Cardy states as idempotents of fusion ring in string field theory

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References:

I.K., Yutaka Matsuo, E. Watanabe PRD68(2003)126006 [hep-th/0306189]
I.K., Yutaka Matsuo, E. Watanabe PTP111(2004)433 [hep-th/0312122]
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+ work in progress

Collaboration with Yutaka Matsuo, E. Watanabe, H. Isono (Univ. of Tokyo)

- Cardy states: $(L_n - \tilde{L}_{-n})|B\rangle = 0$

$$\langle B|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|B'\rangle = \sum_i N^i_{BB'}\chi_i(q)$$

- Idempotents in Closed SFT

$$|B\rangle * |B'\rangle = \delta_{B,B'}\,\mathcal{C}\,|B\rangle$$

Introduction

D-brane $\sim$ Boundary state $\leftarrow$ closed string

$$S = \frac{1}{2}\Psi \cdot Q\Psi + \frac{1}{3}\Psi \cdot \Psi * \Psi$$

Witten cubic ***open*** SFT, VSFT

$$|\Xi\rangle * |\Xi\rangle = |\Xi\rangle$$

Projectors (sliver, butterfly,…)

$$\Downarrow$$

$$S = \frac{1}{2}\Phi \cdot Q\Phi + \frac{1}{3}\Phi \cdot \Phi * \Phi \ \left(+\cdots\right)$$

HIKKO cubic ***closed*** SFT (Nonpolynomial CSFT)

$$|B\rangle * |B\rangle = |B\rangle \ (?)$$

Boundary states

$$|\Phi_B(x^\perp,\alpha_1)\rangle * |\Phi_B(y^\perp,\alpha_2)\rangle = \delta(x^\perp - y^\perp)\mathcal{C} c_0^+ |\Phi_B(x^\perp,\alpha_1+\alpha_2)\rangle$$

"idempotency equation" [KMW1]

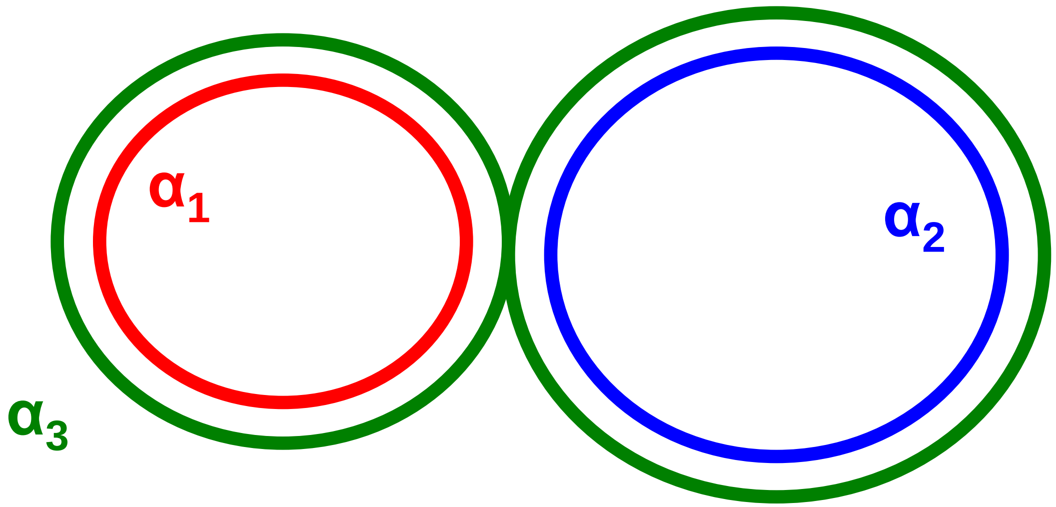
$$\mathcal{C} = [\mu(1,2,3)]^2[\det(1-(\tilde{N}^{33})^2)]^{-\frac{d-2}{2}}$$

Regularization: $\tilde{N}^{33}_{mn} \to \tilde{N}^{33}_{mn} e^{-(m+n)\frac{T}{|\alpha_3|}}$ $\mathcal{C} \sim |\alpha_1\alpha_2\alpha_3| T^{-3}$ for $d = 26$. [KMW2]

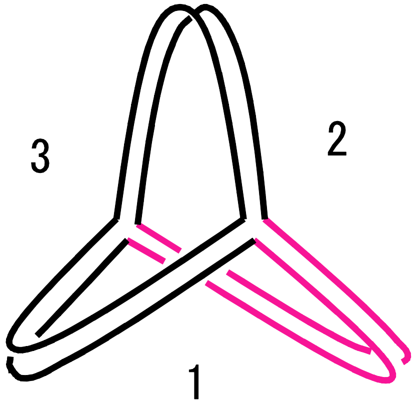
$$|\Phi_B(x^\perp,\alpha)\rangle = c_0^- b_0^+ |B(x^\perp)\rangle \otimes |\alpha\rangle$$

$|B(x^\perp)\rangle$: boundary state for Dp brane

HIKKO closed string $*$ product:



- 3-string vertex in Nonpolynomial CSFT



← closed string version of Witten $*$ product

We can also prove idempotency straightforwardly:

$$|\Phi_B(x^\perp)\rangle * |\Phi_B(y^\perp)\rangle = \delta(x^\perp - y^\perp)\mathcal{C}_W c_0^+ b_0^- |\Phi_B(x^\perp)\rangle$$

(n-string vertices (n≧4) in nonpolynomial CSFT?)

Cardy states and idempotents

- On the flat (R^d) background, we have $*$ product formula for *Ishibashi states* :

$$|p_1^\perp\rangle\rangle_{\alpha_1} * |p_2^\perp\rangle\rangle_{\alpha_2} = \mathcal{C} c_0^+ |p_1^\perp + p_2^\perp\rangle\rangle_{\alpha_1+\alpha_2}.$$

$|p^\perp\rangle\rangle$ satisfies $(L_n - \tilde{L}_{-n})|p^\perp\rangle\rangle = 0,$ but is *not* an idempotent. Its *Fourier transform* $|B(x^\perp)\rangle\rangle$ which is a Cardy state gives an idempotent.

Conjecture

Cardy states $\sim$ idempotents in closed SFT

even on nontrivial backgrounds.

Cardy states $|B\rangle$:

1. $(L_n - \tilde{L}_{-n})|B\rangle = 0.$
2. $\langle B|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-\frac{c}{12})}|B'\rangle = \sum_i N^i_{BB'}\chi_i(q),$
   $N^i_{BB'}$:nonnegative integer.

Closed SFT:

1. $(L_n - \tilde{L}_{-n})|B\rangle = 0, \quad (L_n - \tilde{L}_{-n})|B'\rangle = 0,$
   $\rightarrow \quad (L_n - \tilde{L}_{-n})|B\rangle * |B'\rangle = 0.$
2. idempotency: $|B\rangle * |B'\rangle = \delta_{B,B'}\,\mathcal{C}\,|B\rangle.$

- Orbifold (M/Γ)

twisted sector: $X(\sigma+2\pi)=gX(\sigma) \quad (g\in\Gamma)$

$(g\text{-twisted}) * (g'\text{-twisted}) \sim (gg'\text{-twisted})$

$\rightarrow \quad *$ product of Ishibashi states should be

$$|g\rangle\rangle_{\alpha_1} * |g'\rangle\rangle_{\alpha_2} \sim |gg'\rangle\rangle_{\alpha_1+\alpha_2}$$

Group ring $C^{[\Gamma]}$: $\sum_{g\in\Gamma}\lambda_g e_g \in C^{[\Gamma]}, \ \lambda_g \in C$

$$e_g \star e_{g'} = e_{gg'}$$

Γ:nonabelian $\quad e_g \;\to\; e_i = \sum_{g\in\mathcal{C}_i} e_g \quad$ ($\mathcal{C}_i$: conjugacy class).

Formula: $\quad e_i \star e_j = \mathcal{N}_{ij}{}^{k} e_k$

$$\mathcal{N}_{ij}{}^{k} = \frac{1}{|\Gamma|}\sum_{\alpha:\text{irreps.}} \frac{|\mathcal{C}_i||\mathcal{C}_j|\zeta_i^{(\alpha)}\zeta_j^{(\alpha)}\zeta_k^{(\alpha)*}}{\zeta_1^{(\alpha)}}. \quad (\zeta_i^{(\alpha)} : \text{character})$$

idempotents: $P^{(\alpha)} = \frac{\zeta_1^{(\alpha)}}{|\Gamma|}\sum_{i:\text{class}} \zeta_i^{(\alpha)} e_i, \quad P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha,\beta}P^{(\beta)}.$

orthogonality of characters

↕

Cardy states: $|\alpha\rangle = \frac{1}{\sqrt{|\Gamma|}}\sum_{i:\text{class}} \zeta_i^{(\alpha)}\sqrt{\sigma_i}|i\rangle\rangle, \quad |i\rangle\rangle := \sum_{g\in\mathcal{C}_i} |g\rangle\rangle$,

[cf. Billo et al.(2001)]

$$\sigma_i = \sigma(e,g), g\in\mathcal{C}_i, \quad \chi_h^g(q) = \mathrm{Tr}_{\mathcal{H}_h}(g q^{L_0-\frac{c}{24}}) = \sigma(h,g)\chi_g^{h^{-1}}(\tilde{q})$$

$\to$ $|\alpha\rangle$: idempotents in closed SFT (?)

- <u>Fusion ring of RCFT</u>

$$e_i \star e_j = N_{ij}^{\ \ k} e_k, \quad N_{ij}^{\ \ k} = \sum_l \frac{S_{il} S_{jl} S^*_{kl}}{S_{1l}}$$

[Verlinde(1988)]

idempotents: $P^{(\alpha)} = S^*_{1\alpha} \sum_{i:\text{primary}} S_{i\alpha} e_i, \quad P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha,\beta} P^{(\beta)}.$

[T.Kawai (1989)]

unitarity of S

↕

Cardy states: $|\alpha\rangle = \sum_{i:\text{primary}} \frac{S_{\alpha i}}{\sqrt{S_{1i}}} |i\rangle\rangle$

Suppose $|i\rangle\rangle_{\alpha_1} * |j\rangle\rangle_{\alpha_2} \sim N_{ij}^{\ \ k} |k\rangle\rangle_{\alpha_1+\alpha_2}$,

then Cardy states $|\alpha\rangle \sim$ idempotents in closed SFT

- <u>Comments</u>

General idempotents: $P = \sum_\alpha \epsilon_\alpha P^{(\alpha)},\ \epsilon_\alpha = 0, 1\ :\ P \star P = P$

different from Cardy condition

★ : **associative** in group ring and fusion ring

↕

∗ : **non-associative** in HIKKO *closed* string field theory

→ *associative* among Ishibashi states (on R^D, T^D, T^D/Z_2)

Witten ∗ product: <u>non-commutative</u> in *open* string field theory

→ <u>commutative</u> among wedge states（sliver states）

$T^D, T^D/Z_2$ compactification

Explicit formulation of closed SFT on $T^D, T^D/Z_2$ is known. [HIKKO(1987), Itoh-Kunitomo(1988)]

3-string vertex is modified:

$$(-1)^{p_2 w_2 - p_1 w_3} |V_0(1_u, 2_u, 3_u)\rangle,$$

$$(-1)^{p_1 n_3^f} \delta([n_3^f - n_2^f + w_1]) |V_0(1_u, 2_t, 3_t)\rangle$$

- cocycle factor $\leftarrow$ Jacobi identity,
- matter zero mode part.
- untwisted-twisted-twisted : different Neumann coefficients $\tilde{T}^{rs}_{n_r n_s}$,
- Z_2 projection

We can compute $*$ product of Ishibashi states directly.

Ishibashi states:

$$|\iota(\mathcal{O},p,w)\rangle\rangle_u = e^{-\sum_{n=1}^{\infty}\frac{1}{n}\alpha^i{}_{-n}G_{ij}\mathcal{O}^j{}_k\tilde{\alpha}^k{}_{-n}}|p,w\rangle,$$

$$|\iota(\mathcal{O},n^f)\rangle\rangle_t = e^{-\sum_{r=1/2}^{\infty}\frac{1}{r}\alpha^i{}_{-r}G_{ij}\mathcal{O}^j{}_k\tilde{\alpha}^k{}_{-r}}|n^f\rangle,$$

$\mathcal{O}^T G\mathcal{O} = G$; p_i, w^j:integers such as $p_i = -F_{ij}w^j$,
$F = -(G+B-(G-B)\mathcal{O})(1+\mathcal{O})^{-1}$; $(n^f)^i = 0,1$: fixed point.

∗ products of these states are *not* diagonal.
→ We consider following linear combinations:

Dirichlet type ($\mathcal{O} = -1$)

$$|n^f\rangle_u := (\det(2G_{ij}))^{-\frac{1}{4}}\sum_{p_i}(-1)^{p\,n^f}|\iota(-1,p,0)\rangle\rangle_u,$$

$$|n^f\rangle_t := |\iota(-1,n^f)\rangle\rangle_t.$$

Neumann type ($\mathcal{O} \neq -1$)

$$|m^f,F\rangle_u := \left(\det(2G_O^{-1})\right)^{-\frac{1}{4}}\sum_{w}(-1)^{w\,m^f + wF_u w}|\iota(\mathcal{O},-Fw,w)\rangle\rangle_u,$$

$$|m^f,F\rangle_t := 2^{-\frac{D}{2}}\sum_{n^f\in\{0,1\}^D}(-1)^{m^f n^f + n^f F_u n^f}|\iota(\mathcal{O},n^f)\rangle\rangle_t,$$

where $(m^f)^i = 1,0$, $G_O^{-1} = (G+B+F)^{-1}G(G-B-F)^{-1}$.

- Neumann coefficients in the twisted sector

$$|V_0(1_u,2_t,3_t)\rangle = \mu_t^2\, e^{\frac{1}{2}a^{\dagger r}\tilde{T}^{rs}a^{\dagger s}+\frac{1}{2}\tilde{a}^{\dagger r}\tilde{T}^{rs}\tilde{a}^{\dagger s}}|p_1,w_1;n_2^f;n_3^f\rangle$$

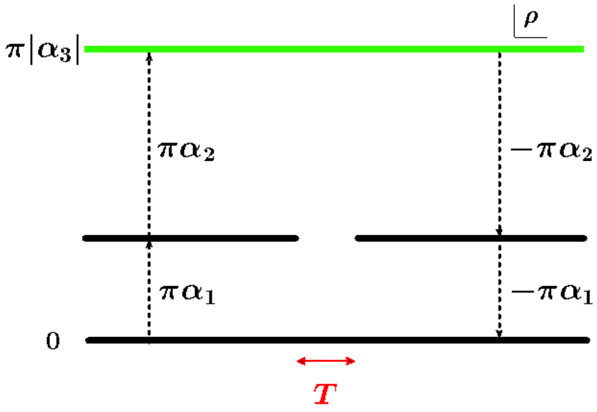
$$\sum_{t,l_t}\tilde{T}^{rt}_{n_r l_t}\tilde{T}^{ts}_{l_t m_s}=\delta_{n_r,m_s}\delta_{r,s},\quad \sum_{t,l_t}\tilde{T}^{1t}_{0l_t}\tilde{T}^{ts}_{l_t m_s}=-\tilde{T}^{1s}_{0m_s},\quad \sum_{t,l_t}\tilde{T}^{1t}_{0l_t}\tilde{T}^{t1}_{l_t 0}=-2T^{11}_{00},$$

$$\tilde{T}^{rs}_{n_r m_s}=\frac{\alpha_1 n_r m_s}{\alpha_r m_s+\alpha_s n_r}\tilde{T}^{r1}_{n_r 0}\tilde{T}^{s1}_{m_s 0}$$

$$T^{11}_{00}-\sum_{r,s=2,3}\tilde{T}^{1r}_{0\cdot}[(1+\tilde{T})^{-1}]^{rs}\tilde{T}^{s1}_{\cdot 0}=-2\sum_{n=1}^{\infty}\frac{\cos^2\left(\frac{\alpha_1}{\alpha_3}n\pi\right)}{n}=-\infty$$

$$\begin{aligned}\mathcal{C} &:= \mu_u^2\,\det{}^{-\frac{d+D-2}{2}}(1-(\tilde{N}^{33})^2)\\ &= \mu_t^2\,\det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_t3_t})^2)\det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2),\\ &\sim |\alpha_1\alpha_2\alpha_3|T^{-3}\end{aligned}$$

follows from *Cremmer-Gervais identity* for $D+d=26$.



$\mathcal{C}' := \mu_t^2\,\det{}^{-\frac{D}{2}}(1-(\tilde{T}^{3_u3_u})^2)\,\det{}^{-\frac{d-2}{2}}(1-(\tilde{N}^{33})^2)$ cannot be evaluated similarly.

Results :

$$|n^f, x^\perp, \alpha\rangle_\pm \;=\; \frac{1}{2}(2\pi\delta(0))^{-D}\left((\det(2G_{ij}))^{\frac{1}{4}}|n^f, x^\perp, \alpha\rangle_u \pm c_t(2\pi\delta(0))^{\frac{D}{2}}|n^f, x^\perp, \alpha\rangle_t\right)$$

are idempotents:

$$|n_1^f, x^\perp, \alpha_1\rangle_\pm * |n_2^f, y^\perp, \alpha_2\rangle_\pm \;=\; \delta^D_{n_1^f, n_2^f}\delta(x^\perp - y^\perp)\,\mathcal{C}\,c_0^+|n_2^f, y^\perp, \alpha_1 + \alpha_2\rangle_\pm,$$

$$|n_1^f, x^\perp, \alpha_1\rangle_\pm * |n_2^f, y^\perp, \alpha_2\rangle_\mp \;=\; 0.$$

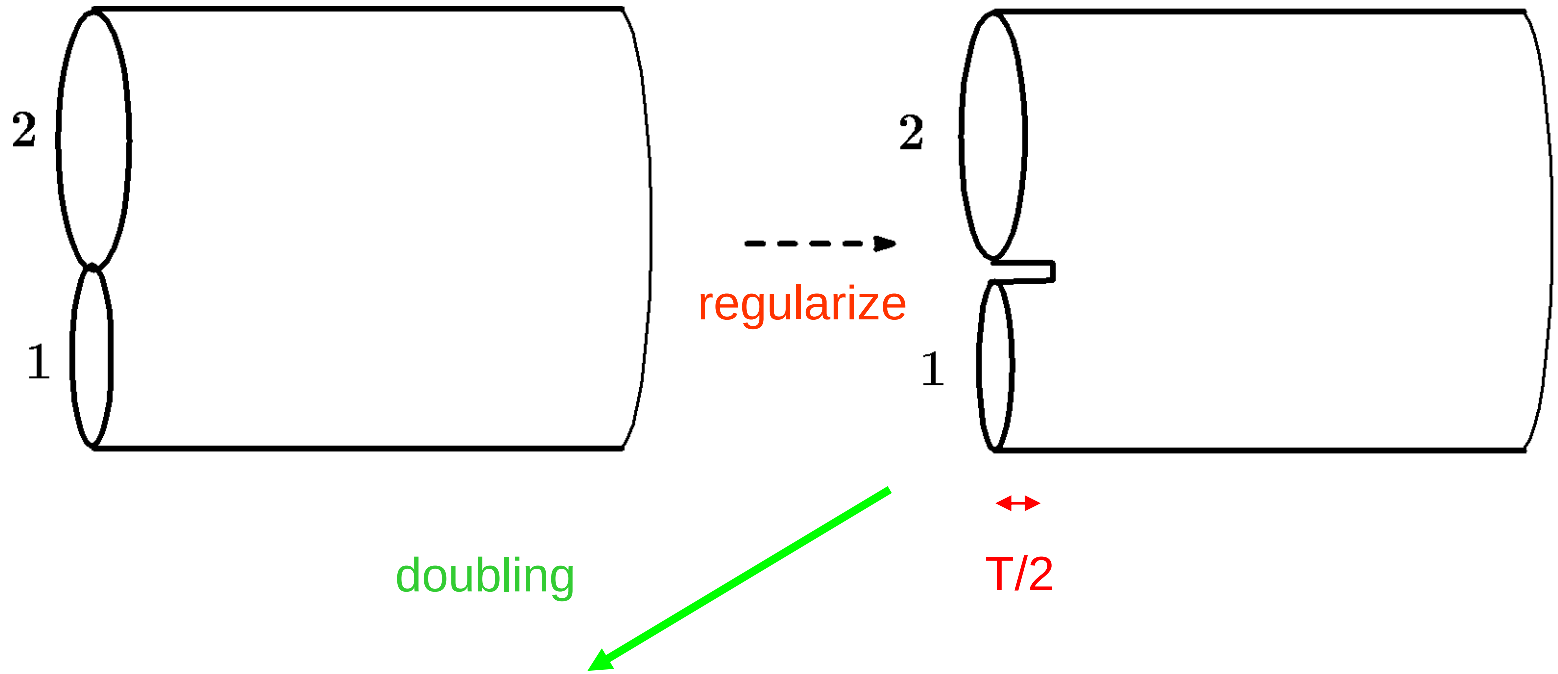
c_t is given by

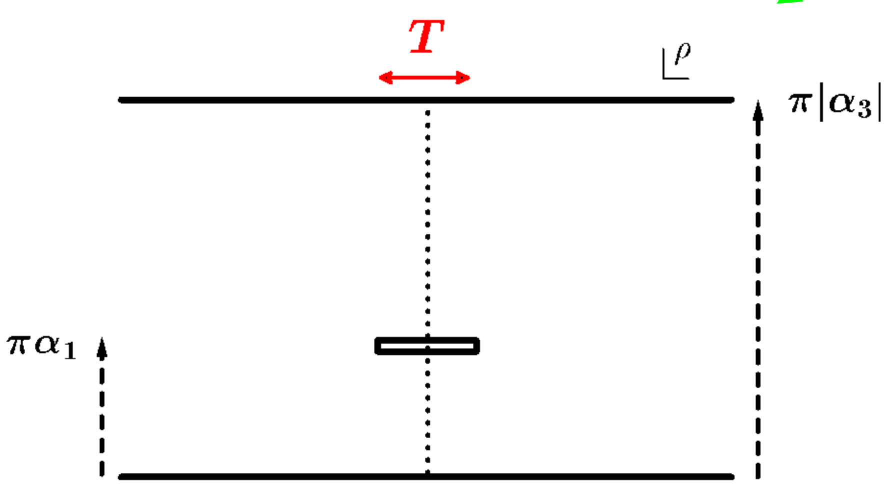
$$c_t \;=\; \sqrt{\frac{\mathcal{C}}{\mathcal{C}'}} = \left(e^{-\frac{\tau_0}{4}(\alpha_1^{-1}+\alpha_2^{-1})}\frac{\det(1-(\tilde{T}^{1_u1_u}(\alpha_3,\alpha_1,\alpha_2))^2)}{\det(1-(\tilde{N}^{33}(\alpha_1,\alpha_2,\alpha_3))^2)}\right)^{\frac{D}{4}},$$

which is evaluated by *open string 1-loop amplitude* as

$$c_t(2\pi\delta(0))^{\frac{D}{2}} = 2^{\frac{D}{4}}(\det(2G))^{\frac{1}{4}} = \sqrt{\sigma(e,g)}(\det(2G))^{\frac{1}{4}}.$$

$\rightarrow$ $|n^f, x^\perp, \alpha\rangle_\pm$: Cardy state for fractional D-brane.





Modulus of torus $\tilde{\tau}$

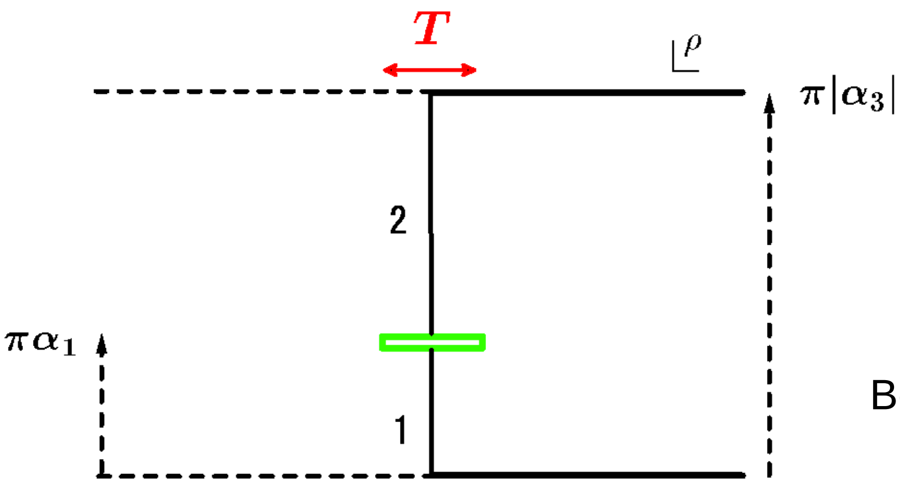
$$e^{-\frac{\pi}{|\tilde{\tau}|}} \sim \frac{T}{8|\alpha_3 \sin(\pi\alpha_1/\alpha_3)|} \quad \text{for } T \to 0$$

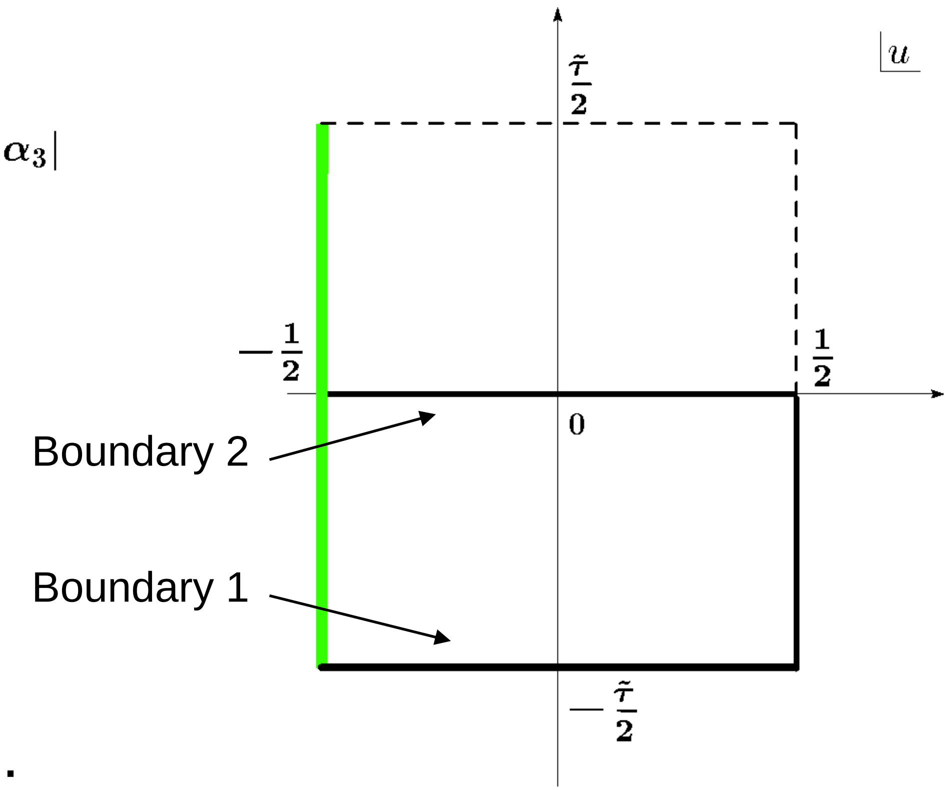
Mandelstam mapping:

$$\rho(u) \;=\; (\alpha_1+\alpha_2)\log\frac{\vartheta_1(u+(\alpha_1\tilde{\tau}/2\alpha_3)|\tilde{\tau})}{\vartheta_1(u-(\alpha_1\tilde{\tau}/2\alpha_3)|\tilde{\tau})} - 2\pi i\alpha_1 u$$

[Asakawa-Kugo-Takahashi(1999)]

$\rightarrow$ Including ghost contribution, we reproduce $\mathcal{C} \sim |\alpha_1\alpha_2\alpha_3|T^{-3}$.





Ratio of 1-loop amplitude:

$$\left(\frac{\eta(\tilde{\tau})}{\vartheta_0(\tilde{\tau})}\right)^{\frac{D}{2}}\left((2\pi\delta(0))^{-D}\eta(\tilde{\tau})^{-D}\sum_{p} e^{i\pi\tilde{\tau}pG^{-1}p/2}\right)^{-1}$$

$$\to\ 2^{-\frac{D}{2}}(2\pi\delta(0))^{D}\det{}^{-\frac{1}{2}}(2G)=\frac{\mathcal{C}'}{\mathcal{C}} \qquad \tilde{\tau}\to +i0$$

Similarly, we obtain Neumann type idempotents:

$$|m^f, F, x^\perp, \alpha\rangle_\pm = \frac{1}{2}\frac{\det^{\frac{1}{4}}(2G_O^{-1})}{(2\pi\delta(0))^D}\left[|m^f, F, x^\perp, \alpha\rangle_u \pm 2^{\frac{D}{4}}|m^f, F, x^\perp, \alpha\rangle_t\right],$$

$$|m_1^f, F, x^\perp, \alpha_1\rangle_\pm * |m_2^f, F, y^\perp \alpha_2\rangle_\pm = \delta^D_{m_1^f, m_2^f}\delta(x^\perp - y^\perp)\mathcal{C}\, c_0^+|m_2^f, F, x^\perp, \alpha_1 + \alpha_2\rangle_\pm,$$

$$|m_1^f, F, x^\perp, \alpha_1\rangle_\pm * |m_2^f, F, y^\perp, \alpha_2\rangle_\mp = 0.$$

（※）Neumann type idempotents are obtained from Dirichlet type by T-duality :

$$\mathcal{U}_g^\dagger|n^f, \alpha\rangle_{\pm,E} \;=\; |m^f = n^f, F, \alpha\rangle_{\pm,g(E)}.$$

In fact, we can prove

$$\mathcal{U}_g^\dagger|A * B\rangle_E = |(\mathcal{U}_g^\dagger A) * (\mathcal{U}_g^\dagger B)\rangle_{g(E)}, \quad g = \begin{pmatrix} -F & 1 \\ 1 & 0 \end{pmatrix} \in O(D, D; \mathrm{Z})$$

for both *uuu* and *utt* 3-string vertices. $(E = G + B)$

$\mathcal{U}_g$ is given by *Kugo-Zwiebach transformation* for the untwisted sector and

$$\mathcal{U}_g^\dagger\alpha_r(E)\mathcal{U}_g = -E^{T-1}\alpha_r(g(E)), \quad \mathcal{U}_g^\dagger\tilde{\alpha}_r(E)\mathcal{U}_g = E^{-1}\tilde{\alpha}_r(g(E)),$$

$$\mathcal{U}_g^\dagger|n^f\rangle_E = 2^{-\frac{D}{2}}\sum_{m^f\in\{0,1\}^D}(-1)^{n^f m^f + m^f F_u m^f}|n^f\rangle_{g(E)},$$

for the twisted sector. $(F_u)_{ij} := F_{ij}\ (i < j),\ 0\ (\text{otherwise}).$

Summary and discussion

- Cardy states satisfy idempotency equation in closed SFT (on $R^D, T^D, T^D/Z_2$). [KMW1, KMW2, KM]
- Variation around idempotents gives open string spectrum (on R^D). [KMW1, KMW2]

- Idempotents ∼ Cardy states
  : detailed correspondence ?

- Closed version of VSFT? (Veneziano amplitude,…)
- Relation to the *original* HIKKO theory?
- More nontrivial background? (other orbifolds,…)
- Super extension? (HIKKO NSR vertex,…) [IKMW work in progress]

“non-commutative” extension

KT operator which was introduced to represent noncommutativety in SFT :

$$V_{\theta,\sigma_c} = \exp\left(-\frac{i}{4}\int_{\sigma_c}^{2\pi+\sigma_c} d\sigma \int_{\sigma_c}^{2\pi+\sigma_c} d\sigma' P_i(\sigma)\theta^{ij}\epsilon(\sigma,\sigma')P_j(\sigma')\right)$$

[Kawano-Takahashi (1999)]

In the Seiberg-Witten limit: $\alpha' \sim \epsilon^{1/2},\ g_{ij} \sim \epsilon,\ \epsilon \to 0$

$$_{\alpha_1+\alpha_2}\langle x| \left[\int dy f_{\alpha_1}(y)\hat{V}_{\theta,\sigma_c}|B(y)\rangle_{\alpha_1} * \int dy' g_{\alpha_2}(y')\hat{V}_{\theta,\sigma_c}|B(y')\rangle_{\alpha_2}\right]$$

$$\sim \ [\det^{-\frac{d}{2}}(1-(\tilde{N}^{33})^2)\,2\pi\delta(0)]\, f_{\alpha_1}(x)\frac{\sin(-\beta\lambda)\sin((1+\beta)\lambda)}{(-\beta)(1+\beta)\lambda^2} g_{\alpha_2}(x)$$

where $\beta = \frac{\alpha_1}{\alpha_3},\quad \lambda = \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x^i}\theta^{ij}\frac{\overrightarrow{\partial}}{\partial x^j}$

By taking the Laplace transformation with an ansatz:
$f_\alpha(x) = \alpha^{\delta-1}f(x)$ the idempotency equation is reduced to

$$f(x)\frac{\sin\lambda}{\lambda}f(x) = f(x)$$

i.e., projector eq. with respect to the ***Strachan product*** which is
commutative and non-associative.

feature of the HIKKO ***closed SFT*** $*$ product