Lecture III: Synthetic Spin-Orbit Coupling for Ultracold Atoms and Majorana fermions

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International School for Topological Science and Topological Matters, Yukawa Institute for Theoretical Physics, Feb 14, 2017

Outline

- Background
- Topological superconductivity by proximity effect
- Properties of Majorana zero modes
- Experiments
- Time-reversal symmetry-protected topological superconductors
- Symmetry protected non-Abelian statistics
- Summary

Introduction

Motivation: Search for Majorana fermions

- Dirac equation: Quantum relativistic description of spin ½ fermions

$$\boxed{(i\gamma^{\mu}\partial_{\mu} - m)\,\Psi = 0}$$

Dirac (1928)
Complex solutions
Ψ (particle) ≠ Ψ * (antiparticle)

Majorana (1937)
Real solutions
Ψ (particle) = Ψ * (antiparticle)

- So Majorana fermion is identical to its own antiparticle:

$$\gamma = \gamma^{\dagger}$$

- In condensed matter physics, Majorana fermion may emerge as a quasiparticle by

$$\gamma = c + c^{\dagger}$$

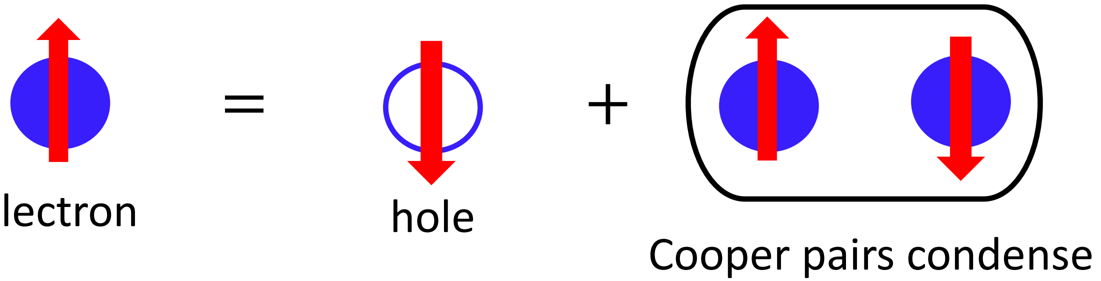
- In particular, Majorana mode can emerge as a zero-energy quasiparticle in p-wave toplogical superconductor.

$$\gamma_E = {\gamma_E}^{\dagger},\ \text{when } E = 0$$

- Majorana zero bound modes obey non-Abelian statistics, and can be applied to topological quantum computation.

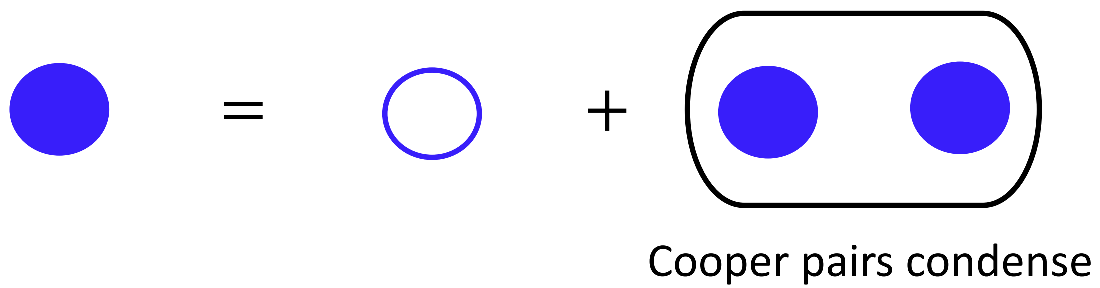
What kind of superconductors hosting Majorana modes?

In an s-wave SC:



Quasi-particle: $b_i = \mu c_\uparrow + \nu c_\downarrow^\dagger$ **not Majorana**

In a spinless p-wave SC:



Quasi-particle: $\gamma_i = \mu c + \nu c^\dagger$ **Is a Majorana if $\mu = \nu^*$**

Intrinsic p-wave pairing systems

Fractional quantum Hall systems at ν=5/2	***(Moore & Read, 1991)***
Chiral p+ip SC	***(Reed & Green, 2000)***
1D p-wave SC	***(Kitaev, 2001)***

Effective p-wave pairing systems by proximity effect

Topological Insulators + s-wave SC	***(Fu & Kane, 2008)*** ***(Nilsson, Akhmerov, & Beenakker, 2008),***
Semiconductors + s-wave SC	***(Sau, Lutchyn, Tewari & Das Sarma, 2010), (Alicea, 2010), (Lutchyn, Sau & Das Sarma, 2010) (Oreg, von Oppen & Refael, 2010) (M. Sato, Takahashi & Fujimoto, 2009)***
Ferromagnetic chains/wires on s-wave SC	***A. Yazidani, A. Bernevig, A. H. MacDonald, F. von Oppen et al. Basel group.***
Symmetry protected topological superconductors Rashba wire + d-wave SC Double Rashba wire + s-wave SC Normal wire + non-centrosymmeric SC Topological crystalline insulator + SC	***DIII class topological superconductors*** ***Topological crystalline superconductors,*** **……**
…….	**………**

Some new developments at PKU groups:

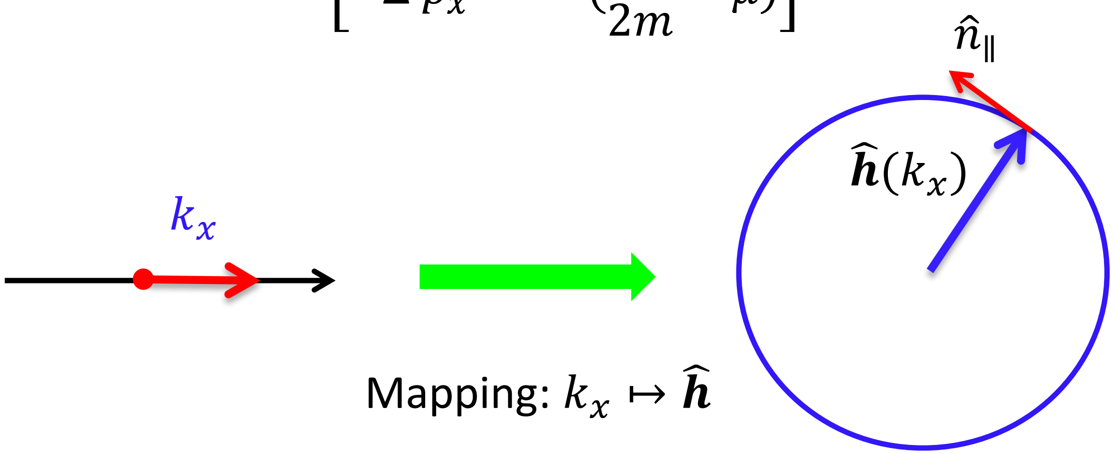
Superconductivity induced on Dirac semimetal Cd3As2: Wang, …, Jia, XJL, Xie, Wei, Wang, *Nature Mater*. *15, 38 (2016).*

Unconventional SC on Weyl semimetal TaAs: Wang, …, Jia, XJL, Wei, Wang, arXiv:1607.00513.

1) 1D example: Intrinsic 1D p-wave superconductor/superfluid

1D $\boldsymbol{p}$-wave topological superconductor (Kitaev, Physics-Uspekhi (2001)). Hamiltonian in the Nambe space $\mathrm{H} = \sum_k (c_k, c_{-k}^{\dagger}) \mathcal{H}_{BdG} (c_k^{\dagger}, c_{-k})^T$, with

$$\mathcal{H}_{BdG} = \begin{bmatrix} \frac{p_x^2}{2m} - \mu & \Delta\, p_x \\ \Delta^* p_x & -(\frac{p_x^2}{2m} - \mu) \end{bmatrix} = h_{\perp}(k_x)\tau_{\perp} + h_z(k_x)\tau_z$$

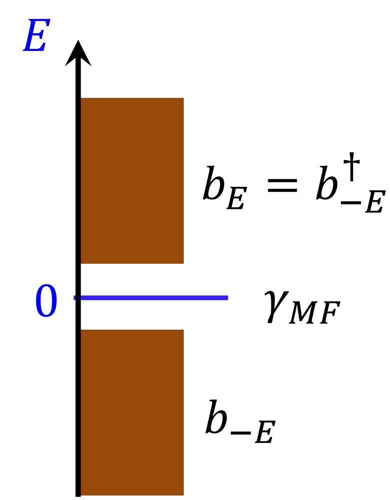


Mapping: $k_x \mapsto \hat{\boldsymbol{h}}$

$$\hat{\boldsymbol{h}}(k_x) = \frac{\boldsymbol{h}}{|\boldsymbol{h}|}$$

1D winding number:

$$\mathcal{N} = \frac{1}{2\pi}\int dk_x\,[\hat{n}_{\parallel} \cdot \partial_{k_x}\hat{\boldsymbol{h}}] = \begin{cases} 0, & \mu < 0; \quad \text{trivial;} \\ 1, & \mu > 0; \quad \text{topological.} \end{cases}$$

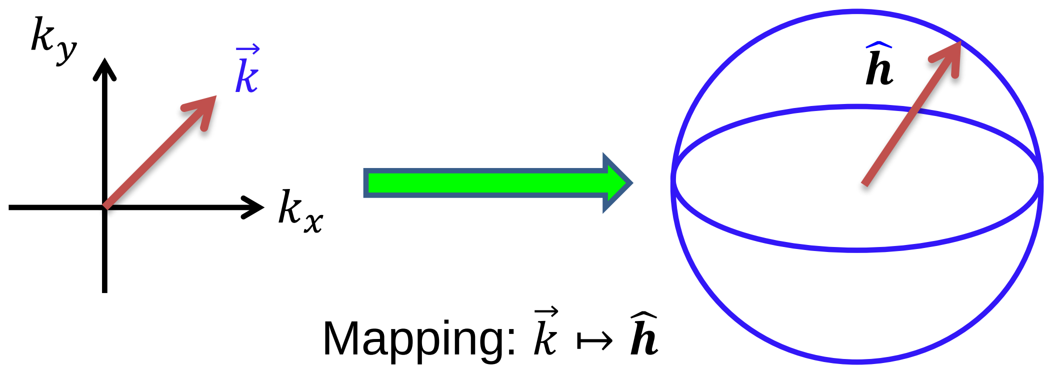


$\mu > 0$

2) 2D example: Intrinsic chiral $p + ip$-wave superconductor/superfluid

Possible candiate: 3He phase, Sr2RuO4 (Reed & Green (2000)); Mean field Hamiltonian in the Nambe space $\mathrm{H} = \sum_k (c_k, c_{-k}^\dagger)\mathcal{H}_{BdG}(c_k^\dagger, c_{-k})^T$, with

$$\mathcal{H}_{BdG} = \begin{bmatrix} \frac{p^2}{2m} - \mu & \Delta\,(p_x - ip_y) \\ \Delta^*\,(p_x + ip_y) & -(\frac{p^2}{2m} - \mu) \end{bmatrix} = h_x(\vec{k})\tau_x + h_y(\vec{k})\tau_y + h_z(\vec{k})\tau_z$$

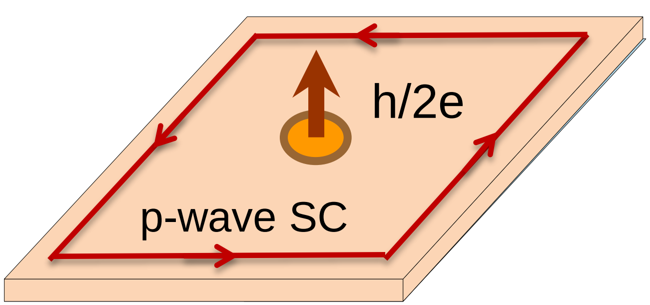


Mapping: $\vec{k} \mapsto \widehat{\boldsymbol{h}}$

$$\widehat{\boldsymbol{h}}(\boldsymbol{k}) = \frac{\boldsymbol{h}}{|\boldsymbol{h}|}$$

1st Chern number:

$$C_1 = \frac{1}{4\pi^2}\int d^2\vec{k}[\widehat{\boldsymbol{h}} \cdot (\partial_{k_x}\widehat{\boldsymbol{h}} \times \partial_{k_y}\widehat{\boldsymbol{h}})] = \begin{cases} 0, & \mu < 0; \quad \text{trivial;} \\ 1, & \mu > 0; \quad \text{topological.} \end{cases}$$



1) There are chiral Majorana chiral modes localized on the edge.

2) A single Majorana bound state $\gamma(E)$ with $E = 0$ exists in the vortex core.

Topological p-wave superconductors by proximity effect

Idea: effective p-wave pairings can be induced in spin-orbit coupled systems in proximity to an s-wave superconductor.

Conditions: 1) Spin should not be fully polarized at Fermi surface;

2) There are odd number of energy bands crossing the Fermi energy.

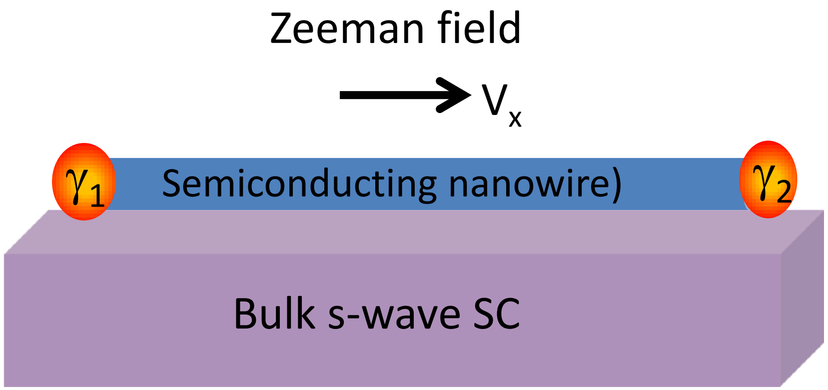
Examples:

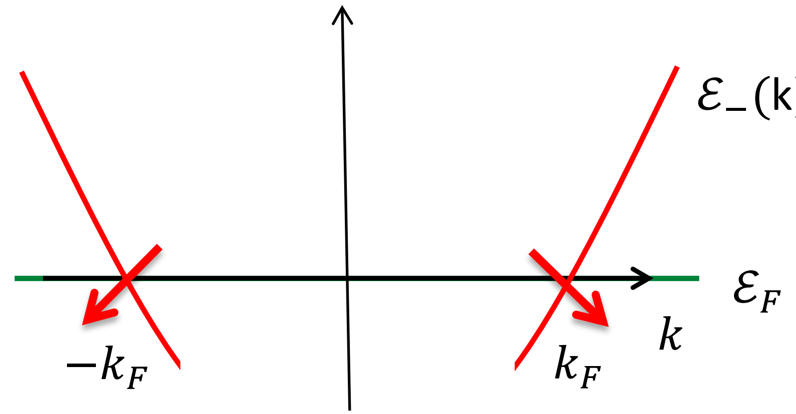
A) Topological insulators + s−wave SC.

B) Semiconductors + Zeeman splitting + s-wave SC

P. W. Anderson: *more is different.*

1) Effective 1D p-wave SC: spin-orbit coupling+s-wave pairing+Zeeman coupling (Lutchyn, Sau, Tudor, Das Sarma, PRL (2010); Oreg, Refael, and von Oppen, PRL (2010))





After projection: effective spinless

$$H = \int dx c_\sigma^\dagger(x)\left(\frac{p_x^2}{2m^*} - \mu + i\lambda_R p_x \sigma_y + V_x \sigma_x\right)_{\sigma\sigma'} c_{\sigma'}(x) + \int dx[\Delta(x) c_\uparrow(x) c_\downarrow(x) + h.c.]$$

projection

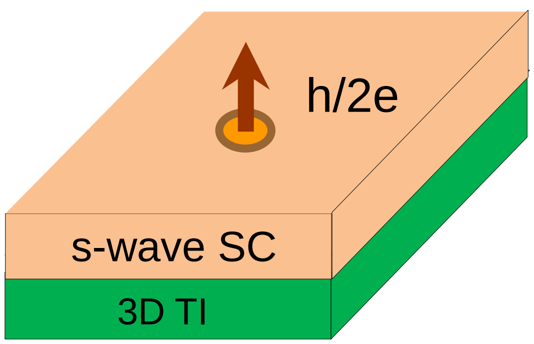
$$\mathcal{H}_{s-wave} = \sum_k \Delta c_{k\uparrow} c_{-k\downarrow} + h.c. \longrightarrow \mathcal{H}_{p-wave} = \sum_k \Delta_k c_{k,-} c_{-k,-} + h.c.$$

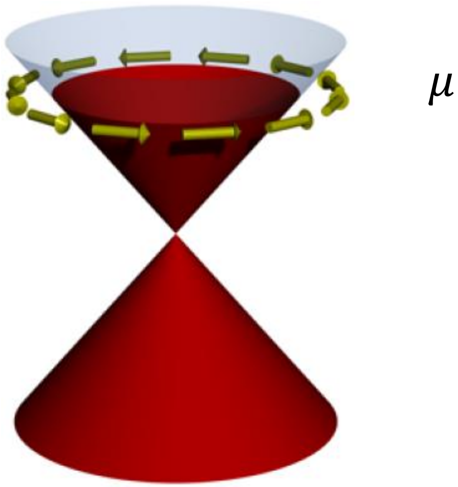
when $V_x > \sqrt{\mu^2 + \Delta^2}$.

2) Effective 2D $p + ip$–wave superconductor: topological insulators + s–wave SC (*Fu & L. Kane, PRL, 2008*).

$$\mathcal{H} = \mathcal{H}_{surf} + \mathcal{H}_{s-wave}$$

$$\mathcal{H}_{surf} = v_f\left(p_x\sigma_y - p_y\sigma_x\right) - \mu$$





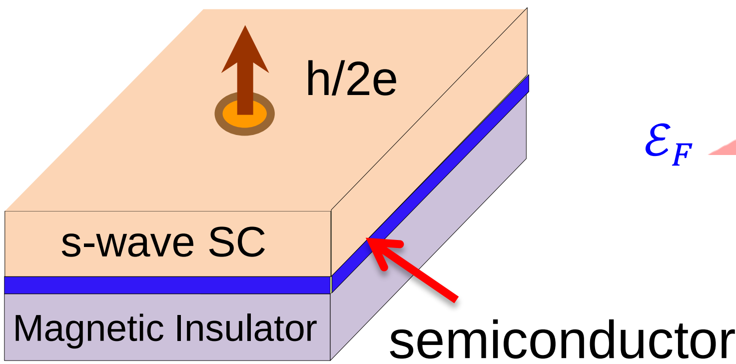
Courtesy of Franz

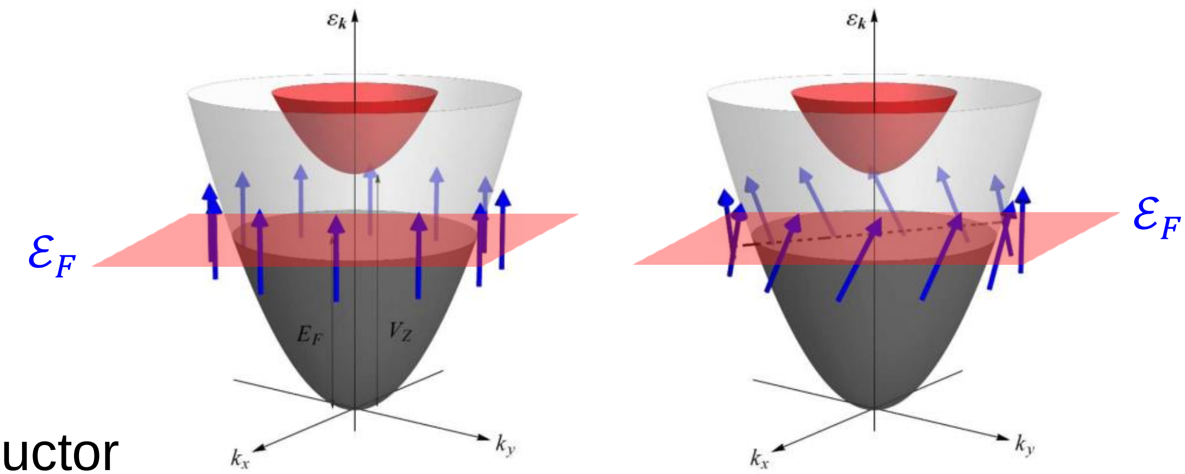
For any μ,

$$\mathcal{H}_{s-wave} = \sum_k \Delta c_{k\uparrow} c_{-k\downarrow} + h.c. \xrightarrow{\text{projection}} \mathcal{H}_{p-wave} = \sum_k \Delta_k c_{k,+} c_{-k,+} + h.c.$$

3) Effective 2D $p + ip$–wave superconductor: SO coupling + magnetizaton + s–wave SC *(M. Sato, Takahashi & Fujimoto, PRL 2009; Sau et al., PRL 2010; Alicea, PRB 2010):*

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{s-wave} \qquad \mathcal{H}_0 = \frac{p^2}{2m^*} - \mu + \lambda_R(p_x\sigma_y - p_y\sigma_x) - V_z\sigma_z$$





$\lambda_R = 0$ $\qquad$ $\lambda_R > 0$

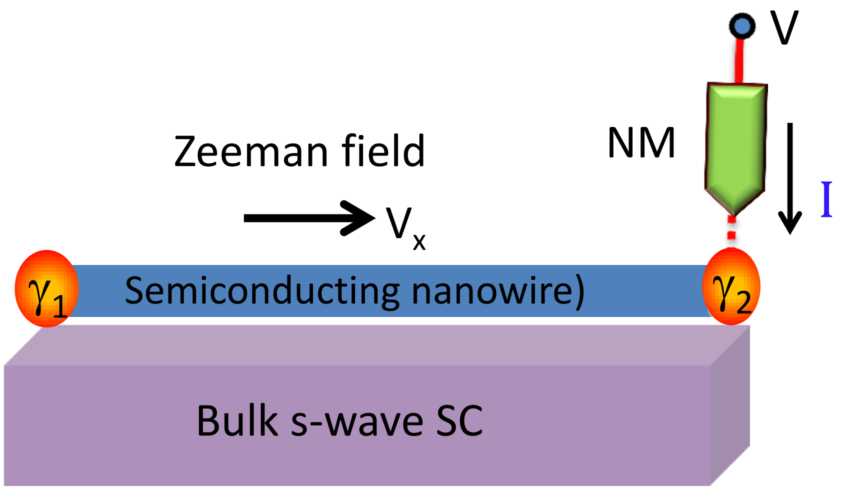
projection

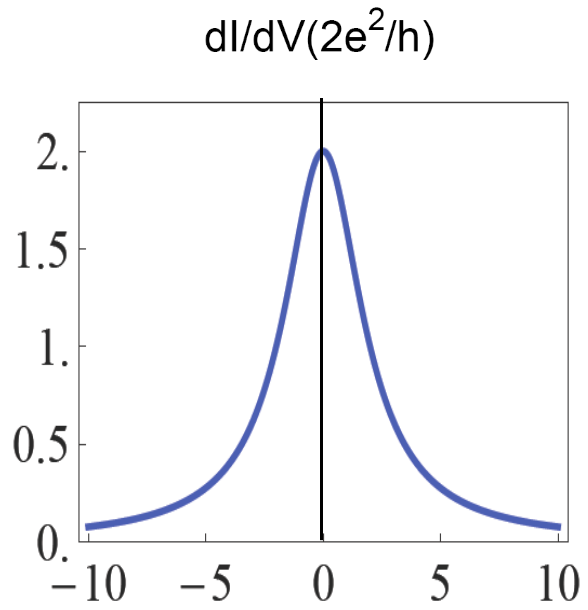
$$\mathcal{H}_{s-wave} = \sum_k \Delta c_{k\uparrow} c_{-k\downarrow} + h.c. \Longrightarrow \mathcal{H}_{p-wave} = \sum_k \Delta_k c_{k,-} c_{-k,-} + h.c.$$

when $V_z > \sqrt{\mu^2 + \Delta^2}$.

Properties and observation of Majorana fermions

1. Zero bias peak quantization in the tunneling charge transport at zero temperature (*K.T. Law etal, PRL(2009), Flensberg, PRB(2010); Wimmer etal, New J. Phys (2011)*)

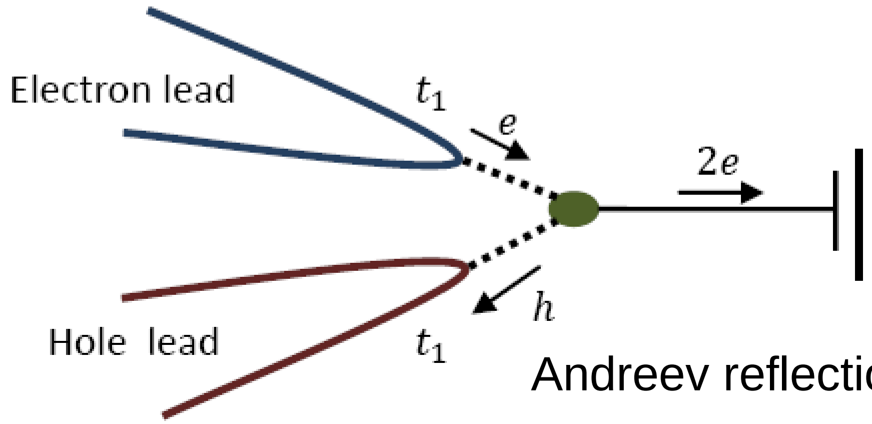




At zero temperature

Self-hermitian: $\gamma = \int dx \left[u(x)c(x) + v(x)c^{\dagger}(x)\right]$with $u(x) = v^{*}(x)$; Effectively charge neutral:

$$e_{\gamma}^{*} = e \int dx \left(|u|^2 - |v|^2\right) = 0$$



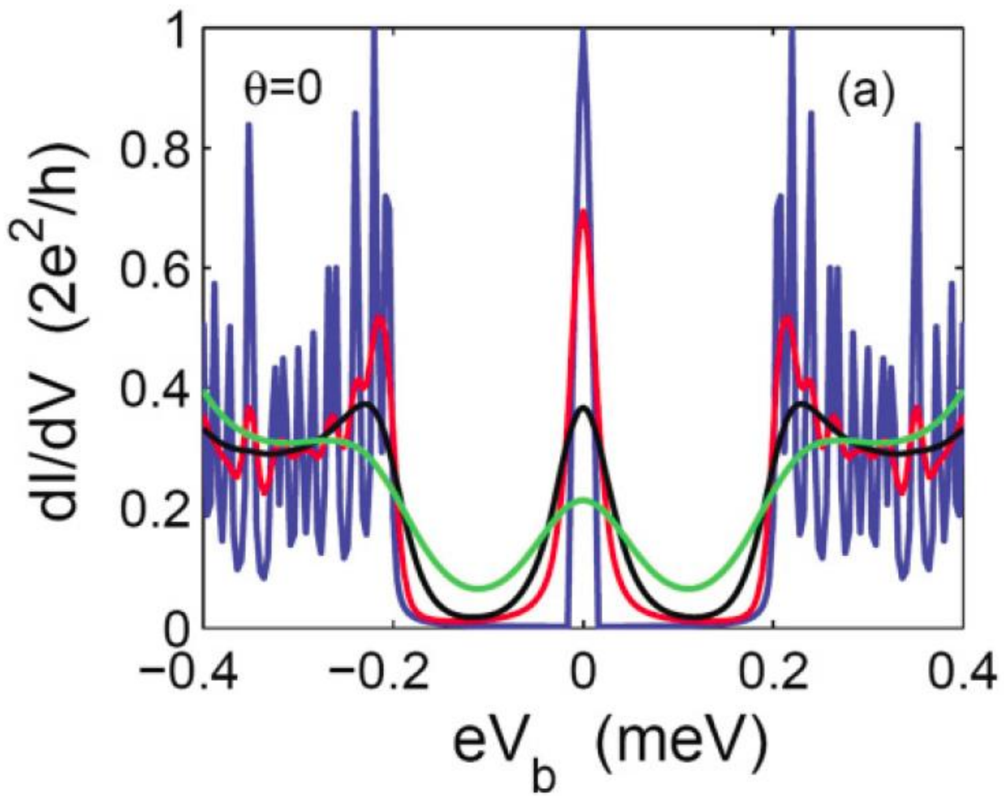
Andreev reflection

The tunneling energies: $t_1 \propto |u|^2 = t_2 \propto |v|^2$ ⟶ Resonant two-lead tunneling:

⟶ $\left(\frac{dI}{dV}\right)_{peak} = \frac{2e^2}{h}$ K. T. Law, T. K. Ng, P. A. Lee, PRL(2009)

Tunneling conductance at finite temperatures

$$I = \frac{e^2}{h}\int d\omega\, \mathrm{Tr}[\Gamma^e \mathcal{G}^R(\omega)\,\Gamma^h \mathcal{G}^A(\omega)]\,[1 - f(\omega - eV_b)]$$
$$+ \frac{e^2}{h}\int d\omega \Gamma(\omega)\, N(\omega)\,[1 - f(\omega - eV_b)]\,,$$

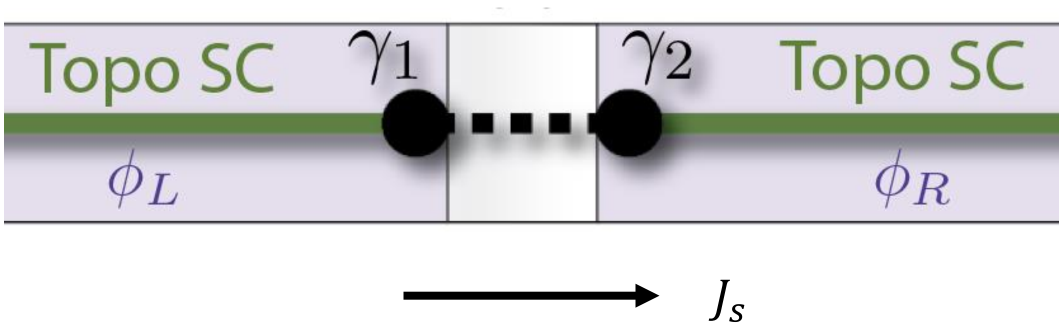


The curves correspond to:

Blue: $\mathrm{T} = 0\mathrm{K}$
Red: $\mathrm{T} = 60\mathrm{mK}$
Black: $\mathrm{T} = 180\mathrm{mK}$
Green: $\mathrm{T} = 360\mathrm{mK}$

XJL, PRL 109, 106404 (2012); XJL & A. Lobos, PRB, 87, 060504 (R) (2013).

2. Fractional Josephson effect:



$$H_{\text{eff}} = -\frac{\Gamma}{2}\cos\left(\frac{\Delta\phi}{2}\right) i\gamma_1\gamma_2$$

$$\Delta\phi = \phi_L - \phi_R$$

$$J_s = J_c \sin\left(\frac{\Delta\phi}{2}\right),$$

→ 4π periodicity

Experiments

1. *V. Mourik et al, Science 336, 1003 (2012):*

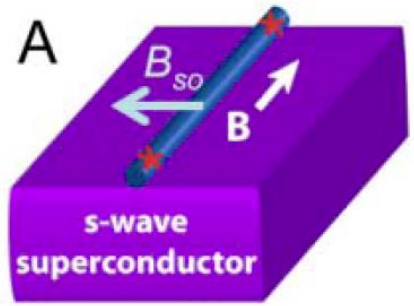
   Nanowire: InSb; Superconductor: NbTiN

2. *H. Xu group: M.T. Deng et al,* Nano Lett. (2012);

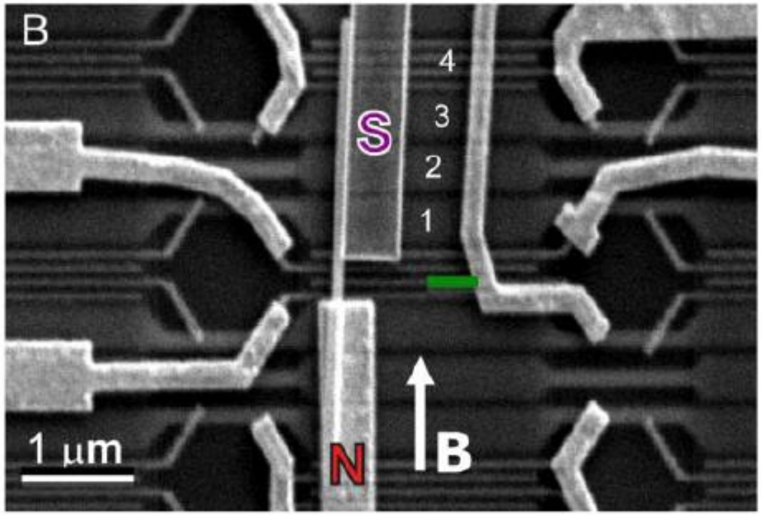
   Nanowire: InSb; Superconductor: Nb

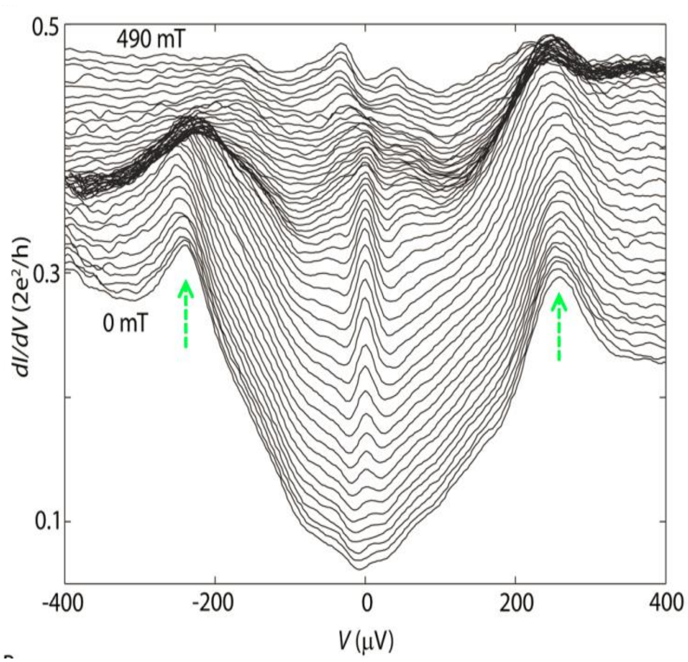
3. *A. Das et al, Nature Physics (2012),*

   Nanowire: InAs; Superconductor: Al



Measuring the **zero bias peak:** *Law, Ng, and Lee, PRL (2009); Flensberg, PRB (R) (2010); Wimmer etal., PRB(2011); XJL, PRL (2012); XJL and Alejandro, PRB(R) (2013).*

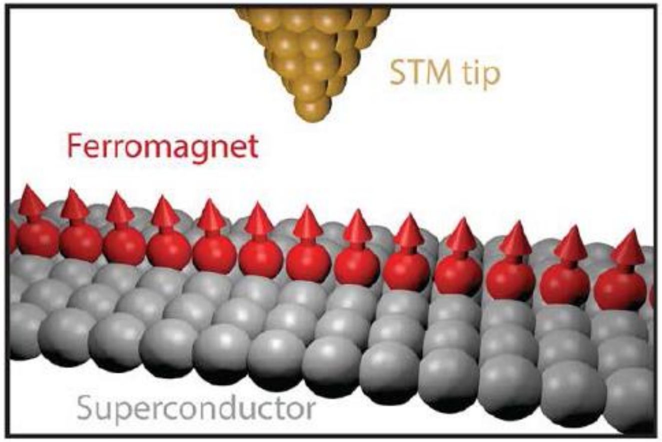


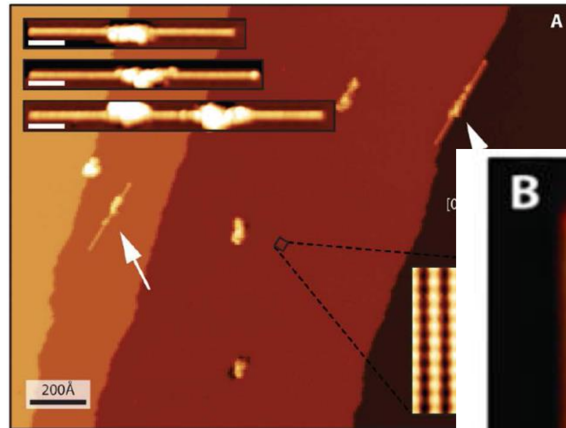


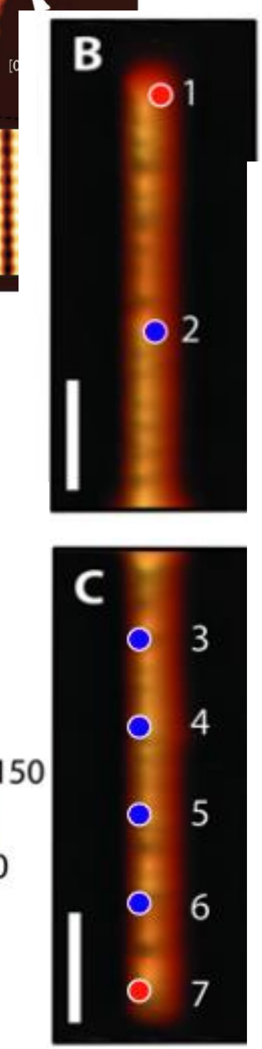
Fe-chains on an s-wave SC

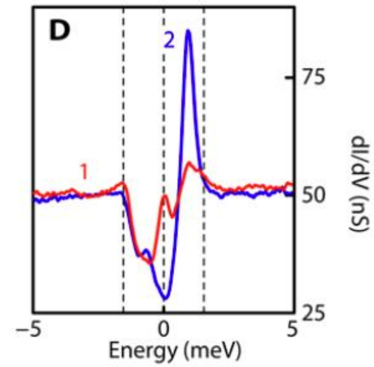
A. Yazdani group, Science, **346**, 602 (2014), related to but different from quite a few earlier theoretical proposals.

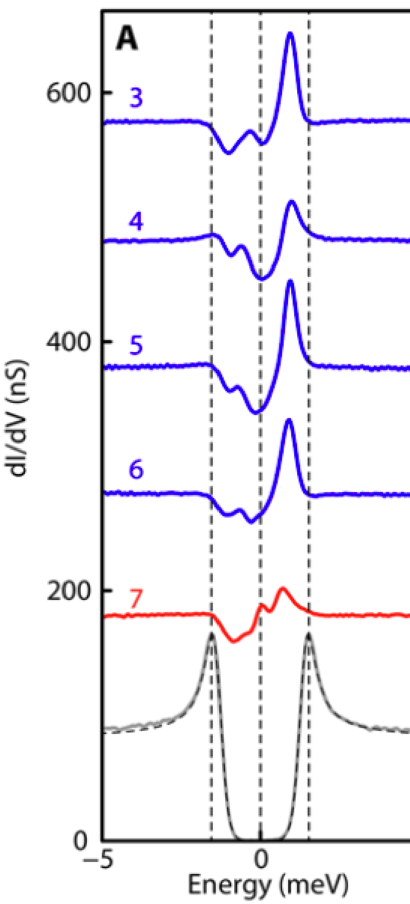
Ferromagnetic Fe-chains on Pb superconductor



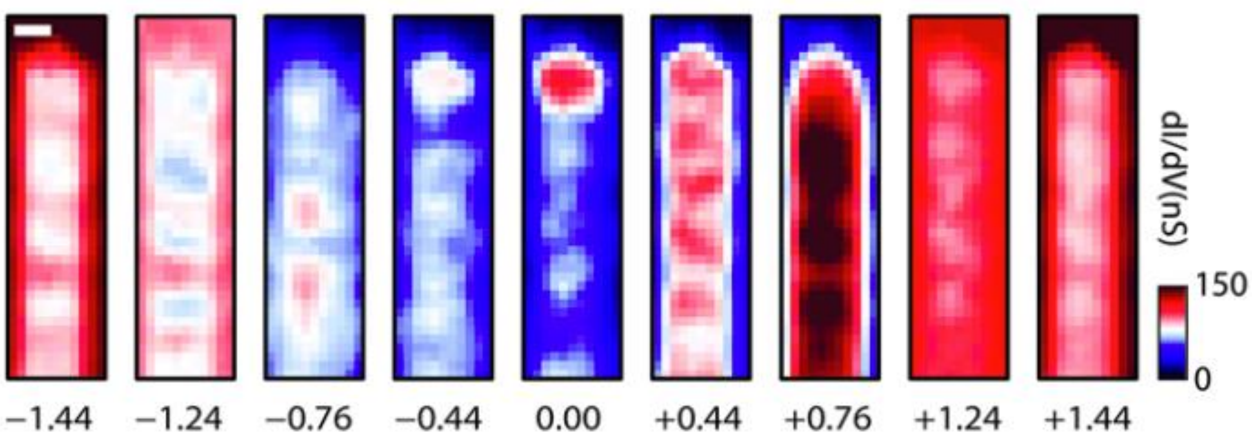








STM tunneling spectra measured on the Fe-chain



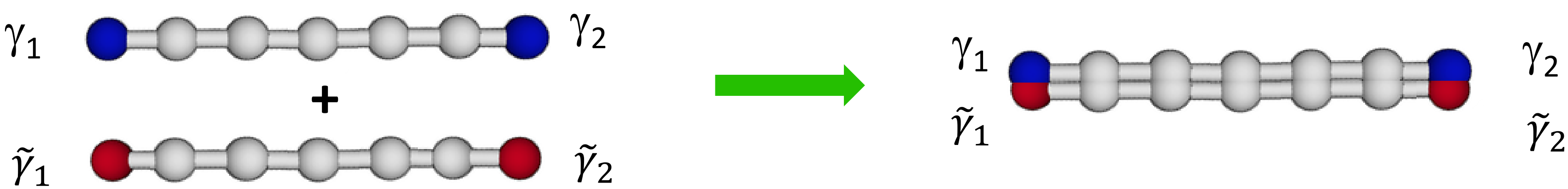
Also lots of improvements have been done in several recent experiments.

Time-reversal invariant (DIII class) topological superconductor

1D TRI topological superconductors (Qi, Hughes, Raghu, Zhang, PRL, 2009; A. P. Schnyder etal., PRB, 2010; Teo, Kane, PRB, 2010; Beenakker etal, PRB, 2011)

Two-copy version of p-wave models:

1D p-wave SC (TR breaking)



time-reversed copy

Two Majoranas exist at each end

Relations: $\mathcal{T}\gamma_{1,2}\mathcal{T}^{-1} = \tilde{\gamma}_{1,2},\ \mathcal{T}\tilde{\gamma}_{1,2}\mathcal{T}^{-1} = -\gamma_{1,2},\ \mathcal{T}^2 = -1$

$\gamma_j, \tilde{\gamma}_j$, Majorana Kramer's pair (MKP)

Time reversal protection:

MKP cannot couple together!

Z_2 invariant

Realization: 1D time-reversal invariant topological superconductor

Theoretical models:

Spin-orbit coupling + d-wave SC: C. L. M. Wong et al., PRB, 2012.

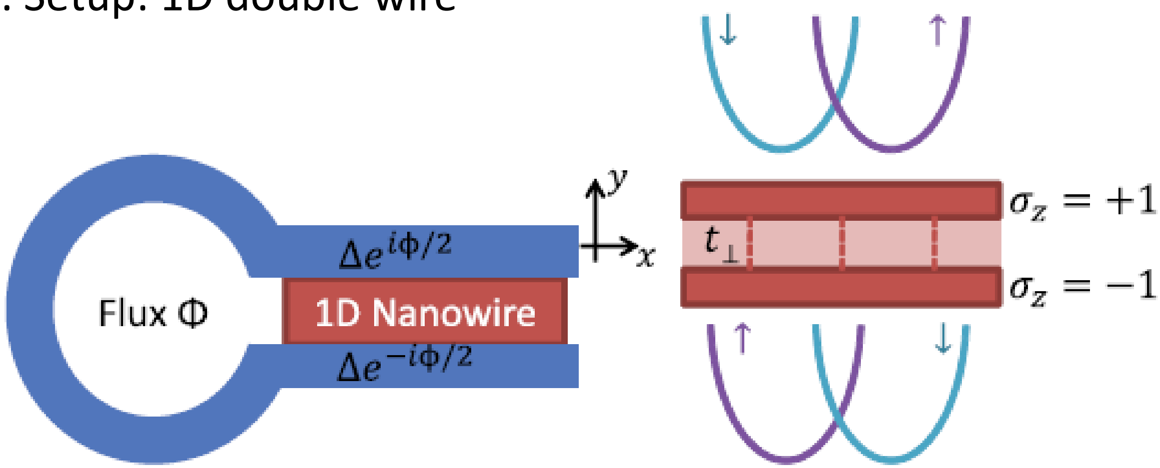
Spin-orbit coupling + $s_{\pm}$ -wave SC: F. Zhang etal., PRL, 2012.

Proximity effect of non-centrosymmetric SC: S. Nakosai et al., PRL, 2013; XJL, Chris L. M. Wong, and K. T. Law, PRX, 2014.

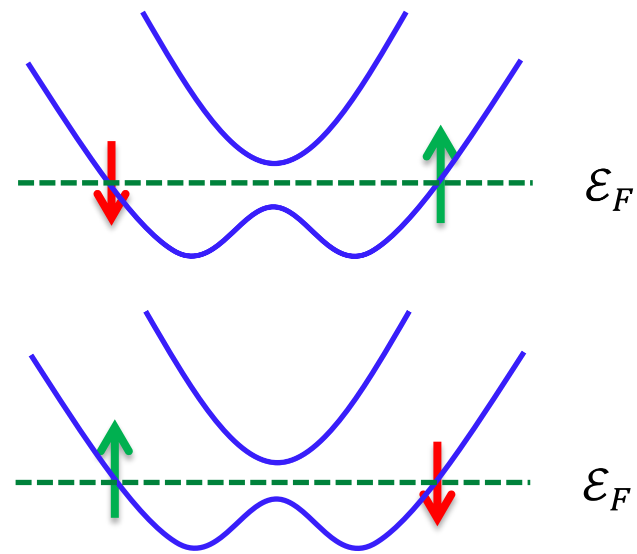
Spin-orbit coupled double wire + s-wave SCs: Keselman, Fu, Stern, and Berg, PRL, 2013.

………

e.g. Setup: 1D double wire



(a) (b)

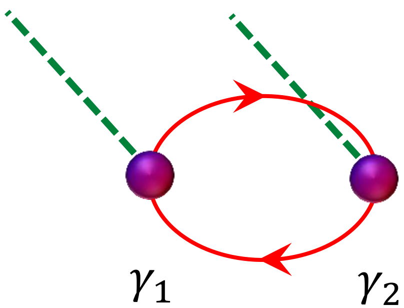


Non-Abelian Statistics

2D system, Ivanov, PRL, 2001

- Abelian statistics: Bosons, Fermions, and Abelian anyons $\quad U_{12}|\psi\rangle = e^{i\phi}|\psi\rangle$

- For Majorana modes, two separated Majorana bound states consist of one usual complex fermion = 1 qubit:



$$c = \gamma_1 + i\gamma_2 \qquad |0\rangle \text{ and } \qquad |1\rangle = c^\dagger|0\rangle$$

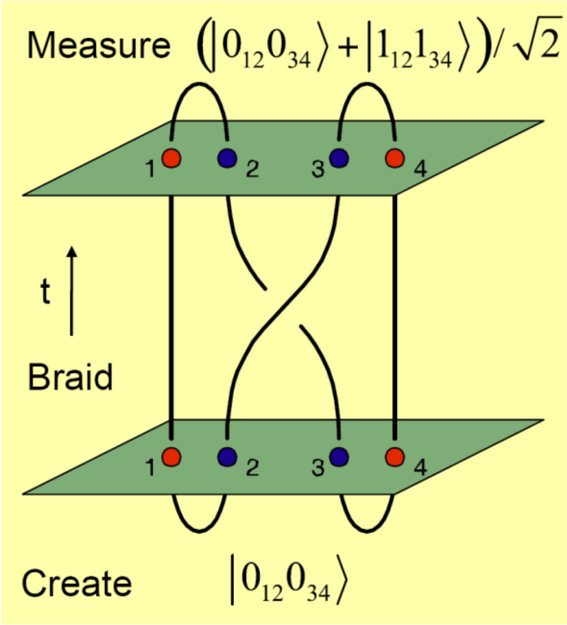
$$\gamma_1 \longrightarrow -\gamma_2$$

$$\gamma_2 \longrightarrow \gamma_1$$

Braiding operator $\quad U_{12} = e^{\frac{\pi}{4}\gamma_1\gamma_2} \rightarrow \begin{bmatrix} 1 & \\ & i \end{bmatrix}$

$$|0\rangle \longrightarrow |0\rangle, \qquad |1\rangle \longrightarrow i|1\rangle$$

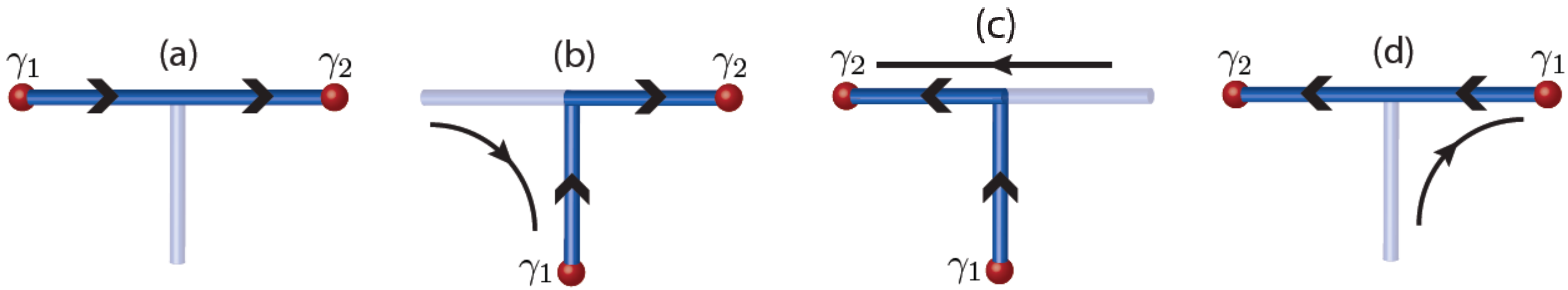
- Consider four Majorana modes, which form 2 qubits:



Topological quantum computation

Figure by Kane

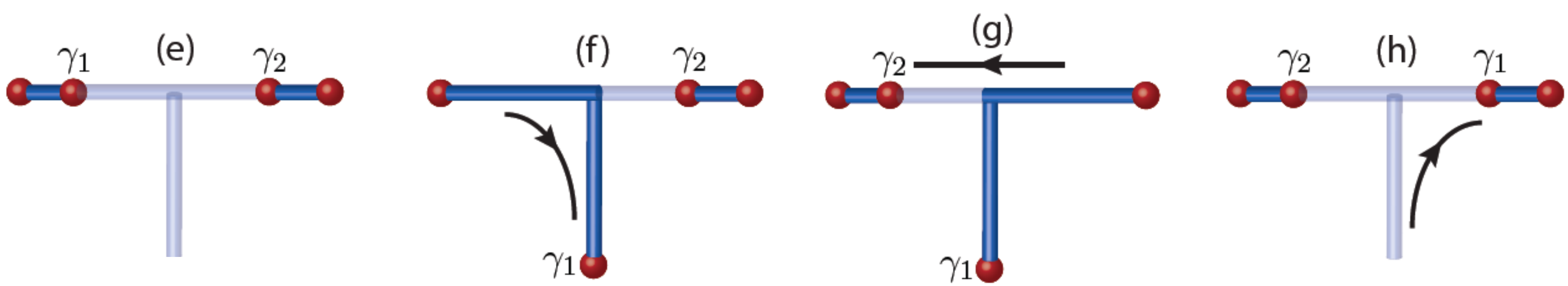
1D system (Alicea et al., Nature Phys. 2011)



t

$$U_{12} = e^{\frac{\pi}{4}\gamma_1\gamma_2}$$

Four Majorana modes



$|00\rangle$

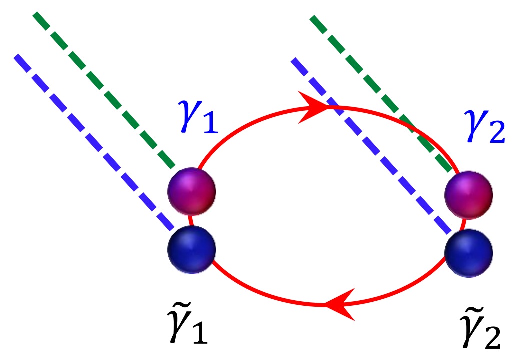
t

$(|00\rangle + i|11\rangle)/\sqrt{2}$

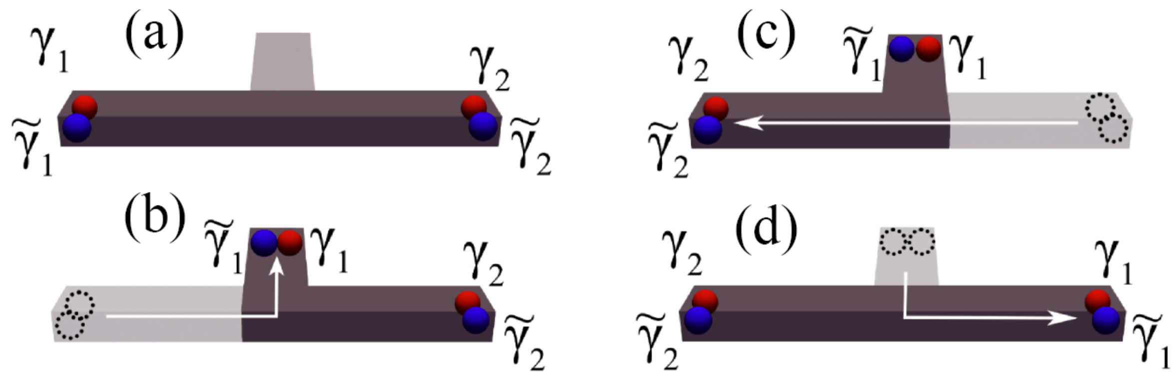
Non-Abelian braiding of Majorana Kramers' pairs?

Definition: the braiding operation for exchanging two Majorana Kramers' pairs, but without local operations of two Majorana modes in a single Kramers' pair.

1) General case



2) 1D version



Braiding operation:

$$U_{12}|0\tilde{0}\rangle \longrightarrow |0\tilde{0}\rangle, \qquad U_{12}|1\tilde{1}\rangle \longrightarrow |1\tilde{1}\rangle$$

$$U_{12}|0\tilde{1}\rangle \longrightarrow e^{i\phi}|0\tilde{1}\rangle, \qquad U_{12}|1\tilde{0}\rangle \longrightarrow e^{-i\phi}|1\tilde{0}\rangle$$

Condition: $U_{12}^4 = 1$, then: $\phi = 0, \pi/2$, or π.

Consider the special case with two decoupled time-reversed copies: $\phi \equiv \pi/2$.

So, generically: $U_{12}=e^{\frac{\pi}{4}\gamma_1\gamma_2}e^{\frac{\pi}{4}\tilde{\gamma}_1\tilde{\gamma}_2}$

XJL, Chris L. M. Wong, and K. T. Law, PRX, 4, 021018 (2014).

Questions

1. What are the sufficient conditions for the non-Abelian braiding of Kramers' pairings?

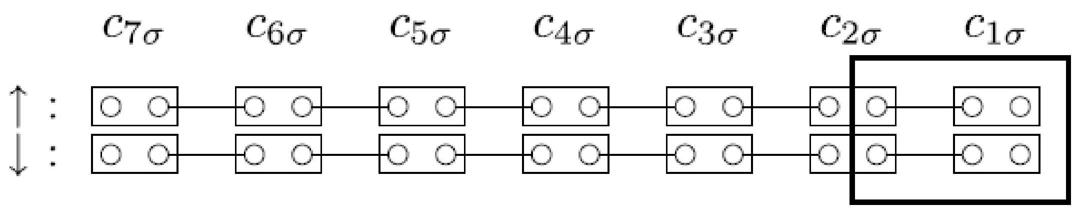
2. Guess the sufficient condition：

   Time-reversal symmetry exists at every time in the braiding?

   Answer: No!

Local mixing by time-dependent perturbations

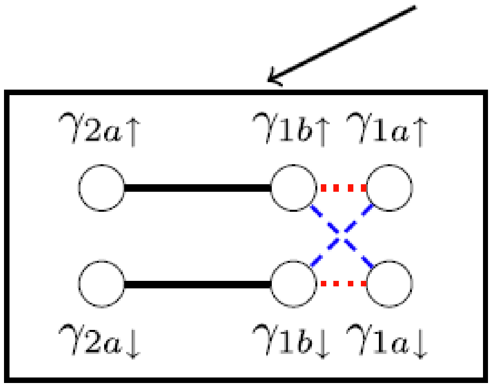
Consider the two time-reversed copies of Kitaev chain. If $t = \Delta_p, \mu = 0$, the Majorana modes are localized in the end site (Kitaev, 2001).



Together with the local couplings in the end site:

$\mu \sum_\sigma c^\dagger_{1\sigma} c_{1\sigma}$

$\Delta[c_{1\uparrow} c_{1\downarrow} + c^\dagger_{1\downarrow} c^\dagger_{1\uparrow}]$



$\mu = B\cos\alpha,$ $\Delta = B\sin\alpha.$ $\tan\theta = B/E_g.$

Consider the parameter varying one loop, $\alpha : 0 \to 2\pi.$

Gives the Berry phase: $\varphi = \oint A_\alpha d\alpha = \pi \sin^2\theta.$. This leads to local mixing of the qubit states

$$U\left|0\tilde{0}\right\rangle \longrightarrow \cos\frac{\varphi}{2}\left|0\tilde{0}\right\rangle + \sin\frac{\varphi}{2}\left|1\tilde{1}\right\rangle,$$

Decoherence effect!

K. Wölms, A. Stern, and K. Flensberg, PRL 113, 246401 (2014); PRL 115, 189902 (2015).

The key issue:

To rule out random local operations in a physical system.

We have to figure out:

1. Sufficient conditions for non-Abelian braiding of Majorana Kramers' pairs.

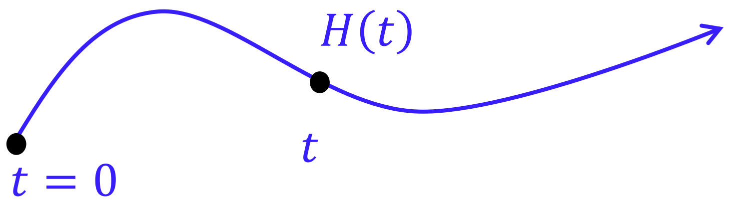
2. How well such conditions can be satisfied in real physical systems?

Braiding conditions

Condition 1: time-reversal symmetry is satisfied at every time of the braiding

$$\hat{\mathcal{T}} H(t) \hat{\mathcal{T}}^{-1} = H(t)$$

where $\hat{\mathcal{T}} = i\sigma_y \mathcal{K}$.



Under this condition, Majorana Kramers' pairs exist at every time in the braiding.

P. Gao, Y.-P. He and XJL, Phys. Rev. B 94, 224509 (2016).

Condition 2: (**adiabatic condition**) Majoranas should not be excited into bulk states, therefore the duration T of operation satisfy:

$$T \gg 1/E_g$$

After a braiding process among $\{\gamma_1, \tilde{\gamma}_1, \gamma_2, \tilde{\gamma}_2\}$,

$$\boxed{e^{-iH_E T}}$$

$$\{\gamma_1, \tilde{\gamma}_1, \gamma_2, \tilde{\gamma}_2\} \xrightarrow{\mathcal{B}=\mathrm{U(T)}=\hat{\mathrm{T}}e^{-i\int_{-T/2}^{T/2} H(t)dt}} \{\gamma_1, \tilde{\gamma}_1, \gamma_2, \tilde{\gamma}_2\},$$

$$\{\gamma_1, \tilde{\gamma}_1\} \xleftrightarrow{\mathcal{B}} \{\gamma_2, \tilde{\gamma}_2\},$$

Under the adiabatic condition, we can introduce an effective Hamiltonian H_E to describe the braiding operation:

$$U(T) = e^{-iH_E T}$$

An irreducible representation space of H_E: $\{\gamma_1, \tilde{\gamma}_1, \gamma_2, \tilde{\gamma}_2\}$

Condition 3: Symmetries of H_E

$$e^{-iH_E T} = U(T) = \lim_{N\to\infty} e^{-iH(T/2)\Delta t} \cdots e^{-iH(\Delta t - T/2)\Delta t} e^{-iH(-T/2)\Delta t}$$

where $\Delta t = T/N$.

1. Particle-hole symmetry

$$\boxed{\hat{\mathcal{P}} H(t) \hat{\mathcal{P}}^{-1} = -H(t)}$$

If $\hat{\mathcal{P}} H_E \hat{\mathcal{P}}^{-1} = -H_E$, $\hat{\mathcal{P}} e^{-iH_E T} \hat{\mathcal{P}}^{-1} = e^{-iH_E T}$.

$$\hat{\mathcal{P}} e^{-iH_E T} \hat{\mathcal{P}}^{-1} = \lim_{N\to\infty} e^{i\hat{\mathcal{P}} H(T/2)\hat{\mathcal{P}}^{-1}\Delta t} \cdots e^{i\hat{\mathcal{P}} H(\Delta t - T/2)\hat{\mathcal{P}}^{-1}\Delta t} e^{i\hat{\mathcal{P}} H(-T/2)\hat{\mathcal{P}}^{-1}\Delta t} = e^{-iH_E T}$$

$$\Longrightarrow \; \hat{\mathcal{P}} H_E \hat{\mathcal{P}}^{-1} = -H_E$$

2. Time-reversal symmetry

$$\boxed{\hat{\mathcal{T}} H(t) \hat{\mathcal{T}}^{-1} = H(t)}$$

If $\hat{\mathcal{T}} H_E \hat{\mathcal{T}}^{-1} = H_E$, $\hat{\mathcal{T}} e^{-iH_E T} \hat{\mathcal{T}}^{-1} = [e^{-iH_E T}]^\dagger$

$$\hat{\mathcal{T}} e^{-iH_E T} \hat{\mathcal{T}}^{-1} = \lim_{N\to\infty} e^{iH(T/2)\,\Delta t} \cdots e^{iH(\Delta t - T/2)\Delta t} e^{iH(-T/2)\Delta t}$$

?

$$[e^{-iH_E T}]^\dagger = \lim_{N\to\infty} e^{iH(-T/2)\Delta t} \cdots e^{iH(T/2 - \Delta t)\Delta t} e^{iH(T/2)\Delta t}$$

$$\Longrightarrow \hat{\mathcal{T}} H_E \hat{\mathcal{T}}^{-1} \;?\!= H_E$$

Symmetry of H_E: Majorana swapping

Our key motivation is let H_E satisfy a new TR like anti-unitary symmetry. If $\hat{S}H(t)\hat{S}^{-1} = H(-t)$, where κ unitary operator,

$$\hat{\kappa}\hat{\mathcal{T}}e^{-iH_ET}\hat{\mathcal{T}}^{-1}\hat{\kappa}^{-1} = \lim_{N\to\infty}\hat{\kappa}[e^{iH(T/2)\,\Delta t}\cdots e^{iH(\Delta t-T/2)\Delta t}e^{iH(-T/2)\Delta t}]\hat{\kappa}^{-1}$$

$$= \lim_{N\to\infty} e^{iH(-\mathrm{T}/2)\,\Delta t}\cdots e^{iH(T/2-\Delta t)\Delta t}e^{iH(T/2)\Delta t}$$

$$= [e^{-iH_ET}]^{\dagger}$$

Then,

$$\hat{\Theta}H_E\hat{\Theta}^{-1} = H_E$$

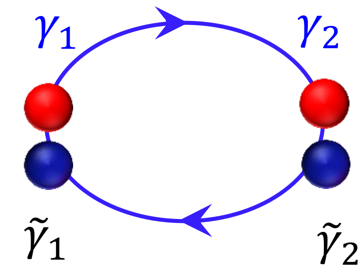
where $\hat{\Theta} = \hat{S}\hat{\mathcal{T}}$.

Lemma: if the braiding Hamiltonian of the 1D-junction satisfies inversion symmetry along braiding direction

$$\pi_x H(x)\pi_x^{-1} = H(x),$$

we can always find an $\hat{S}$ - symmetry.

MKPs’ braiding



Now we have the transformation between these two MKPs,

$$\hat{S}\gamma_1(\tilde{\gamma}_1)\,\hat{S}^{-1} = \gamma_2(\tilde{\gamma}_2)$$

Namely, $\hat{S}$ is a unitary Majorana swapping operator, reflecting the MKPs’ positions.

The braiding Hamiltonian

$$\hat{S}H(-t)\hat{S}^{-1} = H(t), \text{ namely } \widehat{\Theta}H_E\widehat{\Theta}^{-1} = H_E$$

where $\widehat{\Theta} = \hat{S}\hat{\mathcal{T}}$,

$$H_E = i\epsilon_1\gamma_1\tilde{\gamma}_1 - i\epsilon_1\gamma_2\tilde{\gamma}_2 + i\epsilon_2\gamma_1\gamma_2 + i\epsilon_2\tilde{\gamma}_1\,\tilde{\gamma}_2$$

MKPs' braiding

The braiding matrix:

0 0

$$e^{-iH_E T'}\begin{pmatrix}\gamma_1\\ \tilde{\gamma}_1\\ \gamma_2\\ \tilde{\gamma}_2\end{pmatrix}=\begin{pmatrix}\cos\sqrt{\epsilon_1^2+\epsilon_2^2}T' & \frac{\epsilon_1\sin\sqrt{\epsilon_1^2+\epsilon_2^2}T'}{\sqrt{\epsilon_1^2+\epsilon_2^2}} & \frac{\epsilon_2\sin\sqrt{\epsilon_1^2+\epsilon_2^2}T'}{\sqrt{\epsilon_1^2+\epsilon_2^2}} & 0\\ -\frac{\epsilon_1\sin\sqrt{\epsilon_1^2+\epsilon_2^2}T'}{\sqrt{\epsilon_1^2+\epsilon_2^2}} & \cos\sqrt{\epsilon_1^2+\epsilon_2^2}T' & 0 & \frac{\epsilon_2\sin\sqrt{\epsilon_1^2+\epsilon_2^2}T'}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\\ -\frac{\epsilon_2\sin\sqrt{\epsilon_1^2+\epsilon_2^2}T'}{\sqrt{\epsilon_1^2+\epsilon_2^2}} & 0 & \cos\sqrt{\epsilon_1^2+\epsilon_2^2}T' & -\frac{\epsilon_1\sin\sqrt{\epsilon_1^2+\epsilon_2^2}T'}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\\ 0 & -\frac{\epsilon_2\sin\sqrt{\epsilon_1^2+\epsilon_2^2}T'}{\sqrt{\epsilon_1^2+\epsilon_2^2}} & \frac{\epsilon_1\sin\sqrt{\epsilon_1^2+\epsilon_2^2}T'}{\sqrt{\epsilon_1^2+\epsilon_2^2}} & \cos\sqrt{\epsilon_1^2+\epsilon_2^2}T'\end{pmatrix}\begin{pmatrix}\gamma_1\\ \tilde{\gamma}_1\\ \gamma_2\\ \tilde{\gamma}_2\end{pmatrix}$$

Braiding requests:

$$\cos\sqrt{\epsilon_1^2+\epsilon_2^2}T=0 \text{ and } \epsilon_1=0$$

Braiding operator

$$U_{12}=\exp\left(\frac{\pi}{4}\gamma_1\gamma_2\right)\exp\left(\frac{\pi}{4}\tilde{\gamma}_1\tilde{\gamma}_2\right)$$

Condition 3 (Majorana swapping): $\hat{S}H(-t)\hat{S}^{-1}=H(t)$.

Now we have the three conditions which show how to ideally braid the symmetry-protected topological anyons!

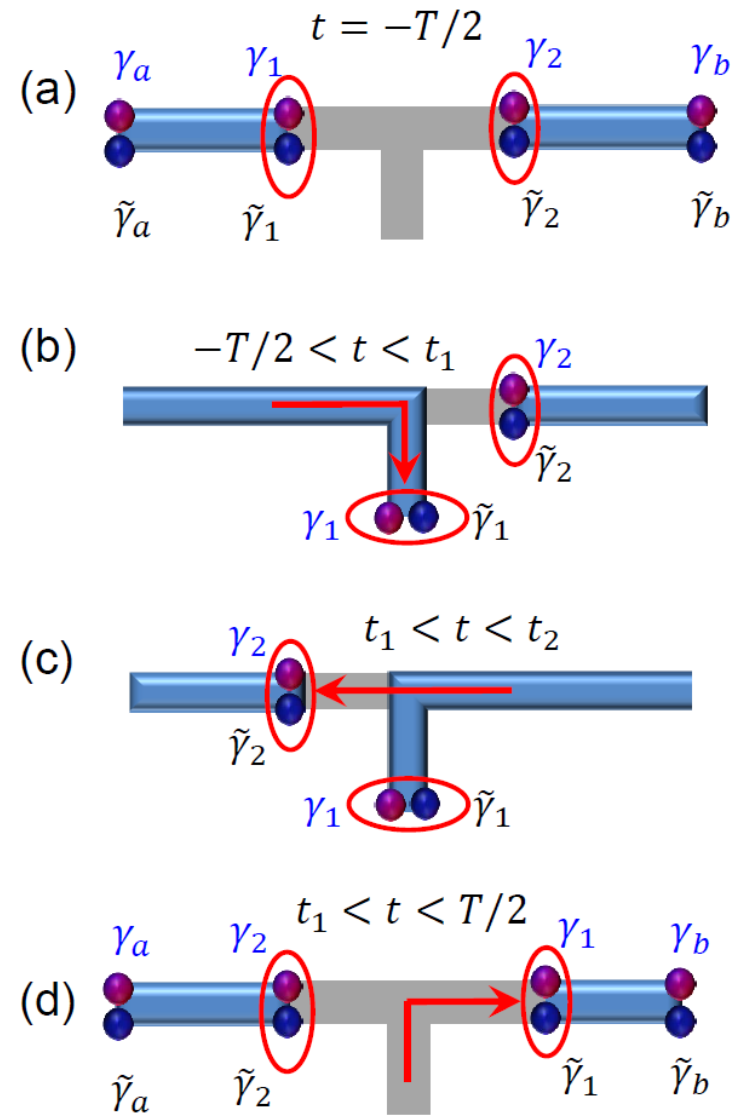
Numerical simulation for MKPs' braiding

Hamiltonian for TR invariant TSC:

$$H_0 = \sum_{\langle i,j\rangle,\sigma} t_0 c^\dagger_{i\sigma} c_{j\sigma} + \sum_j (\pm\alpha_R c^\dagger_{j\uparrow} c_{j\pm1\downarrow} + \Delta_p c_{j\uparrow} c_{j+1\uparrow} + \Delta_p^* c_{j\downarrow} c_{j+1\downarrow} + \Delta_s c_{j\uparrow} c_{j\downarrow} + h.c.) - \mu \sum_{j\sigma} n_{j\sigma}$$

With static random disorder potentials:

$$V_{\rm dis} = \sum_j W_j (n_{j\uparrow} + n_{j\downarrow})$$

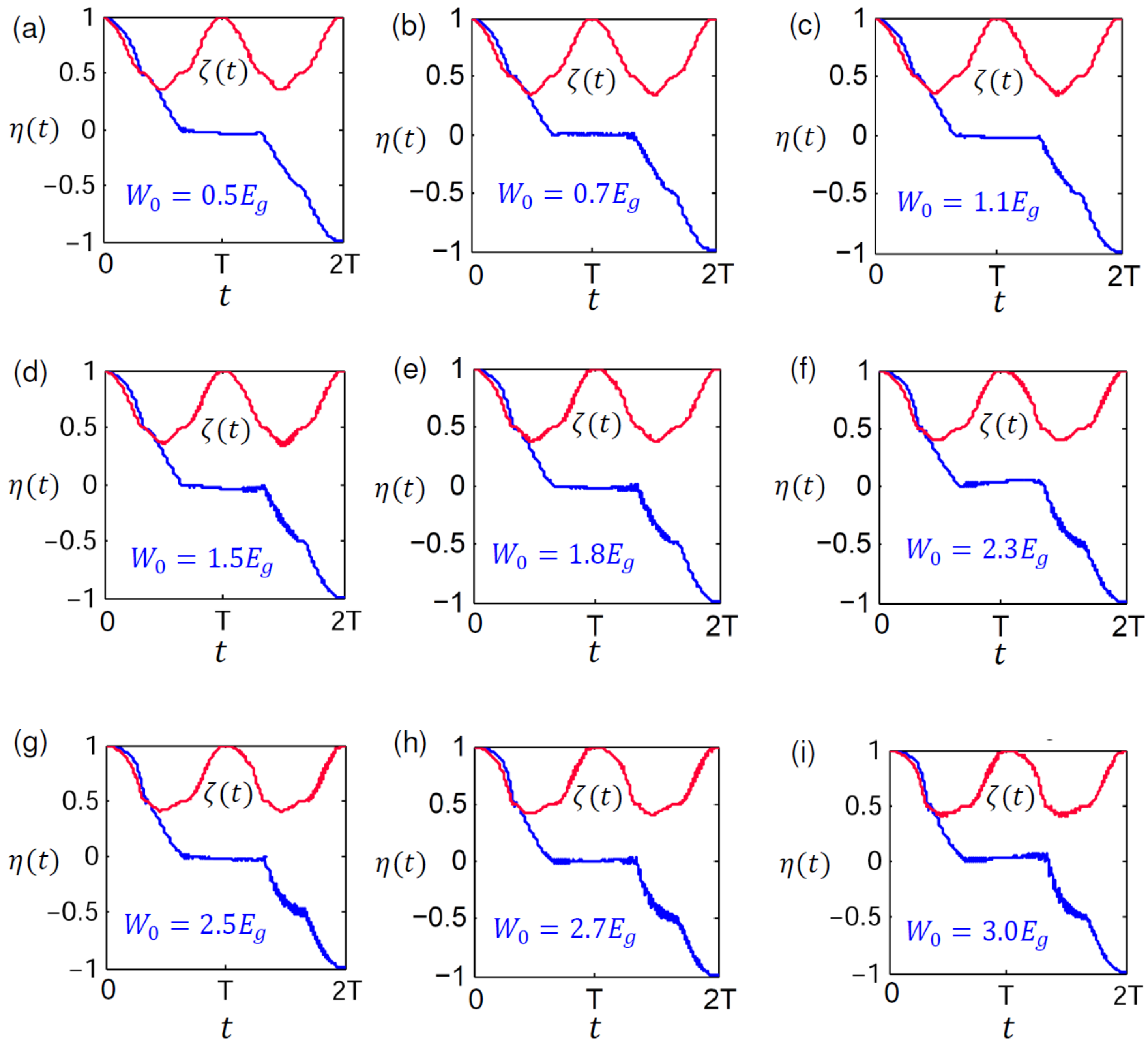


P. Gao, Y.-P. He and XJL, Phys. Rev. B 94, 224509 (2016).

Numerical results

W_0: static disorder strength

E_g: bulk gap



P. Gao, Y.-P. He and XJL, Phys. Rev. B 94, 224509 (2016).

Dynamical noise

Coupling between the Majorana modes and bulk fermionic modes via noise $H = H_0 + H_p$

$$\begin{aligned} H_0 &= \sum_j \epsilon_j (c_j^\dagger c_j + \tilde{c}_j^\dagger \tilde{c}_j) \\ H_p &= \gamma_a \sum_j (V_{j1} c_j - V_{j1}^* c_j^\dagger + V_{j2} \tilde{c}_j - V_{j2}^* \tilde{c}_j^\dagger) + \text{T.P.} \end{aligned}$$

Correlation function:

$$\langle V_{j1}(t_1) V_{j2}(t_2) \rangle_0 = V_0^2 \mathcal{C}_j (t_1 - t_2)$$

Evolution operator:

$$U(t) = U_0(t)[1 - i \int_0^t d\tau U_0^\dagger(\tau) H_{int}(\tau) U_0(\tau) - \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 U_0^\dagger(\tau_1) H_{int}(\tau_1) U_0(\tau_1 - \tau_2) H_{int}(\tau_2) U_0(\tau_2)] \cdots ,$$

Transition amplitude between two Majoranas in a Kramers pair:

$$\begin{aligned} \chi(t) = & \; 2V_0^2 \sum_j \int_{-T/2}^{t} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \, \Re\{[\mathcal{C}_j(\tau_1 - \tau_2) \\ & - \mathcal{C}_j(\tau_2 - \tau_1)] e^{i\epsilon_j(\tau_1 - \tau_2)}\} + \chi^{(4)}(t) + \cdots , \end{aligned}$$

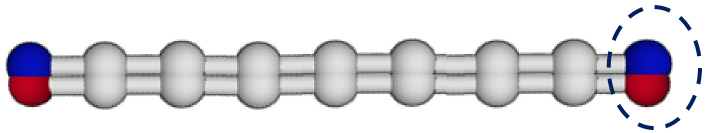
Consequence I: the noise may bring about random local operations only when its correlation function breaks the dynamical time-reversal symmetry in time domain:

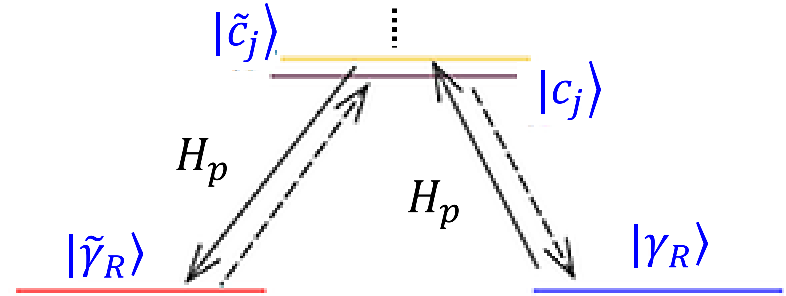
$$\boxed{\mathcal{C}(\tau) \neq \mathcal{C}(-\tau)}$$

Consequence II: the leading contribution by the noise is a 2nd-order transition:

$$\mathcal{D} = |\chi(T)|^2 \propto V_0^4 / E_g^4 + \mathcal{O}(V_0^8 / E_g^8)$$

Simulation for local operations





Leading order transition: 2nd-order

Fluctuations on μ and Δ_s:

$$H_p = \sum_j V_j[\cos(\omega t)(c_{j\uparrow}c_{j\downarrow}+h.c.)-\cos(\omega t+\delta\phi_j)n_j]$$

Break the dynamical time-reversal symmetry,

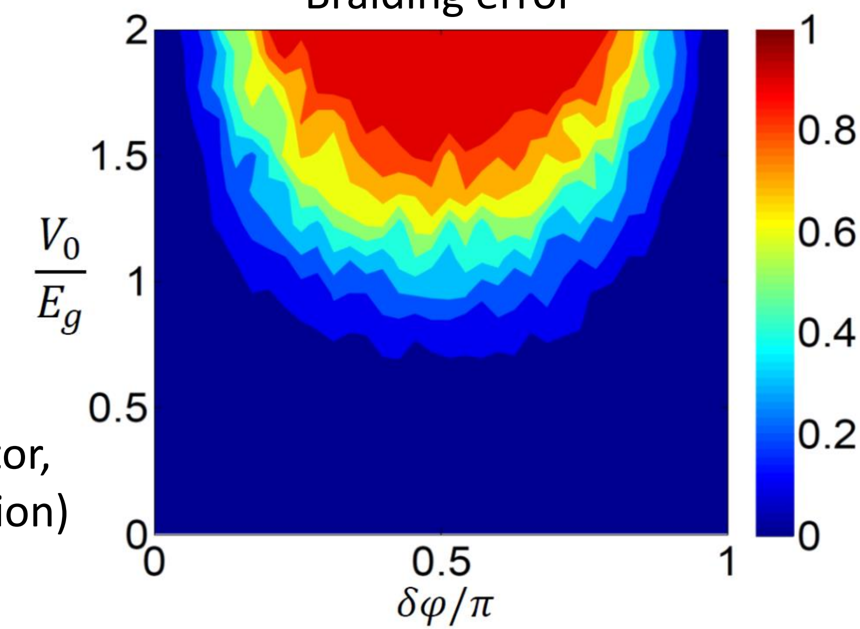
$H_p(t) \neq H_p(-t)$, namely, $\delta\varphi_j \neq 0, \pi$.

Leading to 2nd correction:

$$\langle\tilde{\gamma}_R|e^{-iH_ET}|\gamma_R\rangle$$

$$\approx 1-\cos^2\frac{V_0^2}{4E_g^2}|\langle\sin\delta\varphi_i\rangle|^2 \propto V_0^4/E_g^4$$

Braiding error



Compared with the D-class topological superconductor, the decoherence by dynamic noise (1st order correction)

$$\boldsymbol{\Gamma_{D-calss} \propto V_0^2/E_g^2}$$

Goldstein & Chamon, PRB (2012)

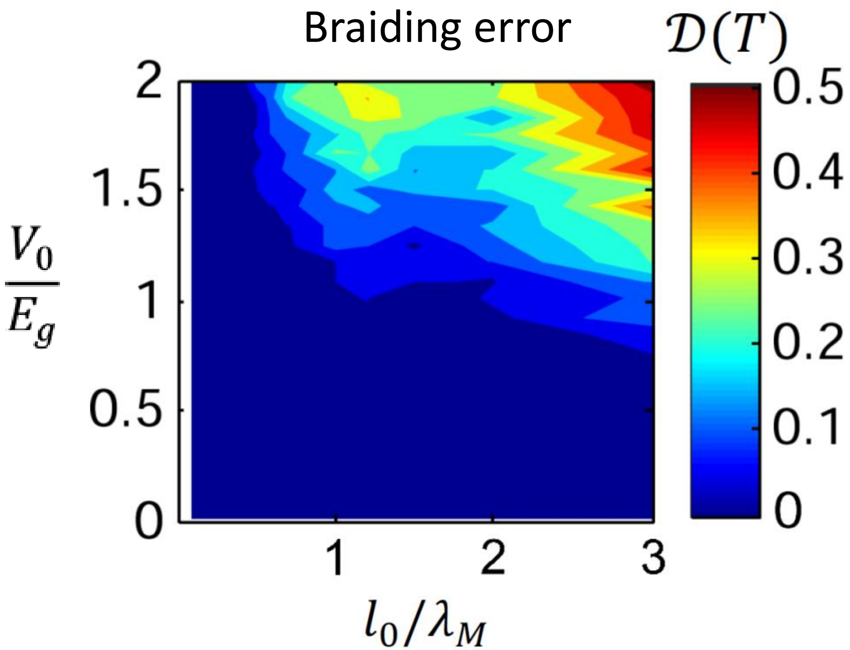
P. Gao, Y.-P. He and XJL, Phys. Rev. B 94, 224509 (2016).

Suppression of error by randomness of the noise

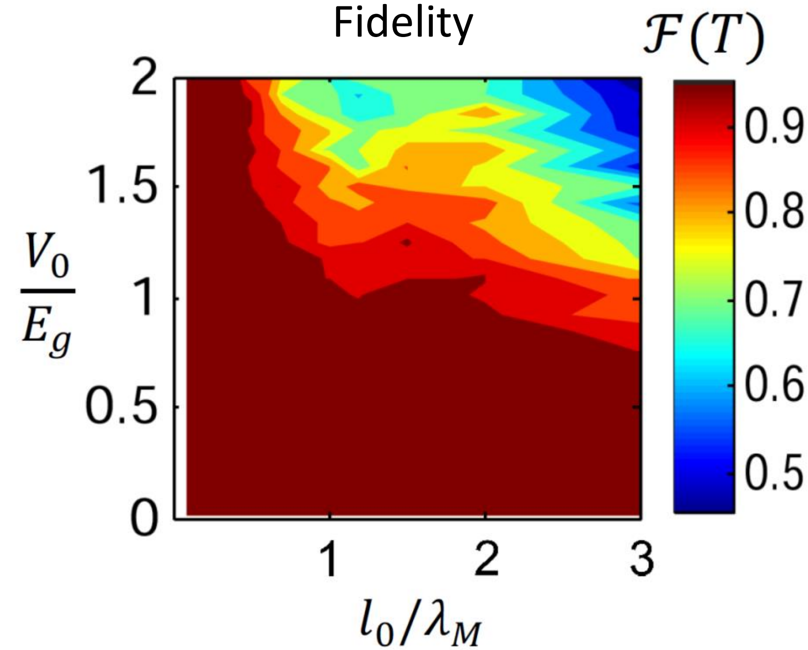
Noise spatial coherence length:

$$\langle \delta\phi_j \delta\phi_{j'} \rangle_{|j-j'|>l_0} = 0$$

Braiding error



Fidelity



λ_M: Majorana localization length

$$\mathcal{D} \approx \frac{V_0^4}{16E_g^4}\langle\gamma_a|\sin\delta\phi_j|\gamma_a\rangle^2 - \frac{V_0^8}{763E_g^8}\langle\gamma_a|\sin\delta\phi_j|\gamma_a\rangle^4.$$

Conclusions

1. We propose the symmetry-protected non-Abelian statistics for Majorana Kramers' pairs.

2. The sufficient conditions for non-Abelian braiding of Majorana Kramers' pairs:

   1) Adiabatic condition
   2) Time-reversal symmetry: $\hat{\mathcal{T}} H(t) \hat{\mathcal{T}}^{-1} = H(t)$
   3) $\hat{S} H(-t) \hat{S}^{-1} = H(t)$

3. Dynamical noise may lead to decoherence, but is only a second-order correction when dynamical TR symmetry is broken: $\mathcal{C}(\tau) \neq \mathcal{C}(-\tau)$.

References: 1) XJL, C. Wong, K.T. Law, Phys. Rev. X 4, 021018 (2014);
2) P. Gao, Y.-P. He, and XJL, Phys. Rev. B 94, 224509 (2016).

Acknowledgement

Thank you for your attention!