Direct Integration of the Lorenz Gauge Equations in the Frequency Domain: Unconstrained Approach to the EHS Method

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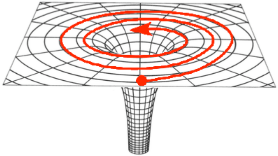
Introduction and Motivation

- EMRI are example of unsolved two-body problem in GR

- EMRI evolution involves Gravitational Self-Force (GSF)

  SCO as point mass → lowest order geodesic motion on background
  Acceleration causes radiation, small μ affects geometry locally
  Radiation/field acts back (locally)
  Corrects motion–conservative and
  nonconservative effects



  Field is divergent locally
  Has "singular" and "regular" parts
  Regularization finds finite Self-Force

  Dirac (1938), DeWitt and Brehme (1960), Mino, Sasaki & Tanaka (1997) and Quinn & Wald (1997)

- EMRIs are potential promising source of GWs for eLISA, NGO, etc. (e.g. Gair)

Calculational Accuracy Requirements

Example: $\epsilon = \mu/M = 10^{-6}$, $\Delta\Phi/\Phi \lesssim 10^{-8}$ - 10^{-7}

$$
\begin{array}{llllllll}
\text{(metric size)} & \mathcal{O}(1) & & \mathcal{O}(10^{-6}) & & \mathcal{O}(10^{-12}) & & \\
 & | & & | & & | & & \\
\text{Metric :} & g_{\mu\nu} & + & {}_1p_{\mu\nu} & + & {}_2p_{\mu\nu} & + & \cdots \\
 & & & | & & | & & \\
\text{Self} - \text{force :} & & & {}_1f_{\mu} & + & {}_2f_{\mu} & + & \cdots \\
 & & & | & & | & & \\
\text{(self} - \text{force size)} & & & \mathcal{O}(1) & & \mathcal{O}(10^{-6}) & & \mathcal{O}(10^{-12})
\end{array}
$$

- Must calculate 1st order $\ll \mathcal{O}(10^{-8}$ - $10^{-7})$ numerical accuracy
- Required accuracy $\Longrightarrow$ compute metric in frequency domain (FD)

Schwarzschild Metric Perturbations in Lorenz Gauge

- SF regularization parameters traditionally calculated in Lorenz gauge; therefore efforts to calculate retarded field have mostly used Lorenz gauge

- Perturbed field equations are:

$$\Box \bar{p}_{\mu\nu} + 2R_{\mu\alpha\nu\beta}\bar{p}^{\alpha\beta} = -16\pi T_{\mu\nu}$$

  along with Lorenz gauge condition

$$\nabla^{\beta}\bar{p}_{\alpha\beta} = 0$$

  where $\bar{p}$ is the trace-reversed metric perturbation

$$\bar{p}_{\alpha\beta} = p_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\ p$$

- see Akcay (2011); Warburton, et. al. (2012); talks at Capra 15

Metric perturbation amplitudes

- Decompose the metric in tensor spherical harmonics (notation from Martel and Poisson, 2005)

$$p_{\alpha\beta} = \left(\begin{array}{cc|cc} p_{tt} & p_{tr} & p_{t\theta} & p_{t\phi} \\ * & p_{rr} & p_{r\theta} & p_{r\phi} \\ \hline * & * & p_{\theta\theta} & p_{\theta\phi} \\ * & * & * & p_{\phi\phi} \end{array}\right)$$

$$p_{\alpha\beta} = \sum_{l,m} \left(\begin{array}{c|c} h_{ab}^{lm} Y^{lm} & j_a^{lm} Y_B^{lm} + h_a^{lm} X_B^{lm} \\ \hline * & r^2 \left(K^{lm} \Omega_{AB} Y^{lm} + G^{lm} Y_{AB}^{lm}\right) + h_2^{lm} X_{AB}^{lm} \end{array}\right)$$

	Odd Parity:	Even Parity:
Harmonics:	X_A^{lm}, X_{AB}^{lm}	$Y^{lm}, Y_A^{lm}, Y_{AB}^{lm}, \Omega_{AB} Y^{lm}$
Amplitudes:	h_t, h_r, h_2	$h_{tt}, h_{tr}, h_{rr}, j_t, j_r, K, G$

Source Terms

- Decompose stress-energy tensor as well to obtain source terms:
- Even-parity:

$$Q^{ab} = 8\pi \int T^{ab} \bar{Y}^{lm} \, d\Omega, \qquad Q^{a} = \frac{16\pi r^2}{l(l+1)} \int T^{aB} \bar{Y}_B^{lm} \, d\Omega$$

$$Q^{\flat} = 8\pi r^2 \int T^{AB} \Omega_{AB} \bar{Y}^{lm} \, d\Omega, \quad Q^{\sharp} = \frac{32\pi r^4}{(l-1)l(l+1)(l+2)} \int T^{AB} \bar{Y}_{AB}^{lm} \, d\Omega$$

- Odd-parity:

$$P^{a} = \frac{16\pi r^2}{l(l+1)} \int T^{aB} \bar{X}_B^{lm} \, d\Omega, \qquad P = \frac{16\pi r^4}{(l-1)l(l+1)(l+2)} \int T^{AB} \bar{X}_{AB}^{lm} \, d\Omega$$

- Source terms have form

$$F(t,r) \equiv f(t)\delta[r - r_p(t)]$$

Even-parity Equations, Frequency Domain

$$-f\left[\tilde{Q}^{rr}+f\left(\tilde{Q}^{\flat}+f\tilde{Q}^{tt}\right)\right]=\omega^2\tilde{h}_{tt}+\frac{d^2\tilde{h}_{tt}}{dr_*{}^2}+\frac{2(r-4M)}{r^2}\frac{d\tilde{h}_{tt}}{dr_*}-\frac{4iM\omega f}{r^2}\tilde{h}_{tr}+\frac{2M(3M-2r)f^2}{r^4}\tilde{h}_{rr}$$
$$+\left[\frac{2M^2}{r^4}-l(l+1)\frac{f}{r^2}\right]\tilde{h}_{tt}+\frac{4Mf^2}{r^3}\tilde{K}$$

$$2f\tilde{Q}^{tr}=\omega^2\tilde{h}_{tr}+\frac{d^2\tilde{h}_{tr}}{dr_*{}^2}+\frac{2f}{r}\frac{d\tilde{h}_{tr}}{dr_*}-\frac{2iM\omega}{fr^2}\tilde{h}_{tt}-\frac{2iM\omega f}{r^2}\tilde{h}_{rr}$$
$$-\left(\frac{4M^2}{r^4}+[l(l+1)+2]\frac{f}{r^2}\right)\tilde{h}_{tr}+\frac{2l(l+1)f}{r^3}\tilde{j}_t$$

$$\frac{1}{f}\left(f\tilde{Q}^{\flat}-f^2\tilde{Q}^{tt}-\tilde{Q}^{rr}\right)=\omega^2\tilde{h}_{rr}+\frac{d^2\tilde{h}_{rr}}{dr_*{}^2}+\frac{2}{r}\frac{d\tilde{h}_{rr}}{dr_*}-\frac{4iM\omega}{fr^2}\tilde{h}_{tr}+\left[\frac{2M}{r^4}(4r-7M)\right.$$
$$\left.-[4+l(l+1)]\frac{f}{r^2}\right]\tilde{h}_{rr}+\frac{2M(3M-2r)}{f^2r^4}\tilde{h}_{tt}+\frac{4l(l+1)f}{r^3}\tilde{j}_r+\frac{4(r-3M)}{r^3}\tilde{K}$$

$$f^2\tilde{Q}^t=\omega^2\tilde{j}_t+\frac{d^2\tilde{j}_t}{dr_*{}^2}-\frac{2M}{r^2}\frac{d\tilde{j}_t}{dr_*}-\frac{2iM\omega f}{r^2}\tilde{j}_r-\frac{f}{r^2}\left[l(l+1)-\frac{4M}{r}\right]\tilde{j}_t+\frac{2f^2}{r}\tilde{h}_{tr}$$

Even-parity Equations, Frequency Domain (cont'd)

$$-\tilde{Q}^{r} = \omega^2 \tilde{j}_r + \frac{d^2 \tilde{j}_r}{d{r_*}^2} + \frac{2M}{r^2}\frac{d\tilde{j}_r}{dr_*} - \frac{2iM\omega}{fr^2}\tilde{j}_t - \frac{f}{r^2}\left[l(l+1)+4f\right]\tilde{j}_r + \frac{2f^2}{r}\tilde{h}_{rr} - \frac{2f}{r}\tilde{K} + \frac{\left[l(l+1)-2\right]f}{r}\tilde{G}$$

$$\tilde{Q}^{rr} - f^2\tilde{Q}^{tt} = \omega^2\tilde{K} + \frac{d^2\tilde{K}}{d{r_*}^2} + \frac{2f}{r}\frac{d\tilde{K}}{dr_*} - \frac{f}{r^2}\left[l(l+1)+2-\frac{8M}{r}\right]\tilde{K} + \frac{2M}{r^3}\tilde{h}_{tt} - \frac{2f^2(3M-r)}{r^3}\tilde{h}_{rr} - \frac{2l(l+1)f}{r^3}\tilde{j}_r$$

$$-\frac{f}{r^2}\tilde{Q}^{\sharp} = \omega^2\tilde{G} + \frac{d^2\tilde{G}}{d{r_*}^2} + \frac{2f}{r}\frac{d\tilde{G}}{dr_*} - \frac{f}{r^2}\left[l(l+1)-2\right]\tilde{G} + \frac{4f^2}{r^3}\tilde{j}_r$$

- Even-parity is a set of seven coupled 2^{nd}-order equations
- Therefore, the system is 14^{th}-order

Odd-parity Equations, Frequency Domain

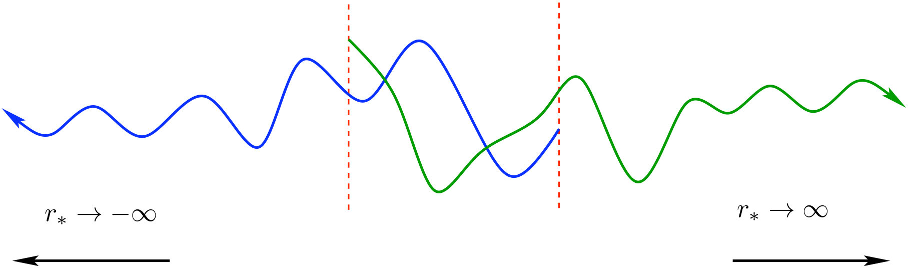
$$f^2 \tilde{P}^t = \omega^2 \tilde{h}_t + \frac{d^2 \tilde{h}_t}{d{r_*}^2} + \frac{f\left[4M - l(1+l)r\right]}{r^3} \tilde{h}_t - \frac{2M}{r^2} \left(\frac{d\tilde{h}_t}{dr_*} + i\omega f \tilde{h}_r \right)$$

$$-\tilde{P}^r = \omega^2 \tilde{h}_r + \frac{d^2 \tilde{h}_r}{d{r_*}^2} + \frac{f\left(l^2 + l - 2\right)}{r^3} \tilde{h}_2 + \frac{f\left[8M - \left(4 + l + l^2\right) r\right]}{r^3} \tilde{h}_r + \frac{2M}{fr^2} \left(f \frac{d\tilde{h}_r}{dr_*} - i\omega \tilde{h}_t \right)$$

$$-2f\tilde{P} = \omega^2 \tilde{h}_2 + \frac{d^2 \tilde{h}_2}{d{r_*}^2} - \frac{f\left[8M + \left(-4 + l + l^2\right) r\right]}{r^3} \tilde{h}_2 + \frac{4f^2 \tilde{h}_r}{r} - \frac{2\left[6M^2 + (f-5)Mr + r^2\right]}{fr^3} \frac{d\tilde{h}_2}{dr_*}$$

- Odd-parity is a set of three coupled 2^{nd}-order equations
- Therefore, the system is 6^{th}-order

Homogeneous Solutions, Boundary Conditions



- Equations are source-free outside libration region
- To find particular solution, we'll need a complete set of independent homogeneous solutions (for method of variation of parameters)
- For even-parity, we need 7 causal downgoing homogeneous solutions and 7 causal outgoing homogeneous solutions
- For odd-parity, 3 causal downgoing and 3 causal outgoing solutions

Algorithmic Roadmap

- Use geodesic motion for point particle; compute source terms
- Find complete set of causal homogeneous solutions
- Use method of variation of parameters to normalize homogeneous solutions
- Use method of extended homogeneous solutions (EHS) (Barack, Ori, Sago 2008) to return to the time domain
- Check constraints
- Check accuracy of modes
  - Use quad-precision (EF), use constrained equations (TO)
- Compute first-order self-force

Variation of Parameters

- Integrate 6 homogeneous odd-parity solutions and 14 homogeneous even-parity solutions through the libration region
- General particular solutions have the form (odd-parity, even-parity):

$$\tilde{\mathcal{H}} = c_0^+(r)\tilde{\mathcal{H}}_0^+ + c_1^+(r)\tilde{\mathcal{H}}_1^+ + \cdots + c_0^-(r)\tilde{\mathcal{H}}_0^- + c_1^-(r)\tilde{\mathcal{H}}_1^- + \cdots$$
$$\tilde{\mathcal{J}} = d_0^+(r)\tilde{\mathcal{J}}_0^+ + d_1^+(r)\tilde{\mathcal{J}}_1^+ + \cdots + d_0^-(r)\tilde{\mathcal{J}}_0^- + d_1^-(r)\tilde{\mathcal{J}}_1^- + \cdots$$

where each of the $\tilde{\mathcal{H}}_i^\pm$, $\tilde{\mathcal{J}}_i^\pm$ is a vector of metric amplitudes for the i^{th} outgoing (+) or downgoing (-) homogeneous solution:

$$\tilde{\mathcal{J}}_i^\pm = \begin{pmatrix} \tilde{h}_{tt} \\ \tilde{h}_{tr} \\ \tilde{h}_{rr} \\ \tilde{j}_t \\ \tilde{j}_r \\ \tilde{K} \\ \tilde{G} \end{pmatrix}_i^\pm \quad \text{(even)} \qquad \tilde{\mathcal{H}}_i^\pm = \begin{pmatrix} \tilde{h}_t \\ \tilde{h}_r \\ \tilde{h}_2 \end{pmatrix}_i^\pm \quad \text{(odd)}$$

and i runs from 0 to 6 for even and 0 to 2 for odd-parity

Variation of Parameters

- Plug into equations, demand that (odd-parity example:)

$$\sum_k c_k^{+\prime} \tilde{\mathcal{H}}_k^{+} + c_k^{-\prime} \tilde{\mathcal{H}}_k^{-} = 0$$

- May then show that

$$\mathcal{W}\mathbf{c}' = \tilde{\mathbf{Z}}$$

where $\mathcal{W}$ is the Wronskian matrix of homogeneous solutions and derivatives, and

$$\mathbf{c} = \begin{pmatrix} c_i^{+} \\ c_i^{-} \end{pmatrix}$$

$$\tilde{\mathbf{Z}} = \begin{pmatrix} \mathbf{0} \\ \tilde{Z}_i \end{pmatrix}$$

for $\tilde{Z}$'s the Fourier amplitudes of the source terms

Variation of Parameters

For odd-parity this looks like:

$$\begin{pmatrix} \tilde{h}_{t\,0}^{\,+} & \tilde{h}_{t\,1}^{\,+} & \tilde{h}_{t\,2}^{\,+} & \tilde{h}_{t\,0}^{\,-} & \tilde{h}_{t\,1}^{\,-} & \tilde{h}_{t\,2}^{\,-} \\ \tilde{h}_{r\,0}^{\,+} & \tilde{h}_{r\,1}^{\,+} & \tilde{h}_{r\,2}^{\,+} & \tilde{h}_{r\,0}^{\,-} & \tilde{h}_{r\,1}^{\,-} & \tilde{h}_{r\,2}^{\,-} \\ \tilde{h}_{2\,0}^{\,+} & \tilde{h}_{2\,1}^{\,+} & \tilde{h}_{2\,2}^{\,+} & \tilde{h}_{2\,0}^{\,-} & \tilde{h}_{2\,1}^{\,-} & \tilde{h}_{2\,2}^{\,-} \\ \partial_r\tilde{h}_{t\,0}^{\,+} & \partial_r\tilde{h}_{t\,1}^{\,+} & \partial_r\tilde{h}_{t\,2}^{\,+} & \partial_r\tilde{h}_{t\,0}^{\,-} & \partial_r\tilde{h}_{t\,1}^{\,-} & \partial_r\tilde{h}_{t\,2}^{\,-} \\ \partial_r\tilde{h}_{r\,0}^{\,+} & \partial_r\tilde{h}_{r\,1}^{\,+} & \partial_r\tilde{h}_{r\,2}^{\,+} & \partial_r\tilde{h}_{r\,0}^{\,-} & \partial_r\tilde{h}_{r\,1}^{\,-} & \partial_r\tilde{h}_{r\,2}^{\,-} \\ \partial_r\tilde{h}_{2\,0}^{\,+} & \partial_r\tilde{h}_{2\,1}^{\,+} & \partial_r\tilde{h}_{2\,2}^{\,+} & \partial_r\tilde{h}_{2\,0}^{\,-} & \partial_r\tilde{h}_{2\,1}^{\,-} & \partial_r\tilde{h}_{2\,2}^{\,-} \end{pmatrix} \begin{pmatrix} \partial_r c_0^+ \\ \partial_r c_1^+ \\ \partial_r c_2^+ \\ \partial_r c_0^- \\ \partial_r c_1^- \\ \partial_r c_2^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{Z}_t \\ \tilde{Z}_r \\ \tilde{Z}_2 \end{pmatrix}$$

Solve via Cramer's rule:

$$c_j{}^+(r) = \int_{r_{\min}}^{r} dr' \frac{\det\left(\mathcal{W}^{j+}\right)}{\det\left(\mathcal{W}\right)}, \qquad c_j{}^-(r) = \int_{r}^{r_{\max}} dr' \frac{\det\left(\mathcal{W}^{j-}\right)}{\det\left(\mathcal{W}\right)}$$

EHS for a System of Equations

- Extend Barack, Ori, and Sago's (2008) EHS method to a system of coupled equations
- Homogeneous solutions are normalized by the constants

$$C_i^{\pm} = \int_{r_{\min}}^{r_{\max}} dr' \frac{\det\left(\mathcal{W}^{i\pm}\right)}{\det\left(\mathcal{W}\right)}$$

- The normalized homogeneous solutions are

$$\tilde{\mathcal{H}}_{lmn}{}^{\pm}(r) = \sum_i C_i^{lmn\pm} \tilde{\mathcal{H}}_i^{\pm}(r)$$

- Define time-domain homogeneous solutions

$$\mathcal{H}_{lm}^{\pm}(t,r) \equiv \sum_n \tilde{\mathcal{H}}_{lmn}{}^{\pm}(r) e^{-i\omega_{mn}t}$$

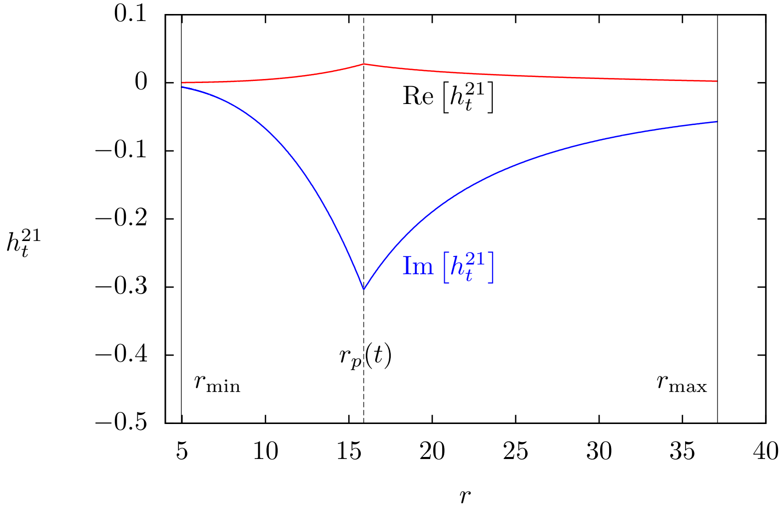
- For any t and r the actual solution to the inhomogeneous equation is

$$\mathcal{H}_{lm}^{\mathrm{EHS}}(t,r) \equiv \mathcal{H}_{lm}^{+}(t,r)\theta\left[r - r_p(t)\right] + \mathcal{H}_{lm}^{-}(t,r)\theta\left[r_p(t) - r\right]$$

Example Lorenz Gauge Solution (Odd-Parity)

$l = 2,\ m = 1,\ p = 8.75455,\ e = 0.764124,\ t = 93.58$

$n = -40$ to $+40$



Lorenz Gauge: Constrained vs. Unconstrained

- The Lorenz gauge condition yields one odd-parity constraint:

$$0 = (l+2)(l-1)f\tilde{h}_2 - 4(r-M)f\tilde{h}_r - 2r^2\left(f\partial_{r_*}\tilde{h}_r + i\omega\tilde{h}_t\right)$$

and three even-parity constraints:

$$0 = 2l(l+1)f\tilde{j}_t + 4f(M-r)\tilde{h}_{tr} - r^2\left[2f\partial_{r_*}\tilde{h}_{tr} + i\omega\left(f^2\tilde{h}_{rr} + 2f\tilde{K} + \tilde{h}_{tt}\right)\right]$$

$$0 = 4(r-M)f\tilde{h}_{rr} + \frac{4Mr}{f}\partial_{r_*}\tilde{K} - 4fr\tilde{K} - 2l(l+1)f\tilde{j}_r$$
$$+ r^2\left[f\partial_{r_*}\tilde{h}_{rr} - f\left(i\omega f\tilde{h}_{rr} + 2i\omega\tilde{K} + 2\partial_{r_*}\tilde{h}_{tr}\right)\right]$$

$$0 = fr^3\left[(l+2)(l-1)\tilde{G} + f\tilde{h}_{rr}\right] + 4(M-r)rf\tilde{j}_r - r^3\left(\tilde{h}_{tt} + 2f\partial_{r_*}\tilde{j}_r\right) - 2i\omega r^3\tilde{j}_t$$

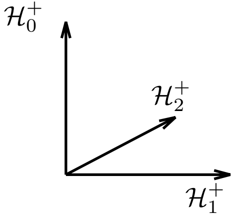
Lorenz Gauge: Unconstrained Approach

- The space of homogeneous solutions to the unconstrained system is larger than the space of gauge-constrained solutions
- An arbitrary causal homogeneous solution will not necessarily “know” about gauge conditions
- But the source does!
- Integration through the source region constrains the normalized particular solution so that the gauge condition *is* satisfied

Example: Unconstrained Odd-Parity as $r_* \to +\infty$

Asymptotic analysis led to choice of boundary conditions for three independent, outgoing homog. solutions:

$$\mathcal{H}_0^+ = \begin{pmatrix} h_t \\ h_r \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{i\omega r_*},\ \mathcal{H}_1^+ = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{i\omega r_*},\ \mathcal{H}_2^+ = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} e^{i\omega r_*}$$



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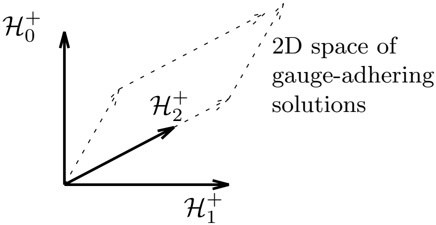
- $\mathcal{H}_2^+$ lies in space of gauge-adhering solutions



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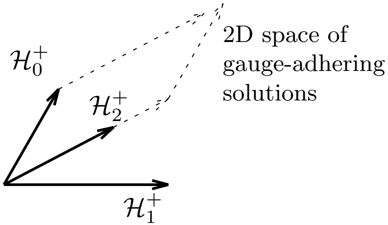
- $\mathcal{H}_2^+$ lies in space of gauge-adhering solutions
- As it turns out, so does $\mathcal{H}_0^+$ (we chose wisely!)



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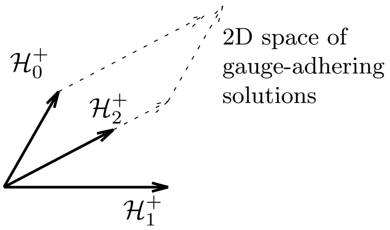
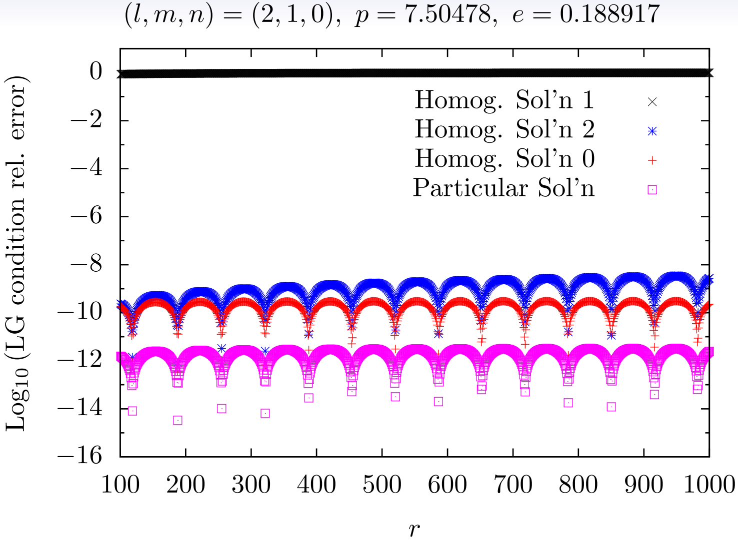


- So $\mathcal{H}_0^+$ and $\mathcal{H}_2^+$ want to satisfy LG condition, $\mathcal{H}_1^+$ does not

Satisfaction of the Lorenz Gauge Condition



LG Condition and the Particular Solution

- So what happens to the second homogeneous solution?
- Well, it is nulled out by the source integration
- For $(l, m, n) = (2, 1, 0),\ p = 7.50478,\ e = 0.188917$, EHS normalization coefficients are:

$$|C_0^+| = 4.0602424246563079 \text{ e} - 02$$
$$|C_1^+| = 0.0000000000000006 \text{ e} - 02$$
$$|C_2^+| = 4.0368840944840946 \text{ e} - 02$$

- Note: quad precision

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$$|C_2^+| = 4.0368840944840946\mathrm{0\ e} - 02$$

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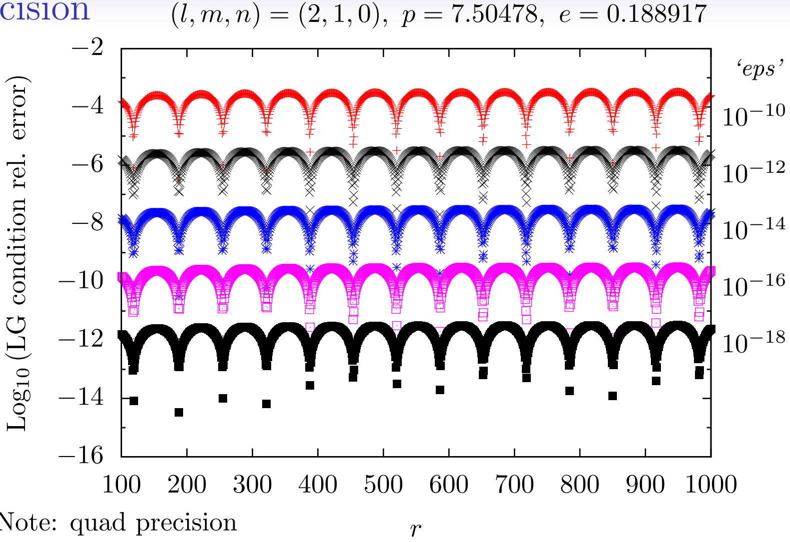
Satisfaction of Gauge Condition vs. Integration Precision



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Near-static Modes and Ill-Conditioned Wronskian

- Two-fold frequency spectrum:

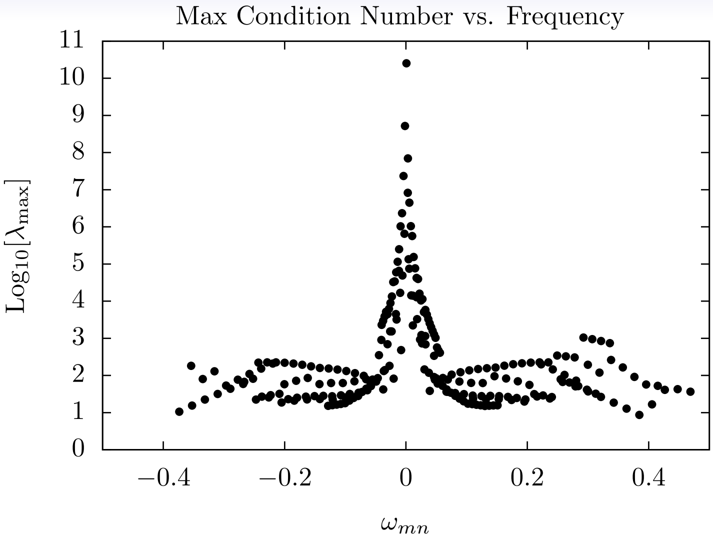
$$\omega_{mn} = m\Omega_\phi + n\Omega_r$$

- Values of m and n can conspire to bring the frequency near zero, especially for larger values of semi-latus rectum p

- For such modes, matrix inversion of the Wronskian becomes numerically inaccurate

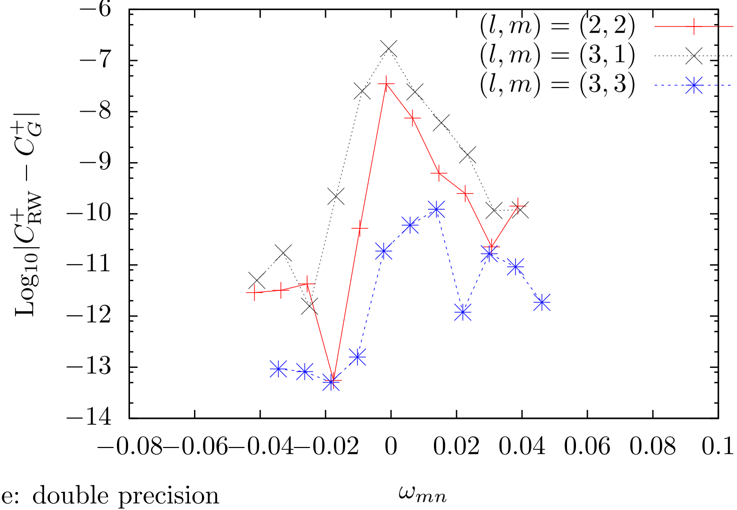
- Runtimes increase, but more importantly we lose digits of accuracy!

Ill-Conditioning of the Wronskian



Agreement with Regge-Wheeler Code

How does the double precision Lorenz gauge code compare with Regge-Wheeler for near-static modes? (p=8.75455, e=0.764124)



Note: double precision

Agreement with Regge-Wheeler Code

- Now, how about the quad precision Lorenz gauge code?
- Comparing our Regge-Wheeler code with our quad precision Lorenz code for two of the worst-case modes:

- For (2,2,-3):

$$|C^{+}_{\mathrm{RW}} - C^{+}_{G}| = 1 \times 10^{-15}$$

- For (3,1,-2):

$$|C^{+}_{\mathrm{RW}} - C^{+}_{G}| = 5 \times 10^{-12}$$

Conclusions

- Demonstrated extension of EHS method to coupled system of equations
- Solved directly for metric perturbation in Lorenz gauge, unconstrained
- Problem is more subtle than it seems: gauge constraints, near-static modes
- May "brute-force" through static modes problem with high precision code
  - Quad code is slow!
  - Call as routine from double precision code when needed