais identity* for $D+d=26$.

$$\mathcal{C}' := \mu_t^2 \det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_u 3_u})^2) \det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2)$$

cannot be evaluated similarly → *other method*

$$|n^f, x^\perp, \alpha\rangle_\pm = \frac{1}{2}(2\pi\delta(0))^{-D}\left((\det(2G_{ij}))^{\frac{1}{4}}|n^f, x^\perp, \alpha\rangle_u \pm c_t(2\pi\delta(0))^{\frac{D}{2}}|n^f, x^\perp, \alpha\rangle_t\right)$$

are idempotents:

$$|n_1^f, x^\perp, \alpha_1\rangle_\pm * |n_2^f, y^\perp, \alpha_2\rangle_\pm = \delta^D_{n_1^f, n_2^f}\delta^{d-p-1}(x^\perp - y^\perp)\,\mathcal{C}\,c_0^+|n_2^f, y^\perp, \alpha_1+\alpha_2\rangle_\pm,$$

$$|n_1^f, x^\perp, \alpha_1\rangle_\pm * |n_2^f, y^\perp, \alpha_2\rangle_\mp = 0.$$

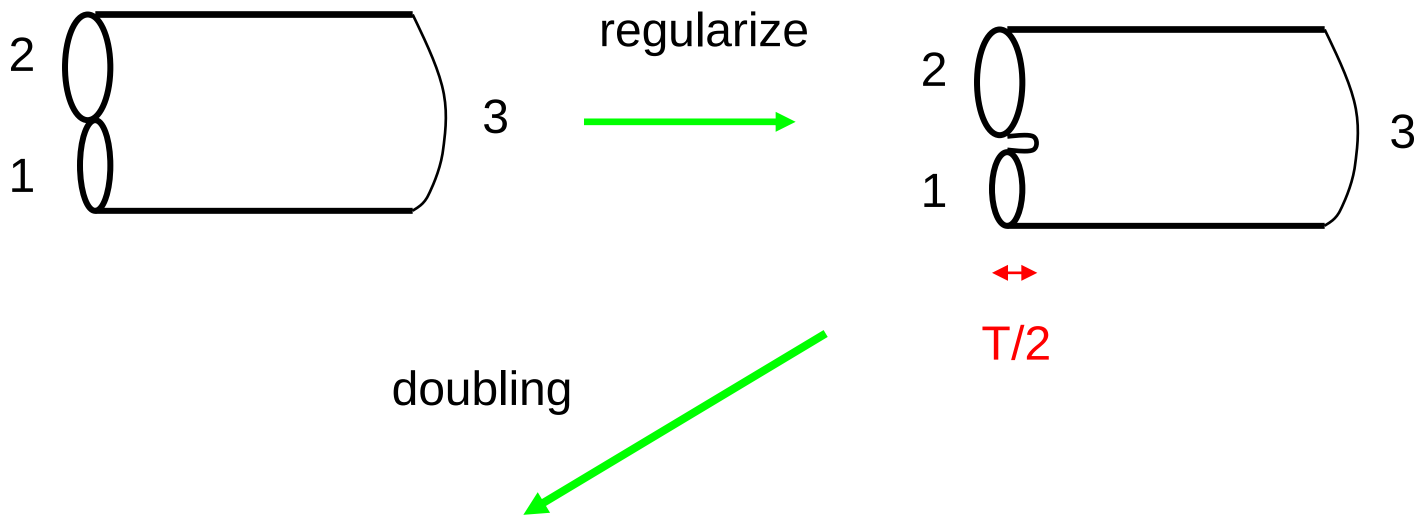
c_t is given by

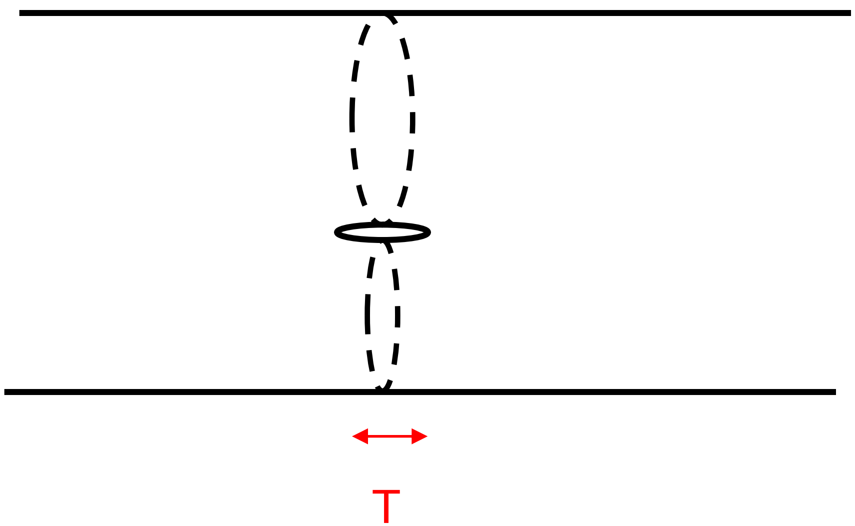
$$c_t = \sqrt{\frac{\mathcal{C}}{\mathcal{C}'}} = \left(e^{-\frac{\tau_0}{4}(\alpha_1^{-1}+\alpha_2^{-1})}\frac{\det(1-(\tilde{T}^{1_u 1_u}(\alpha_3,\alpha_1,\alpha_2))^2)}{\det(1-(\tilde{N}^{33}(\alpha_1,\alpha_2,\alpha_3))^2)}\right)^{\frac{D}{4}},$$

which is evaluated by *1-loop amplitude* as

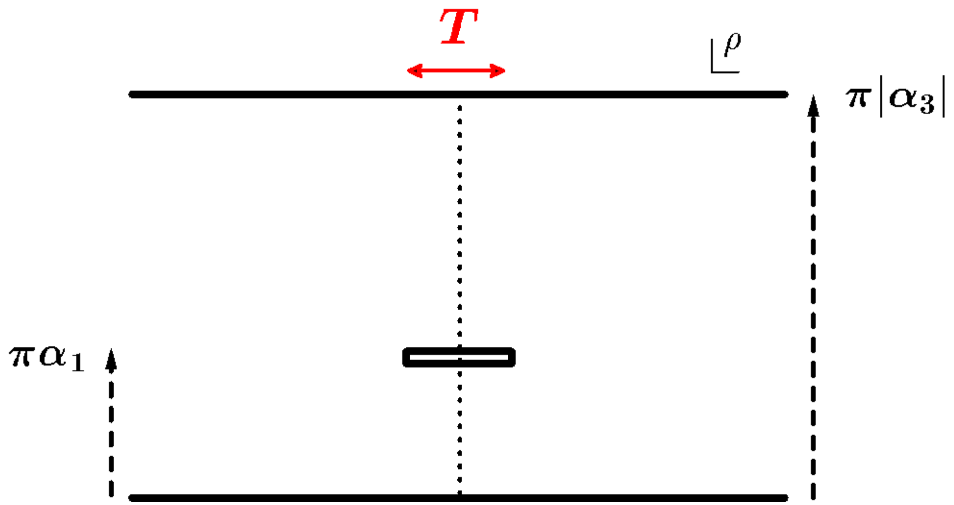
$$c_t(2\pi\delta(0))^{\frac{D}{2}} = 2^{\frac{D}{4}}(\det(2G))^{\frac{1}{4}} = \sqrt{\sigma(e,g)}(\det(2G))^{\frac{1}{4}}.$$

$\rightarrow$ $|n^f, x^\perp, \alpha\rangle_\pm$: Cardy state for fractional D-brane.





∼ doubling of *open string* 1-loop

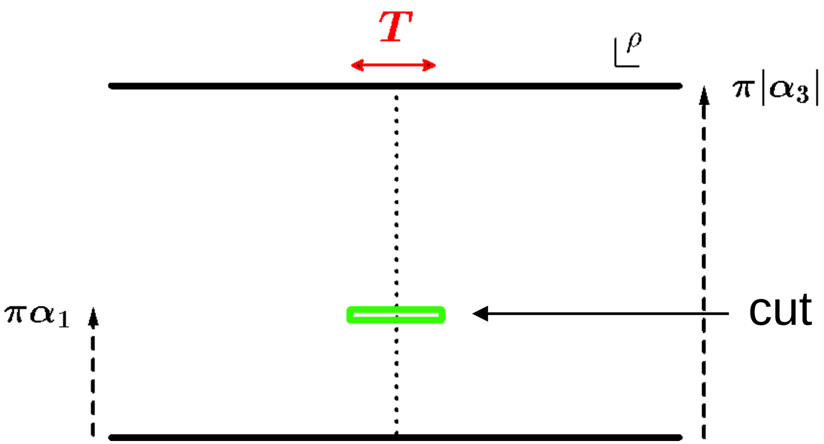


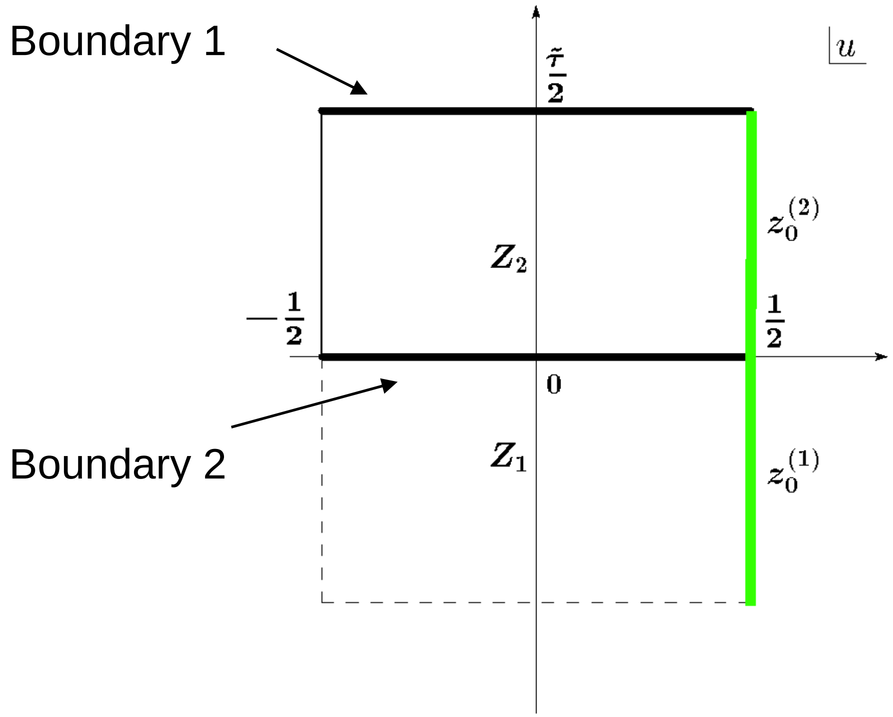
Modulus of torus $\tilde{\tau}$: $\leftarrow$ Mandelstam mapping using ϑ-function

[Asakawa-Kugo-Takahashi(1999)]

$$e^{-\frac{\pi}{|\tilde{\tau}|}} \sim \frac{T}{8|\alpha_3 \sin(\pi\alpha_2/\alpha_3)|} \quad \text{for } T \to 0$$

$\rightarrow$ Including ghost contribution, we reproduce $\mathcal{C} \sim |\alpha_1\alpha_2\alpha_3|T^{-3}$.





Ratio of 1-loop amplitude:

$$\left(\frac{\eta(\tilde{\tau})}{\vartheta_0(\tilde{\tau})}\right)^{\frac{D}{2}}\left((2\pi\delta(0))^{-D}\eta(\tilde{\tau})^{-D}\sum_p e^{i\pi\tilde{\tau}pG^{-1}p/2}\right)^{-1}$$

$$\to\ 2^{-\frac{D}{2}}(2\pi\delta(0))^D\det{}^{-\frac{1}{2}}(2G)=\frac{\mathcal{C}'}{\mathcal{C}}\qquad \tilde{\tau}\to+i0$$

Similarly, we obtain Neumann type idempotents:

$$|m^f, F, \alpha\rangle_{\pm} = \frac{1}{2}\frac{\det^{\frac{1}{4}}(2G_O^{-1})}{(2\pi\delta(0))^D}\left[|m^f, F, x^{\perp}, \alpha\rangle_u \pm 2^{\frac{D}{4}}|m^f, F, x^{\perp}, \alpha\rangle_t\right],$$

$$|m_1^f, F, \alpha_1\rangle_{\pm} * |m_2^f, F, \alpha_2\rangle_{\pm} = \delta^D_{m_1^f, m_2^f}\mathcal{C}\, c_0^+|m_2^f, F, \alpha_1+\alpha_2\rangle_{\pm},$$

$$|m_1^f, F, \alpha_1\rangle_{\pm} * |m_2^f, F, \alpha_2\rangle_{\mp} = 0.$$

（※）Neumann type idempotents are obtained from Dirichlet type by T-duality :

$$\mathcal{U}_g^{\dagger}|n^f, \alpha\rangle_{\pm,E} \;=\; |m^f = n^f, F, \alpha\rangle_{\pm, g(E)}.$$

In fact, we can prove

$$\mathcal{U}_g^{\dagger}|A * B\rangle_E = |(\mathcal{U}_g^{\dagger}A) * (\mathcal{U}_g^{\dagger}B)\rangle_{g(E)}, \quad g = \begin{pmatrix} -F & 1 \\ 1 & 0 \end{pmatrix} \in O(D, D; \mathrm{Z})$$

for both *uuu* and *utt* 3-string vertices. $(E = G + B)$

$\mathcal{U}_g$ is given by *Kugo-Zwiebach's transformation* for the untwisted sector and

$$\mathcal{U}_g^{\dagger}\alpha_r(E)\mathcal{U}_g = -E^{T-1}\alpha_r(g(E)), \quad \mathcal{U}_g^{\dagger}\tilde{\alpha}_r(E)\mathcal{U}_g = E^{-1}\tilde{\alpha}_r(g(E)),$$

$$\mathcal{U}_g^{\dagger}|n^f\rangle_E = 2^{-\frac{D}{2}}\sum_{m^f \in \{0,1\}^D}(-1)^{n^f m^f + m^f F_u m^f}|n^f\rangle_{g(E)},$$

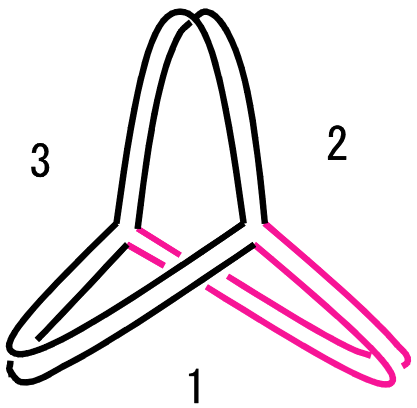
for the twisted sector. $(F_u)_{ij} := F_{ij}\ (i < j),\ 0\ (\text{otherwise}).$

Summary and discussion

- Cardy states satisfy idempotency equation in closed SFT (on $R^D, T^D, T^D/Z_2$).
- Variation around idempotents gives open string spectrum.
- Idempotents ∼ Cardy states

  : detailed correspondence?
- Closed version of VSFT? (Veneziano amplitude,…)
- Relation to the original HIKKO theory?
- More nontrivial background? (other orbifolds,…)
- Supersymmetric extension? (HIKKO's NSR vertex,…)

- 3-string vertex in Nonpolynomial CSFT



← closed string version of Witten's $*$ product

We can also prove idempotency straightforwardly:

$$|\Phi_B(x^\perp)\rangle * |\Phi_B(y^\perp)\rangle = \delta(x^\perp - y^\perp)\mathcal{C}_W c_0^+ b_0^- |\Phi_B(x^\perp)\rangle$$

(Computation is simplified by closed sting version of MSFT. [Bars-Kishimoto-Matsuo])

- n-string vertices (n≧4) in nonpolynomial CSFT?