This is the lecture note taken by Hiroyuki Fujita (a grad student at ISSP). I thank him for allowing me to upload this note.
Of course, it's me who is responsible for any possible errors. If you have any questions, send me an email at haruki.watanabe@ap.t.u-tokyo.ac.jp

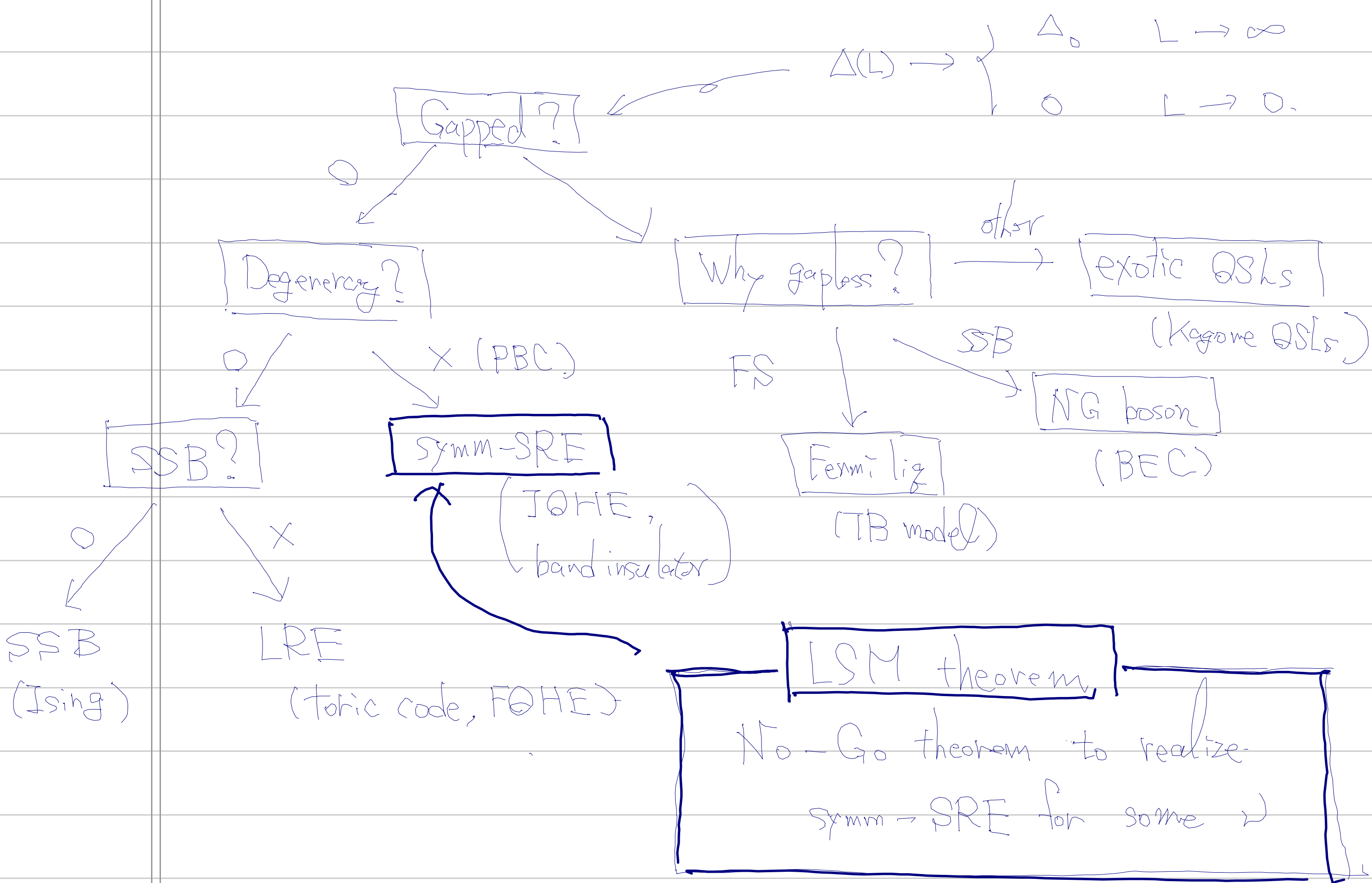
☆ Extension of LSM and its application to $\begin{cases} \text{QSL} \\ \text{TI} \end{cases}$

I. Intro

It is difficult (impossible) to solve quantum many-body models

Instead of solving them exactly,
$\longrightarrow$ let's try to find general theorems.

e.g. Nambu – Goldston theorem
SSB of continuous, global excitations
$\longrightarrow$ gapless bosonic excitations



Magnetic translation でも ok.

Assume U(1) symmetry + translational symm $\mathbb{R}^{d=2}$

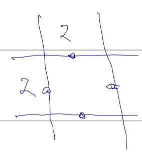
$\hat{N}|\psi_0\rangle = N|\psi_0\rangle$, $\nu = \frac{N}{N_{uc}}$: filling

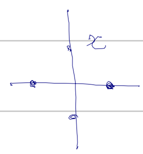
Number of unit cells

→ When symm-SRE (sSRE) prohibited?

• Examples of non-sSRE systems

2D toric code (LRE)





$B_\square = \prod_{i\in\square} Z_i$ $\quad B_\square = \pm 1$

→ $H = -J_1 \sum_\square B_\square - J_2 \sum_+ A_+$

$A_+ = \prod_{j\in +} X_j$ $\quad A_+ = \pm 1$

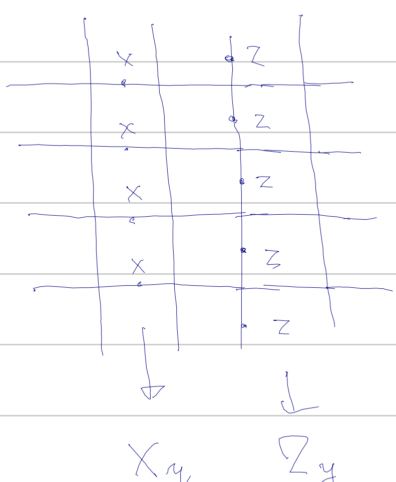
[1] Hilb. space dim = $2^{2L_1L_2}$

[2] $A_+, B_\square$ commute w/ H → $2L_1L_2$ conserved quantities

$\begin{pmatrix} A_{+_1}=1, A_{+_2}=-1, \dots \\ B_{\square_1}=-1, B_{\square_2}=1, \dots \end{pmatrix}$

[3] Two global constraints

$\prod_\square B_\square = 1$, $\prod_+ A_+ = 1$ → 4-fold degeneracy



product of operators along lines

X_x, X_y, Z_x, Z_y

commute w/ $A_+, B_\square, H$.

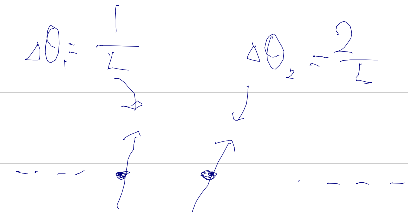
→ additional conserved quantity

two constraint → (in total $2L_1L_2 + 2$ conserved quantity)

$\{X_x, Z_y\} = 0$ → $\hat{X}_x|Z_x=1, Z_y=1\rangle$

$= -|Z_x=1, Z_y=-1\rangle$

$\{A_+, B_\square, Z_x, Z_y\}$ to fix a state. X_x, X_y = "flux insertion op"



II Original LSM

$$\hat{H} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} \quad \leftarrow S = 1/2$$

$$|v\rangle = \hat{U}|GS\rangle, \quad \hat{U} = \exp\left[\frac{2\pi i}{L}\sum_{x=1}^{L} x(\hat{S}_x^z - S)\right]$$

(Nonuniform spin twist)

$$\langle v|\hat{H}|v\rangle - E_{GS} \sim \sum \left(\frac{1}{L}\right)^2 \sim \frac{1}{L}$$

$\therefore$ the variational state has the excess energy $\propto \frac{1}{L} \to 0$

$\longrightarrow$ Degeneracy.

Are $|v\rangle$ and $|GS\rangle$ different?

Momentum different?

$$\begin{cases} \hat{T}|GS\rangle = e^{iP_0}|GS\rangle \\ \hat{T}|v\rangle = e^{iP_0'}|v\rangle \end{cases}$$

$$\hat{T}S_x^z\hat{T}^{-1} = S_{x+1}^z \longrightarrow \hat{T}\hat{U}\hat{T}^{-1} = \exp\left[\frac{2\pi i}{L}\sum x(\hat{S}_{x+1}^z - S)\right]$$

$$\exp\left[\frac{2\pi i}{L} L(S_{L+1}^z - 1)\right] \longrightarrow \quad = \exp\left[\frac{2\pi i}{L}\sum (x-1)(S_x^z - S)\right]$$

$$\therefore \hat{T}\hat{U}\hat{T}^{-1} = \exp\left[\frac{2\pi i}{L}\sum_{x=1}^{L}(\hat{S}_x^z - S)\right]\hat{U}$$

$$= e^{-2\pi i S}\hat{U} = e^{\pi i}\hat{U}$$

$\sum \hat{S}_x^z = 0$
total magnetization

$$\boxed{\begin{array}{l} \therefore |GS\rangle \text{ has } P_0 = 0 \\ \qquad \Longrightarrow |v\rangle \text{ has } P = \pi \\ \\ \therefore GS \text{ is gapless or degenerate.} \\ \qquad \left(\langle v|\hat{H}|v\rangle - E_{GS} \to 0 \quad (L \to \infty)\right) \end{array}}$$

III Promlem of the original LSM.

- for a specific 1D models (spin chain)

  $\longrightarrow$ extension by Yamanaka - Oshikawa - Affleck.

  (itenerant systems in 1D)

YOA ... assume $\begin{cases} \text{translation} \\ U(1) \to \hat{N} = \sum \hat{n}_x \end{cases}$

$$\hat{U} = \exp\left[\frac{2\pi i}{L} \sum_x x \hat{n}_x\right] \qquad \left(\begin{array}{l} \hat{S}^z_x - 1/2 \\ \text{in the original LSM} \end{array}\right)$$

$$\to \hat{T}\hat{U}\hat{T}^{-1} = \exp[-2\pi i \nu]\,\hat{U}$$

$\therefore$ ν is fractional $\to$ $\begin{cases} \text{degenerate} \\ \text{gapless} \end{cases}$

Still limited to 1D. How to treat $d > 1$?

Remarks

- S^z conserved, spinful electrons $\longrightarrow$ $\nu_\uparrow, \nu_\downarrow$ で上記の議論
- TRS $\longrightarrow$ $\nu_\uparrow = \nu_\downarrow \in \mathbb{Z}$

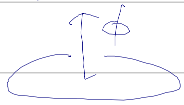
- Higher dimension ?

Statement $\begin{cases} \sum \hat{n}_x = \hat{N} = N, \quad \nu = \dfrac{N}{N_{uc}} \\ \nu \begin{cases} \in \mathbb{Z} & \cdots\cdots \text{unique gapped} \\ \notin \mathbb{Z} & \cdots\cdots \text{gapless or degenerate.} \end{cases} \end{cases}$

still $\hat{T}$, U(1) inv. gauge transf. で equivalent

(1D)

$$H = \sum t e^{i\frac{\phi}{L}} c_i^\dagger c_{i+1} + h.c. \quad \left(\text{or}: \hat{c}_{i+L} = e^{i\phi} c_i\right)$$



$$|GS(\phi=0)\rangle \longrightarrow |GS(\phi=2\pi)\rangle$$

ϕ の値によらず unique gapped と仮定

← ϕ は Boundary condition であり、$L\to\infty$ での Gap の有無は変わらないはず

$$\hat{U} = \exp\left[\frac{2\pi i}{L}\sum_x x\hat{n}_x\right] \longrightarrow \hat{U}^\dagger \hat{H}(\phi=0)\hat{U} = \hat{H}(\phi=2\pi)$$

$$\rightarrow \begin{cases} \hat{H}(\phi=2\pi)|GS(\phi=2\pi)\rangle = E_{GS}|GS(\phi=2\pi)\rangle \\ \hat{H}(\phi=0)\boxed{\hat{U}|GS(\phi=2\pi)\rangle} = E_{GS}\boxed{\hat{U}|GS(\phi=2\pi)\rangle} \end{cases}$$

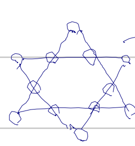
$$\therefore \nu \notin \mathbb{Z} \Rightarrow \hat{U}|GS(\phi=2\pi)\rangle \neq |GS(\phi=0)\rangle$$

$$\left(\begin{aligned} &|GS(\phi=2\pi, t=T)\rangle = \mathcal{T}\exp\left[-i\int_0^T dt\, \hat{H}(\phi(t))\right]|GS(\phi=0, t=0)\rangle \quad \text{(Adiabatic)} \\ &[\hat{H}(\phi(t)), \hat{T}] = 0 \longrightarrow |GS(\phi=2\pi)\rangle \ \&\ |GS(\phi=0)\rangle \\ &\qquad\qquad\qquad\qquad \text{have the same momentum.} \end{aligned} \right)$$

∴ Spinful system. S_z conservation, TRS

$\rightarrow \nu \in 2\mathbb{Z}$ to obtain unique gapped on a torus.

e.g. Kagome AFM



$\longrightarrow \nu = 3$

∴ Symmetric & Gapped ⇒ exotic (top. order)

$$\text{e.g. } \nu = \frac{1}{3} \longrightarrow \begin{cases} |GS\rangle \cdots 1 \\ U|GS\rangle \cdots \exp\left[i\frac{2\pi}{3}\right] \\ U^2|GS\rangle \cdots \exp\left[i\frac{4\pi}{3}\right] \end{cases} \Rightarrow \text{three-fold top. degen.}$$

FQHE

Include SOC

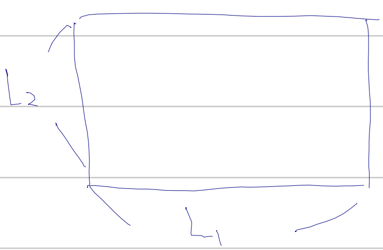
- What if we relax S_z conservation? $\Longrightarrow$ Nothing changes!!

$\Leftrightarrow$ $\nu_\uparrow, \nu_\downarrow$ are ill-defined $\quad$ ($\nu \in 2\mathbb{Z}$ for sSRE)

Assumption … $U(1)$, TRS ($T^2 = -1$ for each particle)

Possible arguments

① Degeneracy shold not depend on B.C.



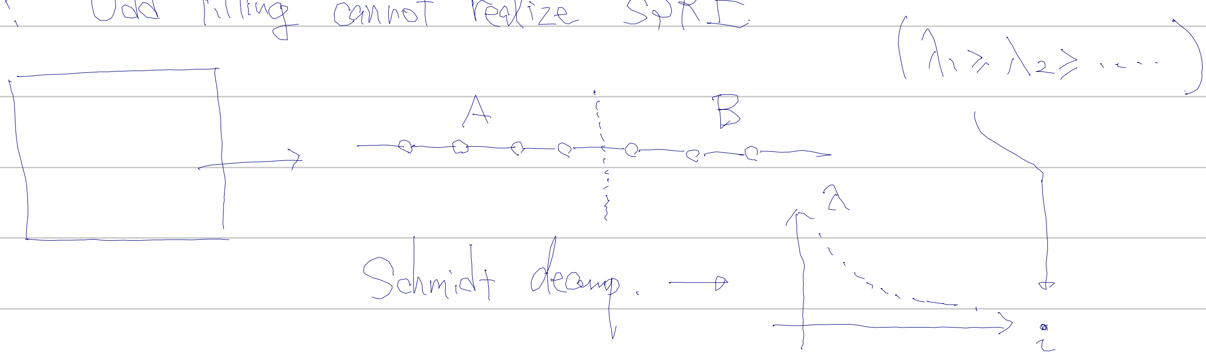
L_1, L_2 : odd

$N_{el} = \nu L_1 L_2$, $N_{uc} = L_1 L_2$.

$T^2 = (-1)^{N_{el}} = -1$

$= \boxed{\text{Kramers degeneracy}}$

∴ Odd filling cannot realize sSRE

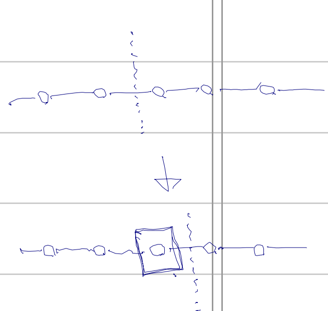


☆ Ent spectrum tells us the symm nature of Schmidt states.

$$|\psi\rangle = \lambda_1 |\uparrow\rangle_A |\downarrow\rangle_B + \lambda_2 |\downarrow\rangle_A |\uparrow\rangle_B$$

$$|\text{singlet}\rangle \longrightarrow |\lambda_1| = |\lambda_2| = \frac{1}{\sqrt{2}}$$

(degenerate spectrum)



Well-defined E. spectrum?

spins enroll in the symm. transformation must be even.

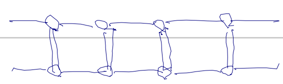
$\boxed{\text{two cuts related by symm (translation) enclose a projective repr } (s=1/2 \text{ for TRS.}) \longrightarrow \text{sSRE impossible.}}$

◎ Flux insertion, projective symm のどちらにせよ、
L_x, L_y の even, odd についての仮定を用いている。

前ページの話で言えば、L_y odd なら
1 site translation で cuts が odd # of els を囲うので
sSRE が禁止されることを示せるが、……

e.g. spin ladder of $S=1/2$

 …. gapless (Bethe)

L_y  …. gapped ($\simeq S=1$ chain)

?

Locality?

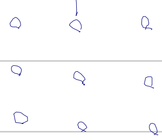
For large $L_y \gg \xi$: corr. fcn
# of ladders shold not affect
the degeneracy (Gapped/Gapless)

IV. Space Group (「応用群論」)

$$g \in G : \underset{\mathbb{R}^3}{\underset{\cap}{\vec{x}}} \longrightarrow \underset{\mathbb{R}^3}{\underset{\cap}{g(\vec{x})}} = \underset{O(3)}{\underset{\cap}{P_g}} \vec{x} + \underset{\mathbb{R}^3}{\underset{\cap}{\vec{t}_g}} \in \mathbb{R}^3$$

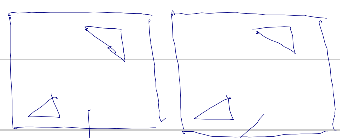
- Translation subgroup $T \to n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3$
- 3D ….. 230 SGs $\quad \vec{x} \to \vec{x}'$
- 2D ….. 80 Layer groups $\quad (x, y, z) \to (x', y', z')$
  17 Wall paper groups $\quad (x, y) \to (x', y')$

1) Symmorphic (73)



$$g(\vec{x}) = P_g \vec{x} + \vec{t}_g$$

2) Non-symmorphic (157)



unit cell

fractional $\vec{t}$ and reflection

V Extension of LSM to NS SGs

Translation

Non symmorphic $\longrightarrow$ Screw $\hat{T}_{\frac{n}{m}\vec{a}_3} \hat{R}^z_{\frac{2\pi}{m}} \equiv \hat{T}'$

then, $$\hat{T}' \hat{U} \hat{T}'^{-1} = \hat{T}' \exp\left[\frac{2\pi i}{L_x} \sum_{x,y,z} \hat{n}_z z\right] \hat{T}'^{-1}$$

$$= \exp\left[\frac{2\pi i}{L_x} \sum \hat{n}_z \left(z - \frac{n}{m}\right)\right] = e^{-\frac{2\pi i}{L_x}\frac{n}{m}N} \hat{U}$$

$$= \exp\left[-2\pi \frac{n}{m} \cdot \nu L_y L_z\right] \hat{U}$$

$\therefore$ L_y, L_z odd $\Rightarrow$ $\nu \in m\mathbb{Z}$ for sSRE

example : diamond lattice (SG 227)

Nonsymmorphic $\longrightarrow$ $\nu = 4\mathbb{Z}$ for sSRE

Action of SGs on real space

$\Rightarrow$ Wyckoff position.

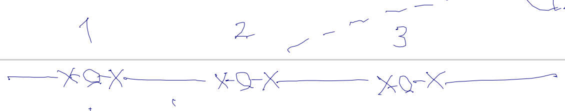
$$g : \vec{x} \mapsto g(\vec{x}) = P_g \vec{x} + \vec{T}_g$$

$\mathcal{O}_{\vec{x}} = \{ g(\vec{x}) \mid g \in G \}$ defines orbit of $\vec{x}$ under G

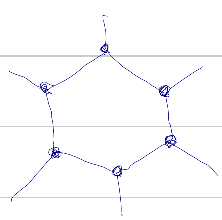
$G_{\vec{x}} = \{ g \in G \mid g(\vec{x}) = \vec{x} \}$ ... little group of G.

- 1D, inversion & translation.

$G_2 = \{ e, T^2 I_1 \} = T G_1 T^{-1}$



$G_1 = \{ e, I_1 \}$ $\quad$ $G_{1.5} = \{ e, T I_1 \}$

- 

Ying Ran PRB (2016)

PEPS

$\downarrow$

Unique gapped GS construction on a honeycomb AFM Heisenberg model.

$S = 3/2 \to$ VBS state

$S = 1/2$?

$\omega(g_1, g_2)$

proj. repr.

$$U(R_x) U(R_y) = \boxed{+} U(R_z)$$

$$U(R_y) U(R_x) = \boxed{-} U(R_z)$$