TBA Equations for Minimal Surfaces in AdS

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Outline

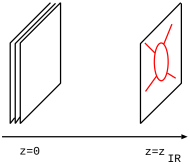
1. Introduction
2. Minimal surface in AdS_3
3. TBA equations and minimal area
4. Remainder Function and Perturbative CFT
5. Outlook

AdS/CFT Correspondence

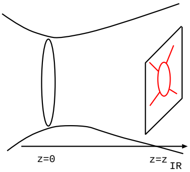
$N = 4\ U(N)$ SYM $\Longleftrightarrow$ type IIB Superstrings on $AdS_5 \times S^5$

Gluon Scattering Amplitudes [Alday-Maldacena 0705.0303]

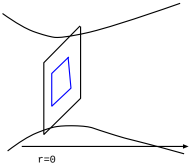
gluon scattering



Disk amplitude in AdS

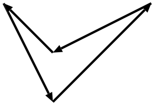


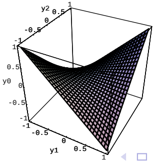
Wilson Loop in AdS



$\Longleftrightarrow$ AdS/CFT $\Longleftrightarrow$ T-dual

vev of Wilson loop = area of minimal surface





Gluon Scattering Amplitudes in $\mathcal{N} = 4$ SYM

the BDS conjecture Bern-Dixon-Smirnov, Anastasiou-Bern-Dixon-Kosower

Planar L-loop, n-point amplitude (recursive structure)

$$A_n^{(L)}(k_1, \cdots, k_n) = A_n^{(0)}(k_1, \cdots, k_n)\mathcal{M}_n^{(L)}(\epsilon)$$

$\mathcal{M}_n(\epsilon) \equiv 1 + \sum_{L=1}^{\infty} \lambda^L \mathcal{M}_n^{(L)}(\epsilon)$

IR divergence: Dimensional regularization ($D = 4 - 2\epsilon$)

$$\begin{aligned}\ln \mathcal{M}_n(\epsilon) &= \frac{A_2}{\epsilon^2} + \frac{A_1}{\epsilon} \\ &-\frac{1}{16} f(\lambda) \sum_{i=1}^{n} \left(\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)\right)^2 - \frac{g(\lambda)}{4} \sum_{i=1}^{n} \ln\left(\frac{\mu^2}{-s_{i,i+1}}\right) + \frac{f(\lambda)}{4} F_n^{(BDS)} + C\end{aligned}$$

$f(\lambda) = 4 \sum_{l=1}^{\infty} f_0^{(l)} \lambda^l$: cusp anomalous dimension

$g(\lambda) = 2 \sum_{l=1}^{\infty} \frac{f_1^{(l)}}{l} \lambda^l$: collinear anomalous dimension

For $n = 4$

$$F_4^{BDS} = \frac{1}{2} \log^2\left(\frac{s}{t}\right) + \frac{2\pi^2}{3}$$

Test of the BDS conjecture (Weak Coupling)

- loop calculations ($n \leq 5$) BDS, ...
- Dual conformal invariance: determines $n \leq 5$ amplitudes
- Gluon amplitude/Wilson loop duality at weak coupling
  Drummond-Henn-Korchemsky-Sokatchev
- Discrepancy in 6-point 2-loop amplitude
  Bern-Dixon-Kosower-Roiban-Spradlin-Volovich
  Drummond-Henn-Korchemsky-Sokatchev

$$\ln M_6^{MHV} = \ln W(C_6) + const., \quad F_6^{WL} = F_6^{BDS} + R_6(u), \quad R_6 \neq 0$$

- remainder function $R_n(u)$: $F_n = F_n^{BDS} + R_n$
  Non-trivial dependence on the cross-ratios ($3n - 15$)

$$u_{ij,kl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}, \quad (x_{ij}^2 = t_i^{[j-i]})$$

For $n = 6$, $u_{13,46}$, $u_{24,15}$, $u_{35,26}$ are independent cross ratios.

Minimal Surface in AdS

AdS_5: embedding coordinates in $\mathbf{R}^{2,4}$: isometry $SO(2,4)$

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$$

Poincaré coordinates: (y^μ, r) $\mu = 0,1,2,3$

$$Y^\mu = \frac{y^\mu}{r}, \quad Y_{-1} + Y_4 = \frac{1}{r}, \quad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r}$$

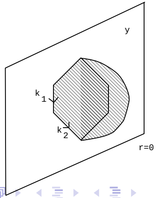
Action in confomal gauge:

$$S = \frac{R^2}{2\pi} \int d^2 z \frac{\partial y^\mu \bar{\partial} y_\mu + \partial r \bar{\partial} r}{r^2}$$

Euler-Lagrange equation+boundary condition

- ends at $r \to 0$ (AdS boundary)
- y^μ: light-like segments

$$\mathcal{M}_n \sim \exp(-S[y,r])$$



Alday-Maldacena's Solution (4pt amplitude)

Alday-Maldacena arXiv:0705.0303

- AdS_3 constraint: $Y_3 = Y_4 = 0$

$$r^2 - (y^0)^2 + (y^1)^2 + (y^2)^2 = 1$$

- static gauge: $r = r(y^1, y^2)$, $y^0 = y^0(y^1, y^2)$
- dimesnional regularization $D = 3 - 2\epsilon$
- conformal boost: $s = t \to$ general (s, t)

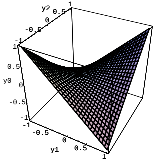
4-point amplitude $(s = t)$: $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$

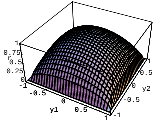
boundary condition:

$r(\pm 1, y_2) = r(y_1, \pm 1) = 0$,

$y_0(\pm 1, y_2) = \pm y_2$, $y_0(y_1, \pm 1) = \pm y_1$

$y_0 = y_1 y_2$, $r = \sqrt{(1 - y_1^2)(1 - y_2^2)}$





$$S_4 = \ln \mathcal{M}_4 \quad f(\lambda) \sim \sqrt{\lambda}$$

We consider the classical string solutions in AdS_3.
$(z, \bar{z})$: worldsheet coordinates (Euclidean)
emmbedding coordinates $\vec{Y} = (Y_{-1}, Y_0, Y_1, Y_2)$ in $R^{2,2}$

$$\vec{Y} \cdot \vec{Y} := -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1$$

action ($\vec{Y}_z = \partial_z \vec{Y}$, $\vec{Y}_{\bar{z}} = \partial_{\bar{z}} \vec{Y}$)

$$S = \int d^2 z \left\{ \vec{Y}_z \cdot \vec{Y}_{\bar{z}} + \lambda(\vec{Y} \cdot \vec{Y} + 1) \right\}$$

equations of motion

$$\vec{Y}_{z\bar{z}} - (\vec{Y}_z \cdot \vec{Y}_{\bar{z}}) \vec{Y} = 0$$

Virasoro constraints

$$\vec{Y}_z^2 = \vec{Y}_{\bar{z}}^2 = 0$$

Pohlmeyer reduction

[Pohlmeyer CMP 46 (1976) 207; de Vega-Sanchez PRD 47 (1993) 3394]
variables $\vec{Y} \to \alpha(z,\bar{z})$(real), $p(z)$(holomorphic), $\bar{p}(\bar{z})$ (anti-hol)

$$e^{2\alpha} = \frac{1}{2}\vec{Y}_z \cdot \vec{Y}_{\bar{z}}$$

$$N_i = \frac{1}{2}e^{-2\alpha}\epsilon_{ijkl}Y^j Y_z^k Y_{\bar{z}}^l\,, \qquad \epsilon_{(-1)012} = +1,$$

$$p = -\frac{1}{2}\vec{N}\cdot\vec{Y}_{zz} \quad \bar{p} = \frac{1}{2}\vec{N}\cdot\vec{Y}_{\bar{z}\bar{z}}$$

$\vec{N}$: normal vector to the surface

$$\vec{N}\cdot\vec{Y} = \vec{N}\cdot\vec{Y}_z = \vec{N}\cdot\vec{Y}_{\bar{z}} = 0, \qquad \vec{N}\cdot\vec{N} = 1$$

Moving-frame basis

$$\vec{q}_1 = \vec{Y}, \qquad \vec{q}_2 = e^{-\alpha}\vec{Y}_{\bar{z}}, \qquad \vec{q}_3 = e^{-\alpha}\vec{Y}_z, \qquad \vec{q}_4 = \vec{N}.$$

evolution of moving-frame basis

$$\partial_z \begin{pmatrix} \vec{q}_1 \\ \vec{q}_2 \\ \vec{q}_3 \\ \vec{q}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & e^{\alpha} & 0 \\ 2e^{\alpha} & -\alpha_z & 0 & 0 \\ 0 & 0 & \alpha_z & -2pe^{-\alpha} \\ 0 & pe^{-\alpha} & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{q}_1 \\ \vec{q}_2 \\ \vec{q}_3 \\ \vec{q}_4 \end{pmatrix}$$

$$\partial_{\bar{z}} \begin{pmatrix} \vec{q}_1 \\ \vec{q}_2 \\ \vec{q}_3 \\ \vec{q}_4 \end{pmatrix} = \begin{pmatrix} 0 & e^{\alpha} & 0 & 0 \\ 0 & \alpha_{\bar{z}} & 0 & -2\bar{p}e^{-\alpha} \\ 0 & 0 & -\alpha_{\bar{z}} & 2e^{\alpha} \\ 0 & 0 & \bar{p}e^{-\alpha} & 0 \end{pmatrix} \begin{pmatrix} \vec{q}_1 \\ \vec{q}_2 \\ \vec{q}_3 \\ \vec{q}_4 \end{pmatrix}$$

Spinor notation: $(a, \dot{a})$: spacetime $SL(2,R) \times SL(2,R)$

$$Y_{a\dot{a}} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} Y_{-1} + Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix}$$

$$W_{\alpha\dot{\alpha},a\dot{a}} = \begin{pmatrix} W_{11,a\dot{a}} & W_{12,a\dot{a}} \\ W_{21,a\dot{a}} & W_{22,a\dot{a}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (q_1 + q_4)_{a\dot{a}} & (q_2)_{a\dot{a}} \\ (q_3)_{a\dot{a}} & (q_1 - q_4)_{a\dot{a}} \end{pmatrix},$$

evolution of $W_{\alpha\dot{\alpha},a\dot{a}}$

$$\partial_z W_{\alpha\dot{\alpha},a\dot{a}} + (B_z^{\mathsf{L}})_\alpha{}^\beta W_{\beta\dot{\alpha},a\dot{a}} + (B_z^{\mathsf{R}})_{\dot{\alpha}}{}^{\dot{\beta}} W_{\alpha\dot{\beta},a\dot{a}} = 0,$$
$$\partial_{\bar{z}} W_{\alpha\dot{\alpha},a\dot{a}} + (B_{\bar{z}}^{\mathsf{L}})_\alpha{}^\beta W_{\beta\dot{\alpha},a\dot{a}} + (B_{\bar{z}}^{\mathsf{R}})_{\dot{\alpha}}{}^{\dot{\beta}} W_{\alpha\dot{\beta},a\dot{a}} = 0,$$

$$B_z^{\mathsf{L}} = B_z(1), \qquad B_z^{\mathsf{R}} = U B_z(i) U^{-1},$$
$$B_{\bar{z}}^{\mathsf{L}} = B_{\bar{z}}(1), \qquad B_{\bar{z}}^{\mathsf{R}} = U B_{\bar{z}}(i) U^{-1}$$

$$B_z(\zeta) = \frac{\alpha_z}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{\zeta}\begin{pmatrix} 0 & e^{\alpha} \\ e^{-\alpha} p & 0 \end{pmatrix},$$
$$B_{\bar{z}}(\zeta) = -\frac{\alpha_{\bar{z}}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \zeta\begin{pmatrix} 0 & e^{-\alpha}\bar{p} \\ e^{\alpha} & 0 \end{pmatrix},$$
$$U = \begin{pmatrix} 0 & e^{\frac{\pi i}{4}} \\ e^{\frac{3\pi i}{4}} & 0 \end{pmatrix}.$$

Each entry of the matrix $W_{\alpha\dot{\alpha},a\dot{a}}$ is a null vector

$$W_{\alpha\dot{\alpha},a\dot{a}} = \psi^{\mathsf{L}}_{\alpha,a}\psi^{\mathsf{R}}_{\dot{\alpha},\dot{a}}$$

evolution of ψ

$$\partial_z\psi^{\mathsf{L}}_{\alpha,a} + (B^{\mathsf{L}}_z)_\alpha{}^\beta\psi^{\mathsf{L}}_{\beta,a} = 0, \qquad \partial_{\bar{z}}\psi^{\mathsf{L}}_{\alpha,a} + (B^{\mathsf{L}}_{\bar{z}})_\alpha{}^\beta\psi^{\mathsf{L}}_{\beta,a} = 0,$$
$$\partial_z\psi^{\mathsf{R}}_{\dot{\alpha},\dot{\alpha}} + (B^{\mathsf{R}}_z)_{\dot{\alpha}}{}^{\dot{\beta}}\psi^{\mathsf{R}}_{\dot{\beta},\dot{\alpha}} = 0, \qquad \partial_{\bar{z}}\psi^{\mathsf{R}}_{\dot{\alpha},\dot{\alpha}} + (B^{\mathsf{R}}_{\bar{z}})_{\dot{\alpha}}{}^{\dot{\beta}}\psi^{\mathsf{R}}_{\dot{\beta},\dot{\alpha}} = 0.$$

compatibility condition

$$\partial B^{\mathsf{L}}_{\bar{z}} - \bar{\partial}B^{\mathsf{L}}_z + [B^{\mathsf{L}}_z, B^{\mathsf{L}}_{\bar{z}}] = 0, \quad \partial B^{\mathsf{R}}_{\bar{z}} - \bar{\partial}B^{\mathsf{R}}_z + [B^{\mathsf{R}}_z, B^{\mathsf{R}}_{\bar{z}}] = 0,$$

$\Longrightarrow$ generalized sinh-Gordon equation

$$\bar{\partial}\partial\alpha(z,\bar{z}) - 2e^{2\alpha(z,\bar{z})} + |p(z)|^2 e^{-2\alpha(z,\bar{z})} = 0$$

The left/right linear problem can be promoted to a family of linear problems with the general spectral parameter ζ:

$$\Big(\partial_z + B_z(\zeta)\Big)\psi(z,\bar{z};\zeta) = 0, \qquad \Big(\partial_{\bar{z}} + B_{\bar{z}}(\zeta)\Big)\psi(z,\bar{z};\zeta) = 0,$$

$\zeta = 1$ (left) $\zeta = i$ (right)

$$B_z(\zeta) = A_z + \frac{1}{\zeta}\Phi_z, \quad B_{\bar{z}}(\zeta) = A_{\bar{z}} + \zeta\Phi_{\bar{z}}$$

$\Longrightarrow$ $SU(2)$ Hitchin equations:

$$D_{\bar{z}}\Phi_z = D_z\Phi_{\bar{z}} = 0, \quad F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0$$

Area

$$A = 2\int d^2z \mathrm{Tr}\Phi_z\Phi_{\bar{z}} = 4\int d^2z e^{2\alpha}$$

change of variable $dw = \sqrt{p(z)}dz$

$$\hat{\alpha} = \alpha - \frac{1}{4}\log p\bar{p}$$

gauge transformation

$$\hat{\psi} = g\psi, \quad g = e^{i\frac{\pi}{4}\sigma^3} e^{i\frac{\pi}{4}\sigma^2} e^{\frac{1}{8}\log\frac{p}{\bar{p}}\sigma^3}$$

$$\left(\partial_w + \hat{B}_w\right)\hat{\psi} = 0, \qquad \left(\partial_{\bar{w}} + \hat{B}_{\bar{w}}\right)\hat{\psi} = 0,$$

$$\hat{B}_w(\zeta) = \frac{\hat{\alpha}_w}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{1}{\zeta}\begin{pmatrix} \cosh\hat{\alpha} & i\sinh\hat{\alpha} \\ i\sinh\hat{\alpha} & -\cosh\hat{\alpha} \end{pmatrix},$$

$$\hat{B}_{\bar{w}}(\zeta) = -\frac{\hat{\alpha}_{\bar{w}}}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \zeta\begin{pmatrix} \cosh\hat{\alpha} & -i\sinh\hat{\alpha} \\ -i\sinh\hat{\alpha} & -\cosh\hat{\alpha} \end{pmatrix}.$$

compatibility cond. $\Longrightarrow$ sinh-Gordon equation

$$\partial_w\partial_{\bar{w}}\hat{\alpha} - e^{2\hat{\alpha}} + e^{-2\hat{\alpha}} = 0$$

solutions with light-like boundary

- $\hat{\alpha} \to 0$ $|w| \to \infty$
- $p(z)$: polynomial of degree $n-2$ ($2n$-gonal boundary)

at large $|w|$

$$(\partial_w - \zeta^{-1}\sigma_3)\hat{\psi} = 0, \qquad (\partial_{\bar{w}} - \zeta\sigma_3)\hat{\psi} = 0,$$

have two independent solutions (big and small)

$$\hat{\eta}_+ = \begin{pmatrix} e^{\left(\frac{w}{\zeta}+\bar{w}\zeta\right)} \\ 0 \end{pmatrix}, \qquad \hat{\eta}_- = \begin{pmatrix} 0 \\ e^{-\left(\frac{w}{\zeta}+\bar{w}\zeta\right)} \end{pmatrix}.$$

$w = |w|e^{i\theta}$, $\zeta = |\zeta|e^{i\alpha}$, $\mathrm{Re}\left(\frac{w}{\zeta} + \bar{w}\zeta\right) = |w|\left(\frac{1}{|\zeta|} + |\zeta|\right)\cos(\theta - \alpha)$

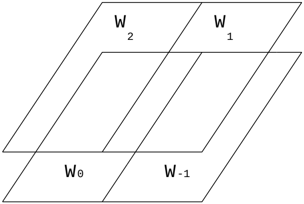
- $\cos(\theta-\alpha) < 0$ $\hat{\eta}_+$: small $\hat{\eta}_-$: large
- $\cos(\theta-\alpha) > 0$ $\hat{\eta}_-$: small $\hat{\eta}_+$: large

Bases of solutions

the Stokes sector

$$\hat{W}_j : \quad (j - \tfrac{3}{2})\pi + \arg\zeta < \arg w < (j - \tfrac{1}{2})\pi + \arg\zeta$$

$\hat{s}_j$: the small solution in $\hat{W}_j$ (unique up to rescaling)



- $\hat{s}_{2k-1} = (-1)^{k-1}\hat{\eta}_-$ for $w \in \hat{W}_{2k-1}$
- $\hat{s}_{2k} = (-1)^{k}\hat{\eta}_+$ for $w \in \hat{W}_{2k}$

In $\hat{W}_j$ one can take $\hat{s}_{j-1}$ (large) and $\hat{s}_j$ (small) as the basis of the solution.

- normalization $\hat{s}_j \wedge \hat{s}_{j+1} \equiv \det(\hat{s}_j, \hat{s}_{j+1}) = 1$
- Stokes data $\hat{s}_{j+1} = -\hat{s}_{j-1} + b_j(\zeta)\hat{s}_j, \quad b_j(\zeta) = \hat{s}_{j-1} \wedge \hat{s}_{j+1}$

small solutions in the z-plane

- the small solution $s_j = g^{-1}\hat{s}_j$
- the Stokes sector in the z-plane

$$W_j: \quad \frac{(2j-3)\pi}{n} + \frac{2}{n}\arg\zeta < \arg z < \frac{(2j-1)\pi}{n} + \frac{2}{n}\arg\zeta$$

- the differential equation is regular for $|z| < \infty$, $W_{j+n} = W_j$
- monodromy factor μ_j: $s_{j+n} = \mu_j s_j$
- $\hat{S}_j = (\hat{s}_j, \hat{s}_{j+1})$, $\hat{S}_{j+1} = \hat{S}_j B_{j+1}, \quad \hat{S}_{j+n} = \hat{S}_j M_j$.

$$B_j = \begin{pmatrix} 0 & -1 \\ 1 & b_j \end{pmatrix}, \quad M_j = \begin{pmatrix} \mu_j & 0 \\ 0 & \mu_{j+1} \end{pmatrix}$$

$\Longrightarrow \mu_{2k-1} = \mu_{2k}^{-1} = \mu$, $b_{j+n} = \mu_j^{-2} b_j$

Two involutions

- holomorphic involution ($\mathbf{Z}_2$ symmetry)

$$\sigma_2[\partial_w+\hat{B}_w(\zeta)]\sigma_2=[\partial_w+\hat{B}_w(-\zeta)],\sigma_2[\partial_{\bar{w}}+\hat{B}_{\bar{w}}(\zeta)]\sigma_2=[\partial_{\bar{w}}+\hat{B}_{\bar{w}}(-\zeta)]$$

$$\sigma_2\hat{s}_j(w,\bar{w};e^{\pi i}\zeta)=i\hat{s}_{j+1}(w,\bar{w};\zeta)$$

$$b_j(e^{\pi i}\zeta)=b_{j+1}(\zeta).$$

- antiholomorphic involution

$$\overline{\partial_w+\hat{B}_w(\zeta)}=\partial_{\bar{w}}+\hat{B}_{\bar{w}}(\bar{\zeta}^{-1}),\quad \overline{\partial_{\bar{w}}+\hat{B}_{\bar{w}}(\zeta)}=\partial_w+\hat{B}_w(\bar{\zeta}^{-1}).$$

$$\overline{\hat{s}_j(w,\bar{w};\bar{\zeta}^{-1})}=\hat{s}_j(w,\bar{w};\zeta)$$

$$\overline{b_j(\bar{\zeta}^{-1})}=b_j(\zeta).$$

TBA Equations and minimal area

Alday-Gaiotto-Maldacena 0911.4708, Hatsuda-Ito-Sakai-Satoh

The functional form $b_j(\zeta)$ (the Riemann-Hilber problem)

- monodromy factor, constraints from involutions
- asymptotic form $|\zeta| \to \infty, 0$ evaluated by the WKB analysis
- $b_j(\zeta)$ has nontrivial asymptotics in each angular (Stokes) sector

Introduce the Fock-Goncharov coordinates $\chi_j(\zeta)$

- cross-ratio
- simple asymptotics in all angular sectors
- discontinuities across the Stokes-line $\Longrightarrow$ TBA equations

Alday-Maldacena-Sever-Vieira 1002.2459

- Plücker relation for s_j+constraints $\Longrightarrow$ T-system $\Longrightarrow$ Y-system (TBA equations)

Decagon solution $(n = 5)$

polynomial: $p(z) = z^3 - 3\Lambda^2 z + u = (z - z_1)(z - z_2)(z - z_3)$
conditions for the Stokes coefficients

$$\begin{aligned} b_1 b_2 &= 1 - \mu b_4, \\ b_2 b_3 &= 1 - \mu^{-1} b_5, \\ b_3 b_4 &= 1 - \mu^{-1} b_1, \\ b_4 b_5 &= 1 - \mu b_2. \end{aligned}$$

$$\beta_{2k-1} = \mu^{-1} b_{2k-1}, \quad \beta_{2k} = \mu b_{2k}$$

$$\beta_j \beta_{j+1} = 1 - \beta_{j+3}, \quad \beta_{j+5} = \beta_j$$

the Fock-Goncharov coordinates

the WKB line: Re $\left(\frac{1}{\zeta}\int\sqrt{p(z)}dz\right)$ largest

the WKB triangulation: $\to$ quadrant $\to$ the Fock-Goncharov coordinates

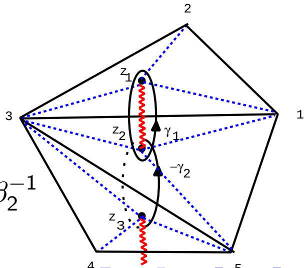
$$\chi_E = \frac{(s_1\wedge s_2)(s_3\wedge s_4)}{(s_2\wedge s_3)(s_4\wedge s_1)}$$

aysmptotic boundary condition $\zeta\to 0,\infty$

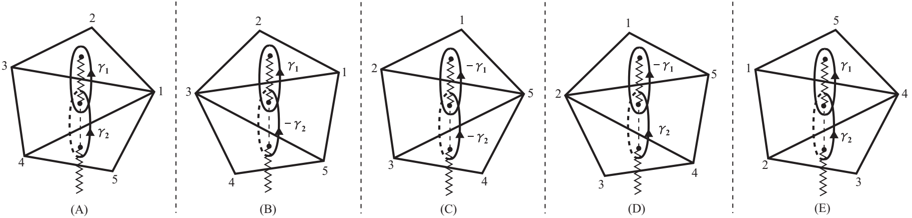
$$\chi_{\gamma_i}(\zeta)\simeq \exp\left(\frac{Z_i}{\zeta}+\bar{Z}_i\zeta\right),\quad Z_i=\oint_{\gamma_i}\sqrt{p(z)}dz$$

$$\chi_{\gamma_1} = -\frac{(s_1\wedge s_2)(s_3\wedge s_5)}{(s_2\wedge s_3)(s_5\wedge s_1)} = -\beta_4,$$

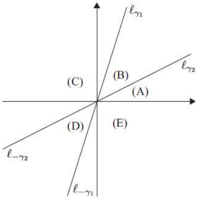
$$\chi_{\gamma_2}^{-1} = \chi_{-\gamma_2} = -\frac{(s_5\wedge s_1)(s_3\wedge s_4)}{(s_1\wedge s_3)(s_4\wedge s_5)} = -\beta_2^{-1}$$



The WKB triangulation jumps discontinuously when argζ crosses the BPS rays.



$\arg\zeta$	(A)	(B)	(C)	(D)	(E)
χ_{γ_1}	$-\beta_5^{-1}$	$-\beta_4$	$-\beta_4$	$-\beta_3^{-1}$	$-\beta_3^{-}$
χ_{γ_2}	$-\beta_2$	$-\beta_2$	$-\beta_1^{-1}$	$-\beta_1^{-1}$	$-\beta_5$



Discontinuities$\Longrightarrow$ Integral equations

$F(\zeta)=\int_0^\infty \frac{dx}{x}\frac{x+\zeta}{z-\zeta}f(x)$

$$F(y+i\epsilon)-F(y-i\epsilon)=\int_0^\infty \frac{4i\epsilon}{(x-y)^2+\epsilon^2}f(x)\to 2\pi i f(y)(\epsilon\to 0)$$

$$\ln \chi_{\gamma_1}(\zeta) = \frac{Z_1}{\zeta} + \bar{Z}_1 \zeta - \frac{1}{4\pi i}\int_{\ell_{\gamma_2}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\ln(1+\chi_{\gamma_2}(\zeta')) + \frac{1}{4\pi i}\int_{\ell_{-\gamma_2}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\ln(1+\chi_{-\gamma_2}(\zeta'))$$

$$\ln \chi_{\gamma_2}(\zeta) = \frac{Z_2}{\zeta} + \bar{Z}_2 \zeta + \frac{1}{4\pi i}\int_{\ell_{\gamma_1}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\ln(1+\chi_{\gamma_1}(\zeta')) - \frac{1}{4\pi i}\int_{\ell_{-\gamma_1}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\ln(1+\chi_{-\gamma_1}(\zeta'))$$

We can rewrite the integral equations in the form TBA equations:

$$\epsilon_1(\theta) = 2|Z_1|\cosh\theta - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta'+i\tilde{\alpha})}\log(1+e^{-\epsilon_2(\theta')}),$$

$$\epsilon_2(\theta) = 2|Z_2|\cosh\theta - \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta'-i\tilde{\alpha})}\log(1+e^{-\epsilon_1(\theta')})$$

- $Z_k = |Z|_k e^{i\alpha_k}$, $\zeta = -e^{\theta+i\alpha_k}$, $\quad \tilde{\alpha} = \pi/2 - (\alpha_1 - \alpha_2)$
- $\epsilon_k(\theta) = \epsilon_{\gamma_k}(\theta)$, $\epsilon_{-\gamma_k}(\theta) = \epsilon_{\gamma_k}(\theta)$ ($\mathbf{Z}_2$-symmetry)
- $\chi_{\gamma_k}(\zeta = -e^{\theta+i\alpha_k}) = e^{-\epsilon_{\gamma_k}(\theta)}$,

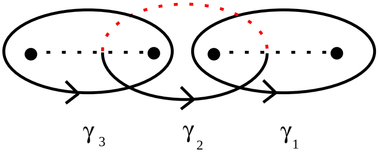
Integral equation and TBA equations for general n

Gaiotto-Moore-Neitzke

$$\log \chi_{\gamma_k}(\zeta) = \frac{Z_{\gamma_k}}{\zeta} + \bar{Z}_{\gamma_k}\zeta - \frac{1}{4\pi i}\sum_{\gamma'} \langle \gamma_k, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 + \chi_{\gamma'}(\zeta'))$$

$\gamma' = \pm\gamma_1, \pm\gamma_2, \cdots, \pm\gamma_{n-2}$

$\ell_{\gamma'}$: $\frac{Z_{\gamma'}}{\zeta'} \in \mathbf{R}_-$



TBA equations

$$\epsilon_k(\theta) = 2|Z_k| \cosh\theta - \sum_{l=1}^{n-3} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{-i\langle \gamma_k, \gamma_l \rangle}{\sinh(\theta - \theta' + i\alpha_k - i\alpha_l)} \log(1 + e^{-\epsilon_l(\theta')})$$

$$Z_k = Z_{\gamma_k}$$

Homogeneous sine-Gordon model

This TBA equations are identified with those of the homogeneous sine-Gordon model Fernandez-Pouza-Gallas-Hollowood-Miramontes 9912196

- perturbed CFT: generalized parafermions $SU(N)_2/U(1)^{N-1}$
- S-matrix Castro-Alvaredo-Fring 0010262

$$S_{ab}(\theta) = (-1)^{\delta_{ab}} \left[c_a \tanh \frac{1}{2}(\theta + \sigma_{ab} - \frac{\pi}{2} i) \right]^{I_{ab}}$$

- TBA eqs

$$\epsilon_a(\theta) = m_a R \cosh \theta - \sum_b \int \frac{d\theta'}{2\pi} \frac{i I_{ab}}{\sinh(\theta - \theta' + \sigma_{ab} + \frac{\pi}{2} i)} \log(1 + e^{-\epsilon_b}).$$

minimal surface	HSG model
$n-2$	N
$2\lvert Z_a\rvert$	$m_a R$
$\langle \gamma_a, \gamma_b \rangle$	$\epsilon_{ab} I_{ab}$
$i(\alpha_1 - \alpha_b)$	$\sigma_{ab} + \frac{\pi}{2} i$

Regularized area and Free energy

cross ratios $\frac{x^{\pm}_{ij}x^{\pm}_{kl}}{x^{\pm}_{ik}x^{\pm}_{jl}} = \frac{(s_i \wedge s_j)(s_k \wedge s_l)}{(s_i \wedge s_k)(s_j \wedge s_l)}$

- $x^{\pm} = \frac{Y_1 \pm Y_0}{Y_{-1} + Y_2}$ light-cone coordinates at the AdS boundary
- $\zeta = 1$ left (+), $\zeta = i$ right (-)

Area (finite+divergent)

$$A = 4\int d^2 z e^{2\alpha} = A_{Sinh} + 4\int d^2 w, \quad A_{Sinh} = 4\int d^2 w(e^{2\alpha} - \sqrt{p\bar{p}})$$

finite part=TBA free energy F

regular $2n$-gon solution (CFT limit)

$$A_{sinh} = F + c_n, \quad c_n = \frac{7}{12}(n-2)\pi$$

The constant c_n is determined by the superposition of hexagon solutions where zeros of $p(z)$ become far apart from each other.

$$F = \frac{\pi}{6}c = \frac{\pi}{6n}(n-2)(n-3) \to A_{Sinh} = \frac{\pi}{4n}(3n^2 - 8n + 4)$$

agrees with regular polygon solutions Alday-Malacena

Remainder Function and Perturbative CFT

6-point gluon scattering amplitudes

- minimal surface with 6 cusps in AdS_5
- $SU(4)$ Hitchin equations with $\mathbf{Z}_4$-automorphism
- Stokes data $\rightarrow$ TBA equations for A_3-integrable ($\mathbf{Z}_4$-symmetric) field theory [Alday-Gaiotto-Maldacena]
- Y-function $Y_a(\theta)$ ($a=1,2,3$) $Y_1=Y_3$

$$Y_1(\theta+\frac{i\pi}{4})Y_1(\theta-\frac{i\pi}{4})=1+Y_2(\theta)$$
$$Y_2(\theta+\frac{i\pi}{4})Y_2(\theta-\frac{i\pi}{4})=(1+\mu Y_1(\theta))(1+\mu^{-1}Y_1(\theta))$$

periodicity $Y_a(\theta+\frac{3i\pi}{2})=Y_a(\theta)$, reality: $\overline{Y_a(\theta)}=Y_a(-\bar{\theta})$
$\mu=e^{i\phi}$: monodromy ($s_{i+6}=\mu^{(-1)^j}s_i$)

TBA equations (6pt amplitude)

cross-ratio

$$U_1 = b_2 b_3 = \frac{x_{14}^2 x_{36}^2}{x_{13}^2 x_{46}^2}, \quad U_2 = b_3 b_1 = \frac{x_{25}^2 x_{14}^2}{x_{24}^2 x_{15}^2}, \quad U_3 = b_1 b_2 = \frac{x_{36}^2 x_{25}^2}{x_{35}^2 x_{26}^2},$$

$$b_k = Y_1\left(\frac{(k-1)\pi i}{2}\right), \quad U_k = 1 + Y_2\left(\frac{(2k+1)\pi i}{4}\right)$$

monodromy

$$b_1 b_2 b_3 = b_1 + b_2 + b_3 + \mu + \mu^{-1}$$

TBA equations: $\epsilon(\theta) = \log Y_1(\theta + i\varphi)$, $\tilde{\epsilon}(\theta) = \log Y_2(\theta + i\varphi)$ $(Z = |Z| e^{i\varphi})$

$$\epsilon = 2|Z| \cosh\theta + K_2 * \log(1 + e^{-\tilde{\epsilon}}) + K_1 * \log(1 + \mu e^{-\epsilon})(1 + \mu^{-1} e^{-\epsilon})$$
$$\tilde{\epsilon} = 2\sqrt{2}|Z| \cosh\theta + 2K_1 * \log(1 + e^{-\tilde{\epsilon}}) + K_2 * \log(1 + \mu e^{-\epsilon})(1 + \mu^{-1} e^{-\epsilon})$$

$$K_1(\theta) = \frac{1}{2\pi \cosh\theta} \quad K_2(\theta) = \frac{\sqrt{2}\cosh\theta}{\pi \cosh 2\theta}$$

$f * g(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta') g(\theta')$: convolution

Remainder function (6pt amplitude)

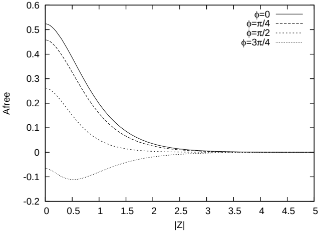
$$Area = 4\int d^2z(e^{2\alpha} - (P\bar{P})^{\frac{1}{4}}) + 4\int d^2z(P\bar{P})^{\frac{1}{4}} - 4\int_{\Sigma_0} d^2w + 4\int_{\Sigma_0} d^2w$$

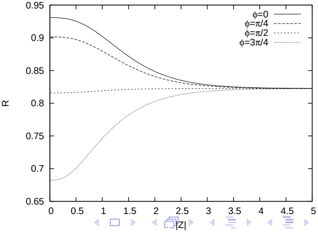
$$R = A_{BDS} - A_{BDS-like} - A_{period} - A_{free}$$

$$= -\frac{1}{4}\sum_{k=1}^{3}\mathrm{Li}_2(1-U_k) - |Z|^2 - (-F)$$

$$F = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\theta \Big\{ 2|Z|\cosh\theta \log(1+\mu e^{-\epsilon(\theta)}))(1+\mu^{-1}e^{-\epsilon(\theta)})$$

$$+2\sqrt{2}|Z|\cosh\theta\log(1+e^{-\tilde{\epsilon}(\theta)}\Big\}$$





Thermodynamic Bethe Ansatz

Al. Zamoldchikov, NPB 342 (1990 695)

$1+1$ dim Euclidean QFT on a cylinder with periodic B.C.

partiton function

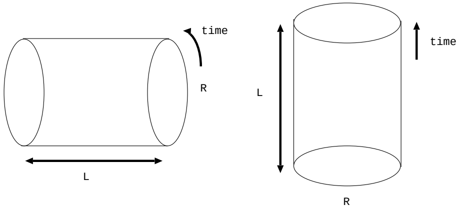
$Z(R,L) = \mathrm{Tr} e^{-RH_L} = \mathrm{Tr} e^{-LH_R}$

$L \to \infty$

$e^{-LRf(R)} \sim e^{-LE_0(R)}$

$f(R)$: free energy per unit length at temperature $1/R$

$E_0(R)$: ground state energy



$$E_0(R) = Rf(R)$$

particle A_a $(a=1,\cdots,n)$ with mass m_a $(E = m\cosh\theta,\ p = m\sinh\theta)$

purely elastic scattering matrix $S_{ab}(\theta) = e^{i\delta_{ab}(\theta)}$

Bethe wave function $\Psi(x_1,\cdots,x_N) = \prod_{i=1}^{N} e^{ip_i x_i}$

Bethe eqs. $e^{ip_i L}\prod_{j\neq i} S_{ij}(\theta_i-\theta_j) = \pm 1$

thermodynamic limit $L \to \infty$, $N_a \to \infty$, N_a/L:fixed
$\rho_a^{(r)}(\theta)$:denisty of roots $\rho_a^{(h)}$:denisty of holes $\rho = \rho^{(r)} + \rho^{(h)}$
total energy

$$E = \sum_{a=1}^{n} \int_{-\infty}^{\infty} m_a \cosh\theta \rho_a^{(r)}(\theta) d\theta$$

entropy

$$S = \sum_{a=1}^{n} \int_{-\infty}^{\infty} [\rho_a \ln \rho_a - \rho_a^{(r)} \ln \rho_a^{(r)} - \rho_a^{(h)} \ln \rho_a^{(h)}]$$

minimizing free energy $F = E - TS$ with constraint $\int d\theta \rho_a^{(r)} = N_a/L \Longrightarrow$ TBA equation

$$m_a R \cosh\theta = \epsilon_a(\theta) + \sum_{b=1}^{n} \varphi_{ab} * \log(1 + e^{-\epsilon_b}), \quad \frac{\rho_a^{(r)}}{\rho_a} = \frac{e^{-\epsilon_a}}{1 + e^{-\epsilon_a}}$$

$\varphi_{ab} = -i \frac{d}{d\theta} \log S_{ab}(\theta)$

Analytic solution in the IR and UV limits

$r = m_1 R$ (m_1:lowest mass gap)

- IR (large mass) limit $r \to \infty$
  $\epsilon_a \sim \hat{m}_a r \cosh\theta$ ($\hat{m}_a = m_a/m_1$)
  free massive field theory (or other CFT)
- UV (small mass) limit $r \to 0$
  ϵ_a:const $\epsilon_a = \sum_b N_{ab} \ln(1+e^{-\epsilon_b})$
  CFT

$E_0(R) = -\frac{\pi}{6R}\tilde{c}(r)$

$$\tilde{c}(r) = \begin{cases} \frac{6}{\pi^2} r \sum_{a=1}^{n} \hat{m}_a K_1(\hat{m}_a r) & r \to \infty \\ \frac{6}{\pi^2} \sum_{a=1}^{n} \mathcal{L}(\frac{1}{1+e^{\epsilon_a}}) = c_{CFT} & r \to 0 \end{cases}$$

$\mathcal{L}(x) = -\frac{1}{2}\int_0^x dt \left(\frac{\ln t}{1-t} + \frac{\ln(1-t)}{t}\right)$: Rogers' dilogarithm function

small mass expansion: perturbed CFT

Klassen-Melzer, NPB338 (1990) 485

$$S = S_{CFT} + \lambda \int d^2 w \epsilon(w, \bar{w})$$

$\epsilon(w, \bar{w})$ has conformal weight (D_ϵ, D_ϵ)
partition function $Z = \langle \exp\left(-\lambda \int d^2 w \epsilon(w, \bar{w})\right) \rangle_0$

$$F = \lim_{L\to\infty} \frac{R}{L} \log Z = RE(R) - \frac{1}{4}(mR)^2$$

$$E(R) = E_0 - R^2 \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \left(\frac{2\pi}{R}\right)^{2+2n(D_\epsilon - 1)} \int d^2 z_2 \cdots d^2 z_n$$

$$\times \langle V(0)\epsilon(1)\epsilon(z_2)\cdots\epsilon(z_n)V(\infty)\rangle_{0,connected} \prod_i |z_i|^{2(D_\varepsilon - 1)}$$

Remainder function for 6-point amplitude

- $\mathbf{Z}_4$ parafermion ($c=\frac{2(N-2)}{N+2}=1$) deformed by enegy operator $\epsilon(z,\bar{z})$ with conformal dims. $(\frac{1}{3},\frac{1}{3})$
- $(2\pi\lambda)^2=\left[m\sqrt{\pi}\gamma(\frac{3}{4})\right]^{\frac{8}{3}}\gamma(\frac{1}{6})$ $(\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)})$ Fateev PLB 324

$R=-\frac{1}{4}\sum_{k=1}^{3}\mathrm{Li}_2(-Y_2(\frac{(2k+1)\pi i}{2}))-|Z|^2-F$

$$F=-\frac{\pi^2}{6}+\frac{\phi^2}{3\pi}+|Z|^2-C_{\frac{8}{3}}\gamma(\frac{1}{3}+\frac{\phi}{2\pi})\gamma(\frac{1}{3}-\frac{\phi}{3\pi})|Z|^{\frac{8}{3}}+\cdots$$

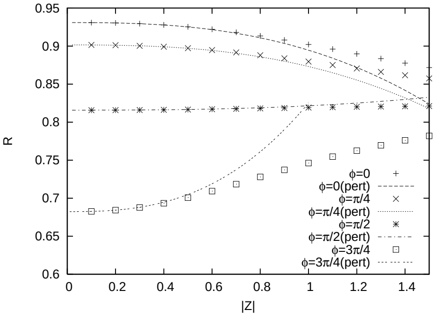
$C_{\frac{8}{3}}=\frac{\pi}{2}\left[\frac{1}{\sqrt{\pi}}\gamma(\frac{3}{4})\right]^{\frac{8}{3}}\gamma(\frac{1}{6})\gamma(\frac{1}{3})$

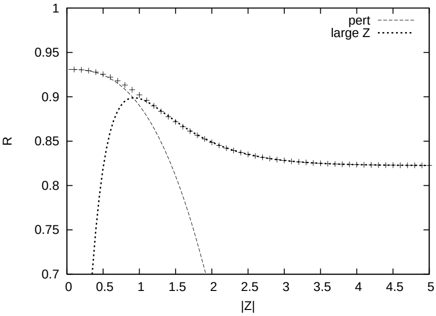
$$Y_2(\theta)=\tilde{Y}^{(0)}(\theta,\phi)+\tilde{Y}_2^{(1)}(\theta,\phi)|Z|^{\frac{4}{3}}+\cdots$$

$$\tilde{Y}_2^{(0)}=1+2\cos(\frac{2\phi}{3}),\quad \tilde{Y}_2^{(1)}=y^{(1)}(\phi)\cos\frac{4(\varphi+i\theta)}{3}$$

$$y^{(1)}(\phi)\simeq 1.31367+2.61136\cos\frac{\phi}{3}+1.55402\cos\frac{2\phi}{3}$$

Numerical results vs CFT perturbation





Outlook

- AdS$_3$ case: $2n$-point amplitudes $SU(n-2)_2/U(1)^{n-3}$ parafermions
- AdS$_5$ case: m-point amplitudes:

$$\frac{SU(m-4)_4}{U(1)^{m-5}} \simeq \frac{[SU(4)_1]^{m-4}}{SU(4)_{m-4}}$$

  Y-system is different from that of HSG model. (new integrable field theory?)

- CFT perturbation of generalized parafermions $\Longrightarrow$ analytical (power series) results for the remainder function
  $m=6$ $SU(2)_4/U(1)$ ($\mathbf{Z}_4$ parafermion)
- form factor Maldacena-Zhiboedov
- Hitchin eqs. / CFT correspondence (ODE/IM correspondence Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov)