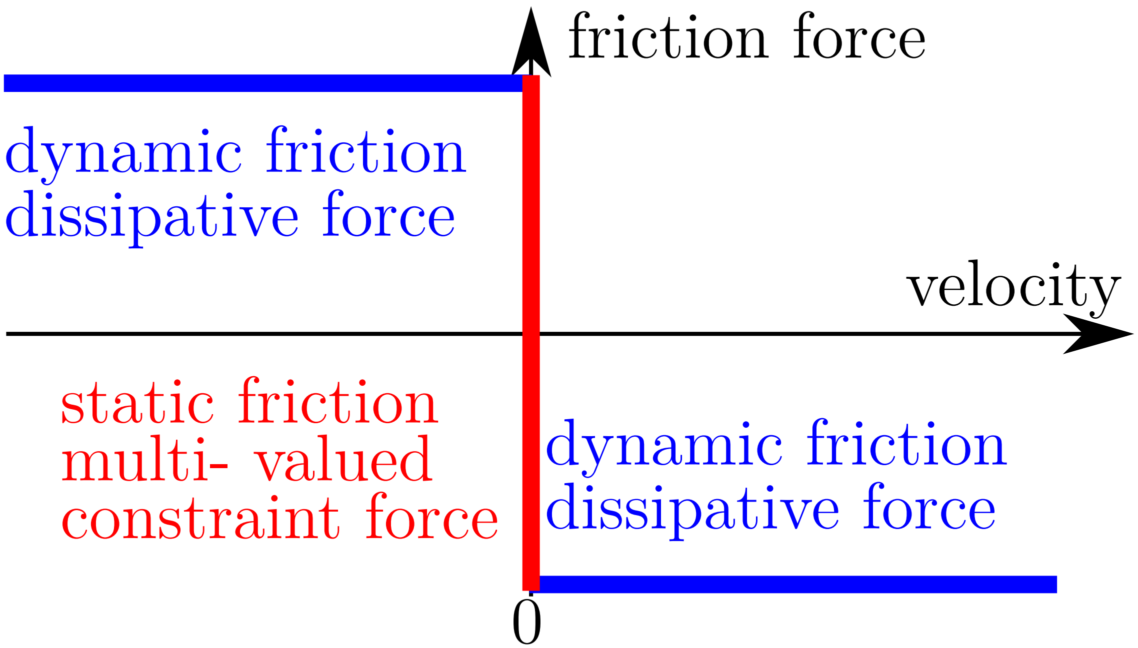
# Numerically exact formulation for Many-Body Coulomb friction

Dominik Krengel, Hans-Georg Matuttis
Department of Mechanical Engineering and Intelligent Systems,
The Graduate School of the University of Electro-Communications

# 1. Dry/Solid/Coulomb friction at a single contact

– tangential contact force between particles
– Friction coefficient $\mu = \mathcal{O}(1) \rightarrow$ no “small perturbation”




**Dynamic friction**: $v \neq 0$
$F^{\text{fric}} = -\mu F^N \text{sgn}(v)$
dissipative force in tangential direction, “the easy part”

**Static friction**: $v = 0$
$-\mu F^N \leqslant F^{\text{static}} \leqslant \mu F^N$
non-dissipative constraint of motion without relative motion, “the hard part”

**Purpose of this work**: Solve the constraint problem “numerically exact” for many-particle-problems for any given normal force $F^N$ and coefficients of friction $\mu$ and arbitrary contact orientations

For this, we first have to discuss the theory of “numerically exact” constraints for ordinary differential equations: Differential Algebraic Equations (DAEs)

## 2. Differential Algebraic Equations

What are Differential Algebraic Equations?

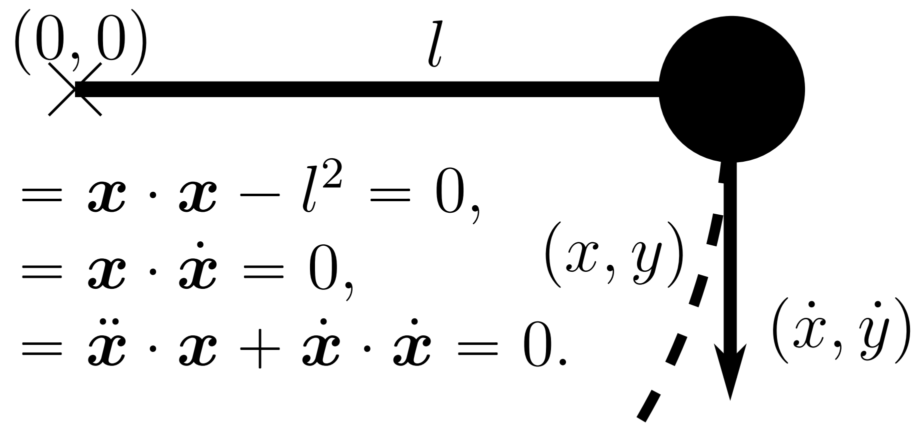
| | | | | |
|---|---|---|---|---|
| Ordinary Differential | | algebraic constraints | | Differential |
| Equation (ODE) | $+$ | bilateral: $g(y_0) = 0$ | $\Longrightarrow$ | Algebraic |
| $\dot{y} = f(y,t) \quad \forall y, t$ | | unilateral: $g(y_0) \leqslant 0$ | | Equation (DAE) |
| $\downarrow$ | | $\downarrow$ | | |
| Dynamical evolution of system | | Only initial conditions $y_0$ consistent with $g(y_0)$ | | |

"Differential Algebraic Equations are not ODEs"[†]

Examples in mechanics:
– Position of point pendulum $\rightarrow$ constraint is pendulum length
– Value of static friction $\rightarrow$ constraint is static Coulomb friction

[†] L. Petzold: SIAM J. Sci. Stat. Comput. 3 (1982) 367: inconsistent initial conditions, error analysis,...

## 2.1. Exemplified: Concept and Terminology




Point pendulum with rotation around origin and length $l$

Bilateral constraint equation $\quad g(x,y) = \boldsymbol{x}\cdot\boldsymbol{x} - l^2 = 0,$

DAE: differentiate as often as necessary for further independent equations

$$\dot{g}(x,y) = \boldsymbol{x}\cdot\dot{\boldsymbol{x}} = 0,$$
$$\ddot{g}(x,y) = \ddot{\boldsymbol{x}}\cdot\boldsymbol{x} + \dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} = 0.$$

Newton's equation of motion $\quad m\ddot{x} = \overbrace{\boldsymbol{f}}^{\text{external forces}} + \overbrace{\widetilde{\boldsymbol{f}}}^{\text{constraint forces}}$

Kinetic energy $\quad \left(\boldsymbol{f} + \widetilde{\boldsymbol{f}}\right)\cdot\boldsymbol{x} = -m\,\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} = 2\,T_{\text{kin}}$

Time derivative $\quad \dot{T}_{\text{kin}} = m\,\ddot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} = -\left(\boldsymbol{f} + \widetilde{\boldsymbol{f}}\right)\cdot\dot{\boldsymbol{x}}$

D'Alembert principle:
no work by constraint forces $\quad \rightarrow \widetilde{\boldsymbol{f}}\cdot\dot{\boldsymbol{x}} = 0 \rightarrow \boldsymbol{x}\perp\dot{\boldsymbol{x}},\ \widetilde{\boldsymbol{f}}\perp\dot{\boldsymbol{x}}$

Lagrange parameter $\lambda$ : $\quad \widetilde{\boldsymbol{f}} = \lambda\boldsymbol{x},\ \lambda = \dfrac{-\boldsymbol{f}\cdot\boldsymbol{x} - m\,\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}}{\boldsymbol{x}\cdot\boldsymbol{x}}$

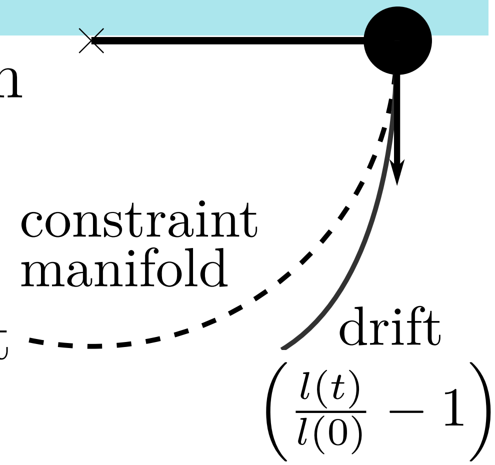
no time evolution for $\lambda$, may vary non-smoothly
$\Rightarrow$ Constraint force $\propto$ kinetic energy,
external forces $\propto$ 1/pendulum length

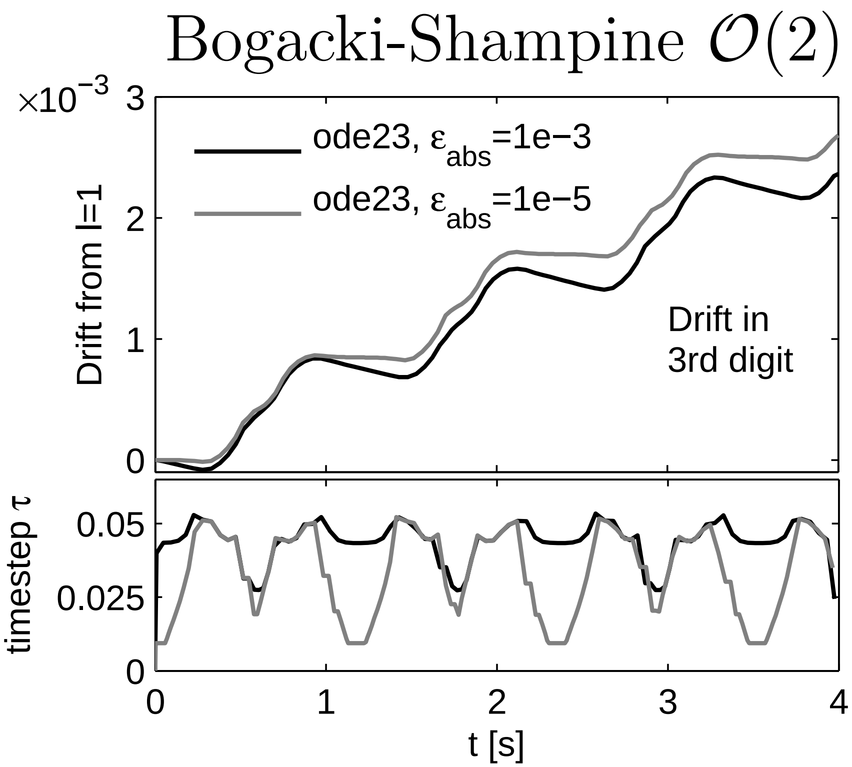
## 2.2. Drift away from the constraint

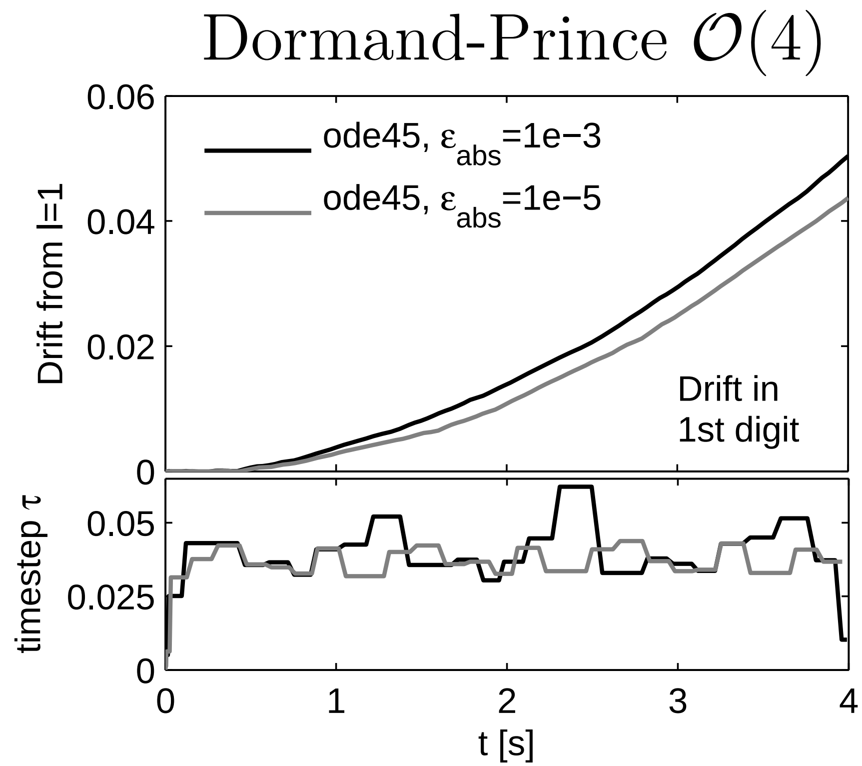
Numerical integration → pendulum length diverges from constraint manifold

Runge-Kutta based on weighted average of several integration points

Integration based on constraint forces, not on constraint itself (Integration may miss constraint manifold)




Drift away from constraint manifold worse for $\mathcal{O}(4)$ than for $\mathcal{O}(2)$

**Not** a problem of **accuracy but** of **stability** → sign of stiff ODE

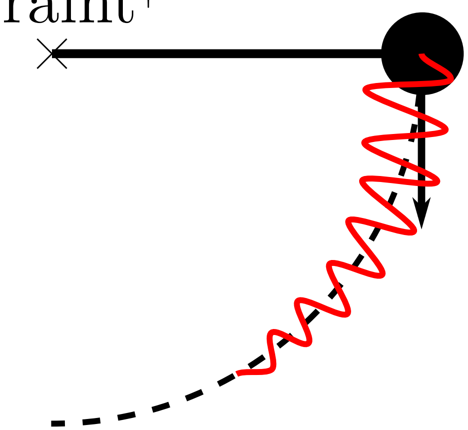
## 2.3. Stiffness, time integrator and stabilization

Stiff ODEs: "problems for which explicit methods don't work"[†]
Constraint problems always "stiff" ODEs

Issue: Not accuracy but stability → solution related to exact solution
Second order BDF integrator → stiffly stable → minimal drift
Stabilize DAE-solution to limit drift away from constraint[†]

**Baumgarte stabilization:**

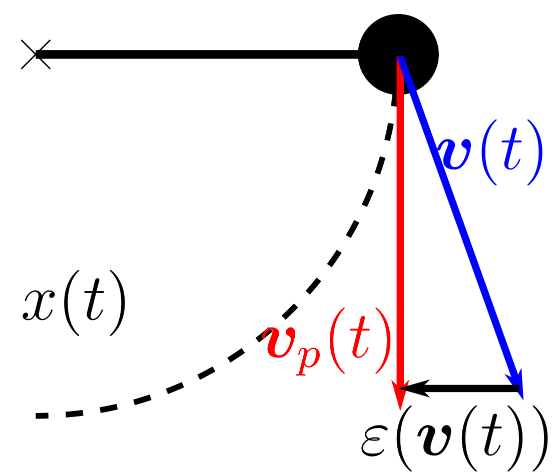
Constraint as linear oscillator $0 = \ddot{g} + 2\alpha\dot{g} + \beta^2 g$
→ damped oscillation back onto constraint manifold
Difficult to find appropriate parameters $\alpha, \beta$



Errors in position constraint $\varepsilon(x(t))$ less critical than errors in velocity constraint $\varepsilon(v(t)) \approx \varepsilon(x(t)/\tau)$

**Stabilization by Projection**[⋆]

Projection on velocity $\boldsymbol{v}_p = (\boldsymbol{v} \cdot \boldsymbol{e}_t)\boldsymbol{e}_t$

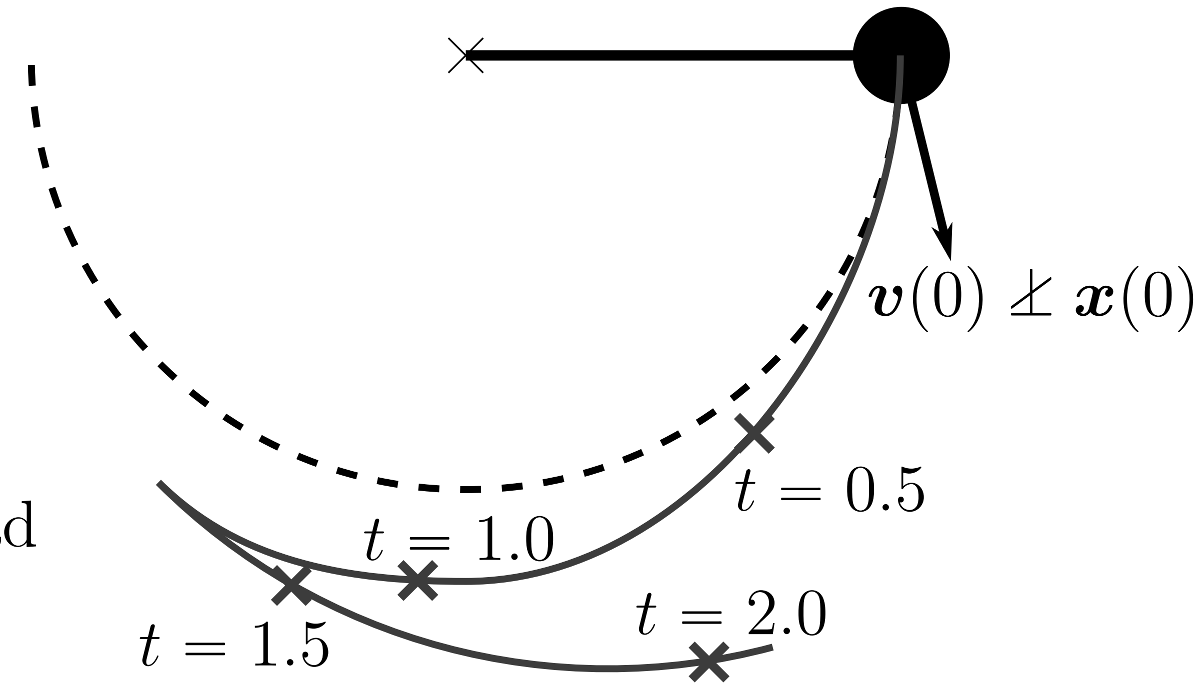



[†] E. Hairer, G. Wanner, Solving Ord. Diff. Eq. II: Stiff and differential algebraic problems, Springer 1996, 2nd ed., p.2

[⋆] Ch. Lubich, On projected Runge-Kutta methods for differential-algebraic equations, BIT Numerical Mathematics, vol. 31, no. 3, pp. 545-550, 1991

## 2.4. Inconsistent initial values in numerical analysis

Violation of orthogonality between velocity and position
$\boldsymbol{v}(0) \not\perp \boldsymbol{x}(0)$
Simulation diverges towards infinity or converges to zero

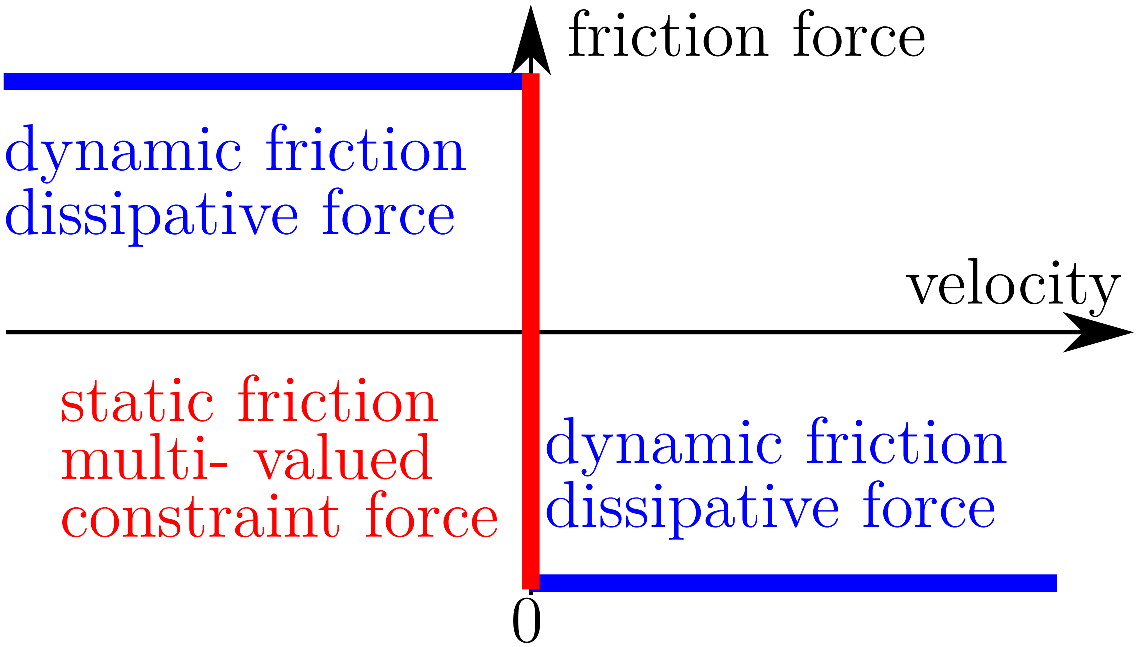



For friction: Painlevé Paradox.
Friction force accelerates instead of decelerates.

In general:
Computation of consistent initial values computationally tedious:
Constraint equations non-linear→ Solution of non-lin. systems necessary
In our case of friction: everything ok with relative finite slipping velocity

# 3. Dry/Solid/Coulomb friction at a single contact

– tangential contact force between particles
– Friction coefficient $\mu = \mathcal{O}(1) \rightarrow$ no 'small perturbation'



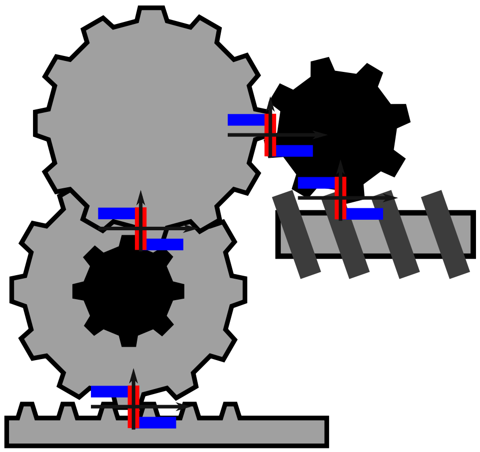

**Dynamic friction**: $v \neq 0$
$F^{\text{fric}} = -\mu F^N \text{sgn}(v)$
dissipative force in tangential direction

**Static friction**: $v = 0$
$-\mu F^N \leqslant F^{\text{static}} \leqslant \mu F^N$
non-dissipative constraint of motion without relative motion

**Purpose of this work**: Introduce a numerically exact evaluation of static friction **for many-particle-problems** for any given normal force $F^N$ and coefficients of friction $\mu$ and arbitrary contact orientations

Find: – Criterion to discriminate static friction and dynamic friction
– Calculation scheme for static friction $< \mu F^N$

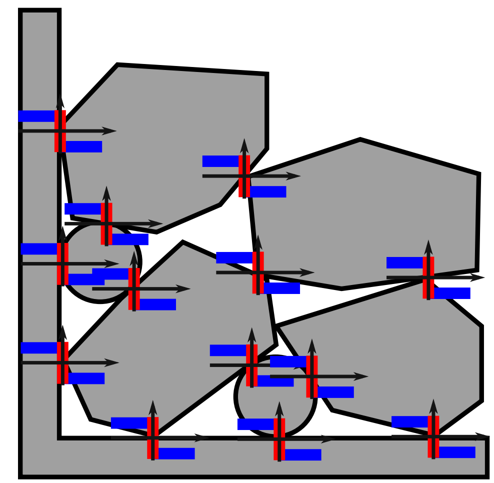
## 4. Many particle problems with dry friction

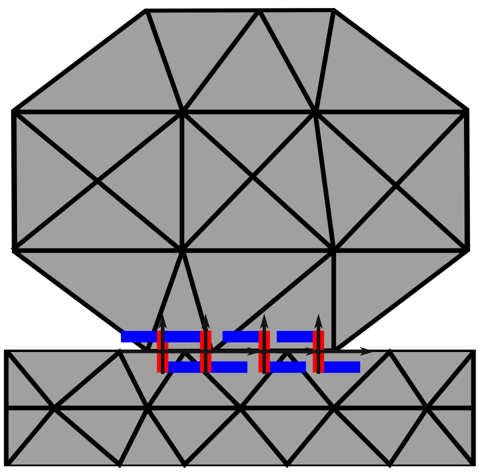


Many Systems in Science and Technology friction dominated

Multi-body mechanism, mechatronics, mechanics of gears

Rock mechanics, soil mechanics, geotechnics





Many-body finite element systems

No analytical formalism for many particle friction with given $\mu_i$, $F_i^N$ and arbitrary contact orientation

Numerical noise:
$v \equiv 0$ does not exist
Numerically stable implementation necessary

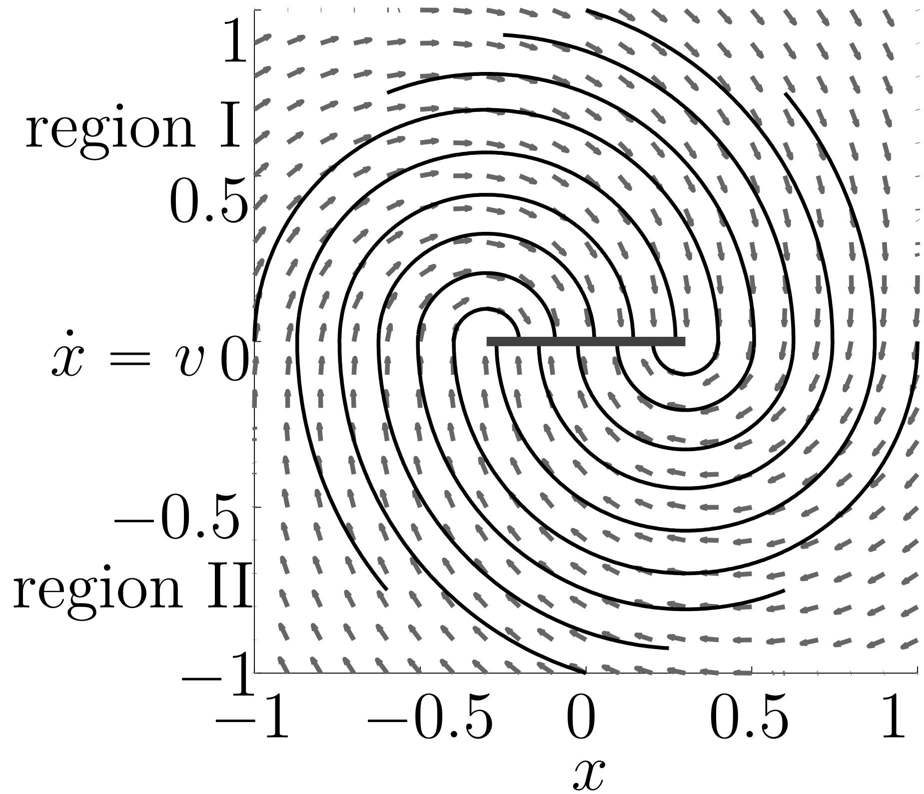
**Up to now**, there was no numerically consistent theory for many-contact static friction for given normal forces $F^N$ and coefficients of friction $\mu$

**This research pioneers such an approach!**

# 5. Friction and constraint: Concept for linear oscillator

Work with dynamical system / phase space $(x(t), \dot{x}(t) = v(t))$ instead of real space $(x(t), t)$

$$m\ddot{x} = \underbrace{-kx - \overbrace{2\delta\dot{x}}^{\text{viscous damping}}}_{f^{\text{ext}}} \quad - \quad \overbrace{\mu F^N \text{sign}(\dot{x})}^{\text{dry friction}}$$




Solve for

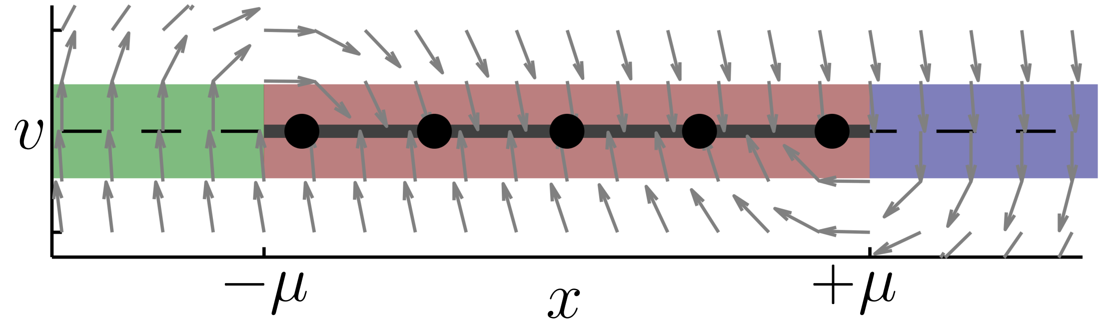
$$\dot{x} = \begin{cases} -\frac{1}{2\delta}\left(m\ddot{x} + kx + \mu F^N\right) & \dot{x} > 0 \\ -\frac{1}{2\delta}\left(m\ddot{x} + kx - \mu F^N\right) & \dot{x} < 0 \end{cases}$$

$\dot{x}$ dependence drops out for $v = 0$

$$f = m\ddot{x} = \begin{cases} f_I = f^{\text{ext}} - \mu F^N & \text{for } \dot{x} > 0, \text{ region I} \\ f_{II} = f^{\text{ext}} + \mu F^N & \text{for } \dot{x} < 0, \text{ region II} \end{cases}$$

What to do at the boundary between region I and region II ? $\Rightarrow$ work with convex hull of the forces $(1-\lambda)f_I + \lambda f_{II}$

$$a_I = \quad f_I/m \; (= a), \qquad \text{region I}$$
$$a_{II} = -f_{II}/m (= -a), \quad \text{region II}$$




Four different scenarios for the auxiliary variables $a_I, a_{II}$

1. $a_{II} < 0 \rightarrow a_I > 0$: dyn. friction, jump from $+\mu F^N$ to $-\mu F^N$ at $\dot{x} = 0$,
2. $a_I < 0 \rightarrow a_{II} > 0$: dyn. friction, jump from $-\mu F^N$ to $+\mu F^N$ at $\dot{x} = 0$,
3. $a_I < 0$, $a_{II} < 0$: flow is pulled into constraint $\dot{x} = 0$ from above and below: static friction solution for $\dot{x} = 0$ (numerically hardly ever $f^{\text{fric}} \in \{0, f_I, f_{II}\}$),

(4. $a_I > 0$, $a_{II} > 0$: both flows pull away from constraint: Painlevé paradox, unfortunate choice of initial conditions at $\dot{x} = 0$.)

| | | |
|---|---|---|
| Dynamic | $f_I = f^{\text{ext}} - \mu F^N \quad (v > 0),$ | $a_I = f_I/m \; (= a)$ |
| friction | $f_{II} = f^{\text{ext}} + \mu F^N \quad (v < 0),$ | $a_{II} = -f_{II}/m (= -a)$ |

Determine static friction in convex hull of dynamic friction values†

$$f(\dot{x}, \lambda) = (1-\lambda) f_I + \lambda f_{II}, \quad (v = 0) \quad 0 \leqslant \lambda \leqslant 1 \quad (*)$$

$0 = g(\cancel{x}, \dot{x}) = v = \dot{x}$ — Indicator function $v = 0$ becomes constraint function $g$

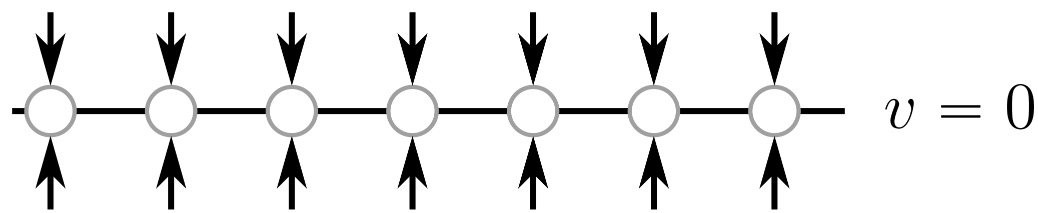
$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(\dot{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\dot{x} = \ddot{x}$ — Differentiate (as for bilateral constraints)

$\ddot{x} = f(\dot{x}, \lambda)/m = 0$ — Independent equations

from $(*)$ $0 = (1-\lambda) a_I - \lambda a_{II}$ — Insert into convex hull, $f_{\dots} \to a_{\dots}$.

Constraint static friction force computed with Lagrange parameter $\lambda$

$$\lambda = \frac{a_I}{a_I + a_{II}}$$



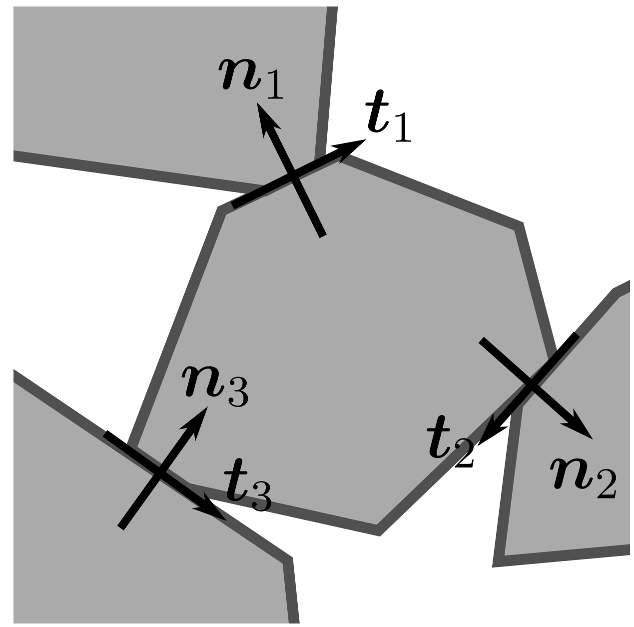

Produces stable fixpoints of the ODE

† E. Hairer, S.P. Norsett, G. Wanner, Solving Ordinary Differential Equations I, Springer (1993)

(A. Filippov, Differential equations with discontinuous righthand sides, Springer (1988))

# 6. Formalism for many particle problems

In this talk so far: only one contact
Conventional dynamical-systems theory: fixed coordinate system
**Computation of static situations in many-particle systems problematic**: necessary to "hit" force equilibrium exactly, irrespective of numerical errors




⇒ Deal with problem at individual contact level
– Work with reduced masses and relative coordinates and velocities for each contact
– Relative velocities and accelerations obtainable for centres of mass and contact point

– Reduced mass for tangential forces: accounts for moments of inertia and distance between centres of mass and contact point

$$\frac{1}{m^{\tan}} = \frac{1}{m_1} + \frac{r_1^2}{I_1} + \frac{1}{m_2} + \frac{r_2^2}{I_2}$$

## 6.1. Velocity conditions for static friction

– Computation of constraint force instead of dynamic friction
– Other contacts formally absorbed as external force
– Deformation of manifold near zero velocity, attractors for $v \neq 0$ possible → more phaseflow patterns possible

☞ either velocity condition necessary for static friction to occur

1. Change of sign of velocity in the **next** timestep
2. Change of sign of velocity in the **previous** timestep
3. Vanishing relative velocity $v_{\mathrm{rel}} \approx 0$ in **current** timestep

## 6.2. Auxiliary variables $a_I$,$a_{II}$ for many-particle approach

Analog to one-particle case with relative accelerations

$$f_I = m^{\tan} a_{\mathrm{eff}}^{\tan} - \mu |F^N| \quad (v > 0) \qquad a_I = a_{\mathrm{eff}}^{\tan} - \mu |F^N| / m^{\tan}$$

$$f_{II} = m^{\tan} a_{\mathrm{eff}}^{\tan} + \mu |F^N| \quad (v < 0) \qquad a_{II} = -a_{\mathrm{eff}}^{\tan} - \mu |F^N| / m^{\tan}$$

Same condition as before:

$a_I < 0$, $a_{II} < 0$ friction forces compensate relative acceleration

⇒ Both conditions are necessary for static friction

## 6.3. Computation of static friction

Conditions for velocity and auxiliary variables fullfilled: use previous approach for static friction

– Convex hull of dynamic friction values

$$f(v_{\text{rel}}^{\tan}, \lambda) = (1-\lambda) f_I + \lambda f_{II}, \quad (v=0) \quad 0 \leqslant \lambda \leqslant 1$$

– Differentiate constraint function to obtain Lagrange parameter $\lambda$

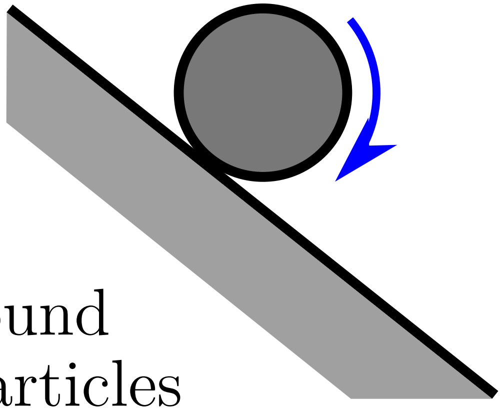
$$\lambda = \frac{a_I}{a_I + a_{II}} = \frac{1}{2} - \frac{m_{\text{rel}}^{\tan} a_{\text{rel}}^{\tan}}{2\mu |F^N|}$$

☞ Static friction computation based on relative accelerations, not based on velocities

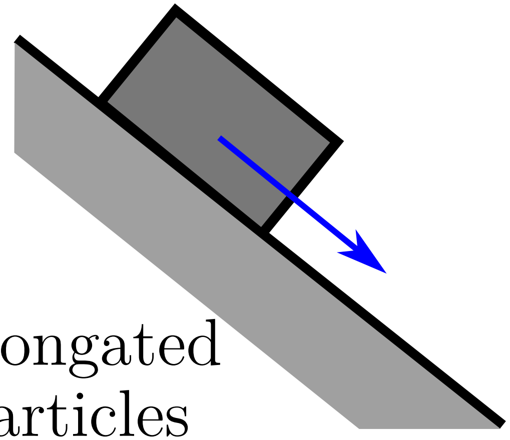
☞ Not required to solve non-linear equation

# 7. Dynamics of granular media

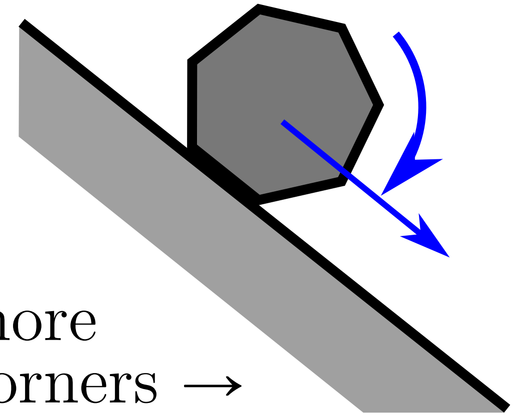
Competition between rolling and sliding determines the dynamics of granular materials, for single particles and for aggregates



round particles will roll

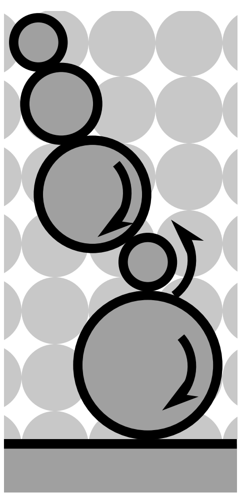


elongated particles will slide.

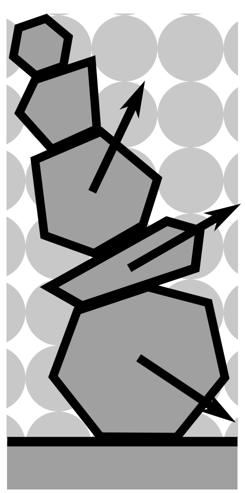


more corners → competition between rolling and sliding

Shape effects:

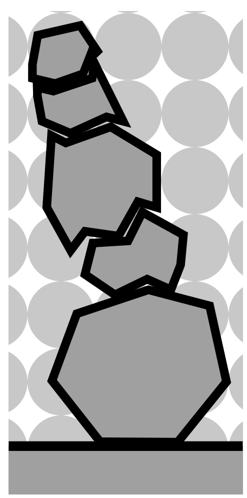


rolling

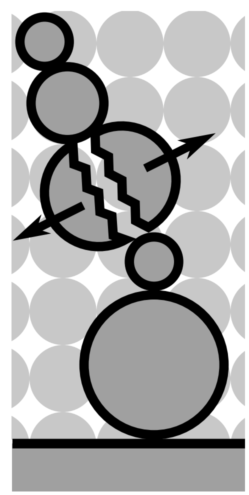


sliding

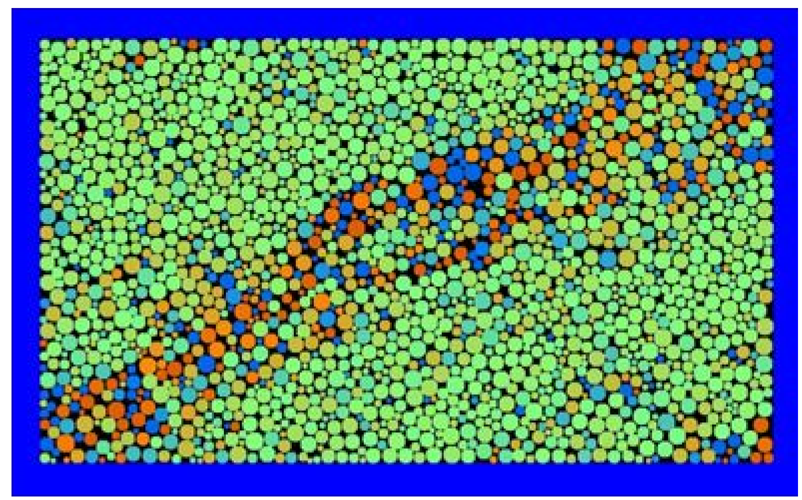
Not considered:



~~interlocking~~



~~fracture~~



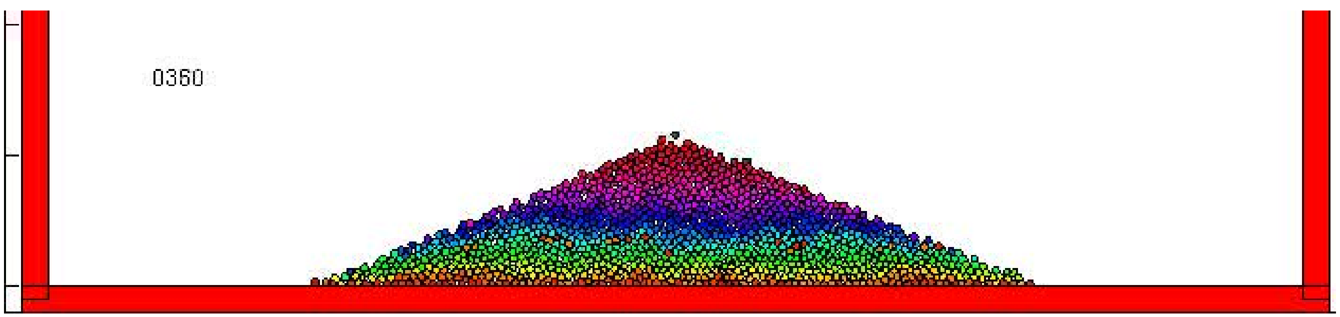
Shearbands: rotation ↺, ↻

S. Luding, Univ. Twente

# 8. Shape effects experimentally

Spheres:
no heap

Polyhedra:
stable heap

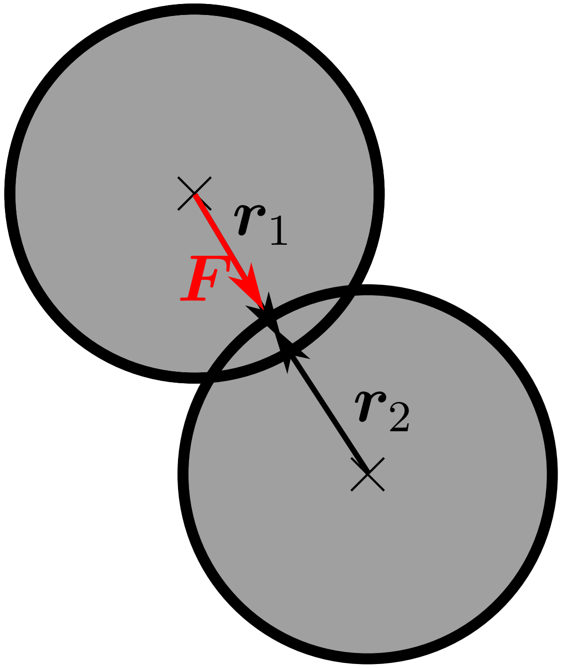


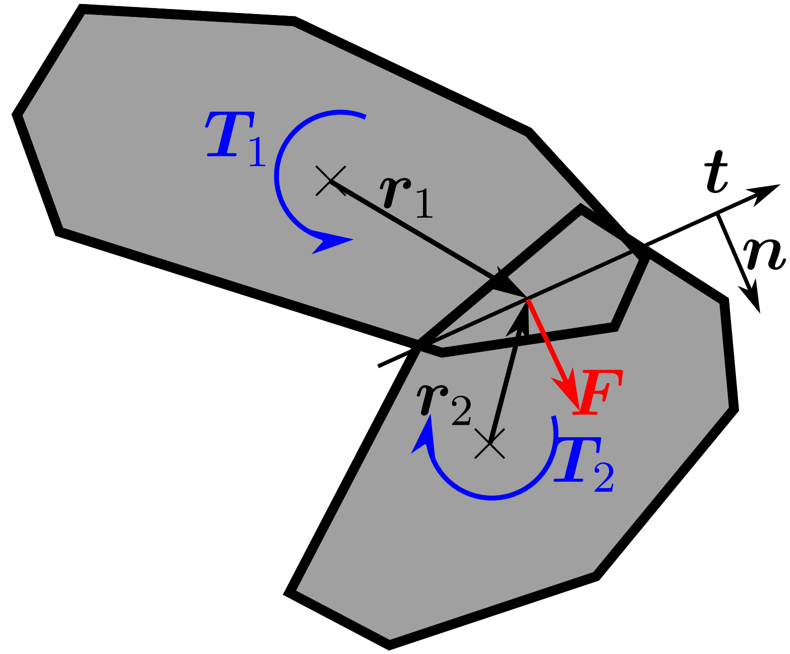

2D simulation:
friction turned of at
420 timesteps

☞ Shape effects can only be simulated if additionally to the dynamic friction also the static friction is modelled correctly!

# 9. DEM: Method shape dependent force laws

Discrete Element Method (DEM): Elastic forces depending on shape




spherical particles:
central forces

$$F^{el} \parallel r_1, r_2$$

non-spherical particles: no central forces

Elastic forces:
shape dependence

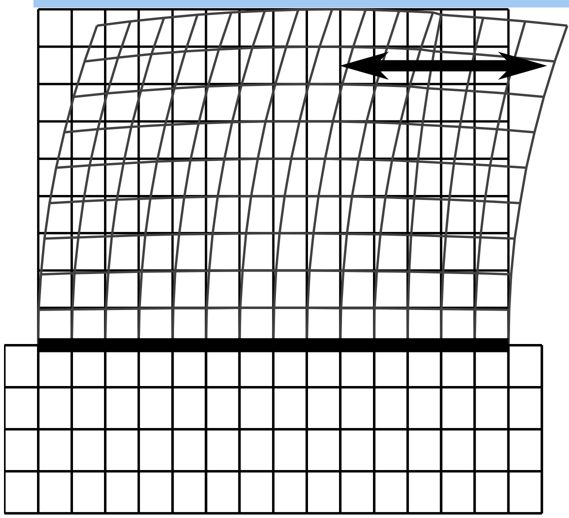
$$|F^{el}| = Y\frac{A}{l}$$

Torques due to shape:

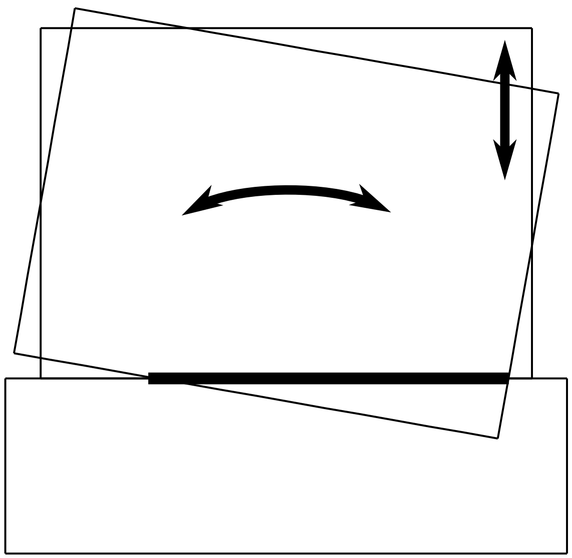
$$\begin{aligned} T_1 &= r_1 \times F \\ T_2 &= r_2 \times -F \end{aligned}$$

With spherical particles: rolling friction coefficients of unphysical magnitude used. Realistic observable values for non-spherical particles.

# 10. Concept for rotational damping



**Physically**: Particles are elastic.
– Particles can deform and vibrate.
– Internal vibration can store and dissipate energy via fixed contacts



**DEM**: Particles are stiff.
– Deformation is modelled as overlap
– Vibration as particle rotation
– Rotational degrees of freedom can store and release energy
– Friction acts only in tangential direction
→ rotation only affected by normal interaction
– Asymmetric contacts don't compensate normal forces completely
→ energy remains in rotational degrees of freedom

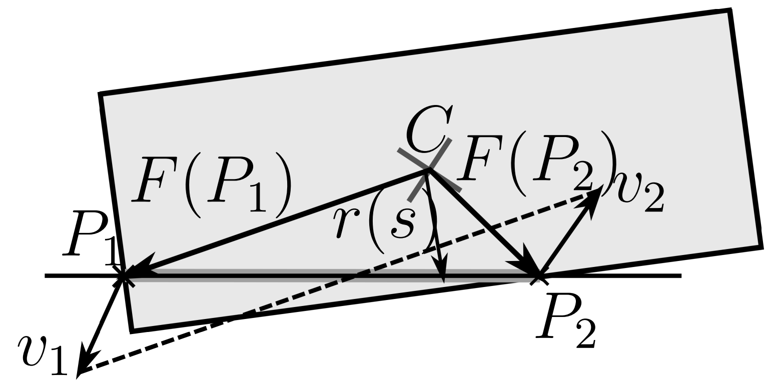
☞ With Cundall-Strack: Tangential interaction modelled as springs, particle rotation can be damped out
☞ With DAE: No tangential oscillator, additional tangential damping required

## 10.1. One particle with a fixed floor

Damping torque $T^{\text{damp}}$ based on interpolated velocity field of contact line $\overline{P_1P_2}$

Approximation of corresponding elastic torque based on endpoints of contact line

$T^{\text{elast}} \approx \frac{1}{2}\left(\boldsymbol{F}(P_2) \times \boldsymbol{CP}_2 + \boldsymbol{F}(P_1) \times \boldsymbol{CP}_1\right)$




Rotational damping as counteracting torque based on velocity field of contact line, approximated

$$T^{\text{damp}} \approx -\gamma^{\tan}\sqrt{Y \cdot m_{\text{eff}}}\frac{1}{2}\Big(\boldsymbol{v}^{\text{cont}}(P_2) \times \boldsymbol{CP}_2 + \boldsymbol{v}^{\text{cont}}(P_1) \times \boldsymbol{CP}_1\Big)$$

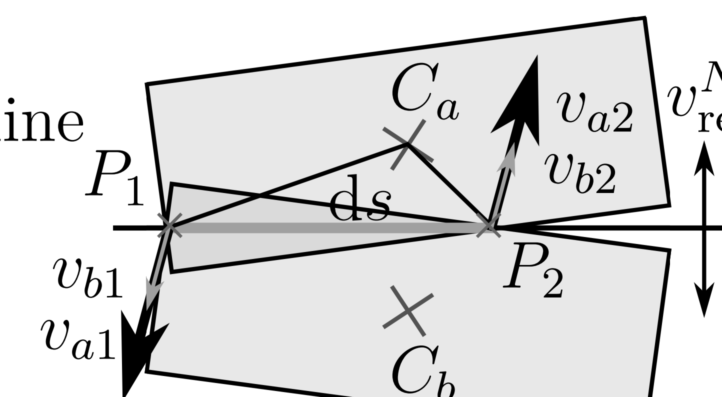
Velocity of contact point $i$: $\boldsymbol{v}^{\text{cont}}(P_i) = \boldsymbol{v}(P_i) - \boldsymbol{v}^N$

## 10.2. Two particles in non-rotated coordinate system

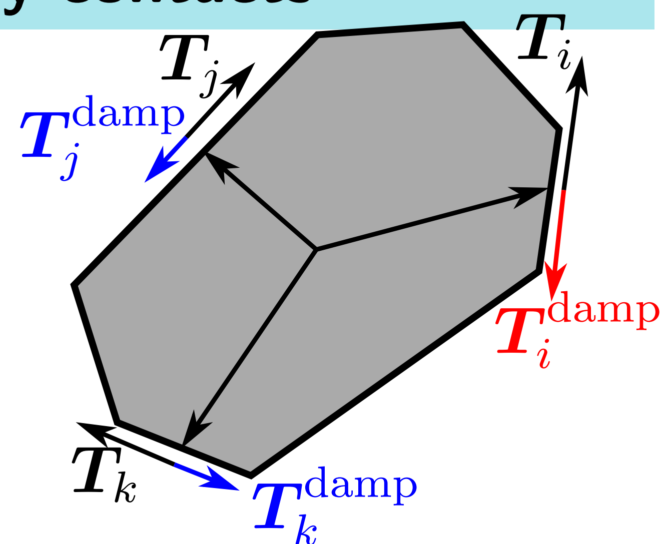
Torque based on relative velocities at contact line

$\boldsymbol{v}^{\text{cont}}(P_i) = \Big(\boldsymbol{v}_{ai}(P_i) - \boldsymbol{v}_{bi}(P_i)\Big) - \boldsymbol{v}^N_{\text{rel}}$

for particles $a$,$b$ and contact points $i = 1, 2$



## 10.3. Conditional damping of torques for many contacts




☞ Particle rotation result from sum of damping torques of all contacts → not possible to compute damping torques individually

☞ Damping torques reactive to elastic torques → not possible to deal with contacts individually

⇒ Extremal limits with respect to (counter-)clockwise rotation

$T^{\min} = \sum T_i^{\text{damp}}, \forall T_i^{\text{damp}} < 0,$ $\qquad T^{\max} = \sum T_i^{\text{damp}}, \forall T_i^{\text{damp}} > 0.$

Minimize angular motion → compensate $\tilde{T} = T + I\frac{\omega}{\tau}$, where

$T$ = external torque, $\omega$ = angular velocity of particle, $\tau$ = timestep

$$\tilde{T} > 0\text{: use } T^{\min} \quad \rightarrow \quad \begin{array}{l} |\tilde{T}| > |T^{\min}|\text{: } T^{\text{damp}} = T^{\min}; \\ |\tilde{T}| < |T^{\min}|\text{: } T^{\text{damp}} = -\tilde{T}\text{, else} \\ \text{overcompensation} \end{array}$$

$$\tilde{T} < 0\text{: use } T^{\max} \quad \rightarrow \quad \begin{array}{l} |\tilde{T}| > |T^{\max}|\text{: } T^{\text{damp}} = T^{\max} \\ |\tilde{T}| < |T^{\max}|\text{: } T^{\text{damp}} = -\tilde{T}\text{, else} \\ \text{overcompensation} \end{array}$$

## 10.4. Coordinate system rotating with $\omega_0$

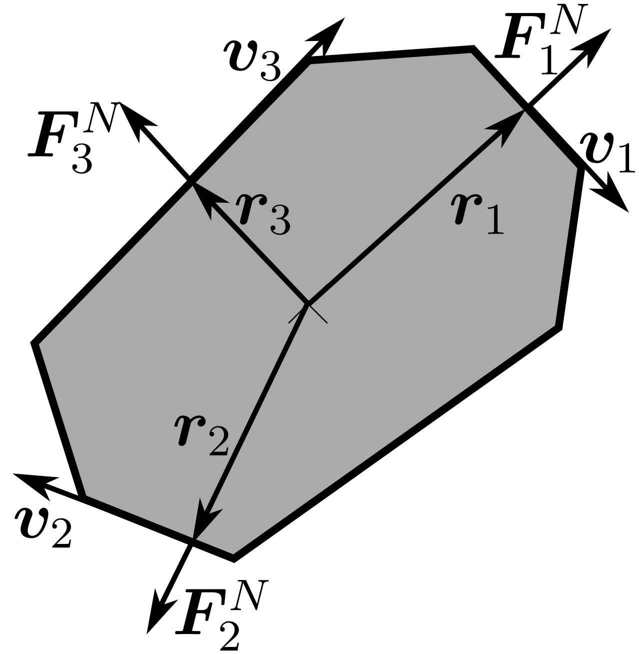
– Surrounding granular matrix influences particle rotation

– Rotational damping has to respect bulk motion

– Consider influence from contacting particles $i$

→ Weighted angular velocity due to environment

$\omega_0 = \dfrac{\sum_i \omega_i |F_i^N|}{\sum_i |F_i^N|}$ from contacts $i$, $\omega_i = |\boldsymbol{r}_i \times \boldsymbol{v}_i|$



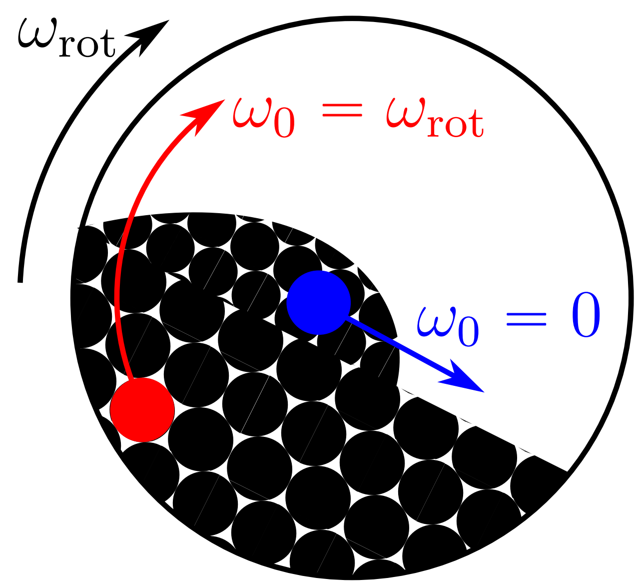

– Adapt damping torque with bulk angular motion

$$\Rightarrow \widetilde{T} = T + I\frac{\omega}{\tau} \quad \rightarrow \quad \widetilde{T} = T + I\frac{\omega - \omega_0}{\tau}$$

–Threshold now based on $\omega_0$

$\widetilde{T} > I\dfrac{\omega_0}{\tau}$: use $T^{\min}$ $\qquad$ $\widetilde{T} < I\dfrac{\omega_0}{\tau}$: use $T^{\max}$

Then, as before test for $|T^{\max}|$,$|T^{\min}|$

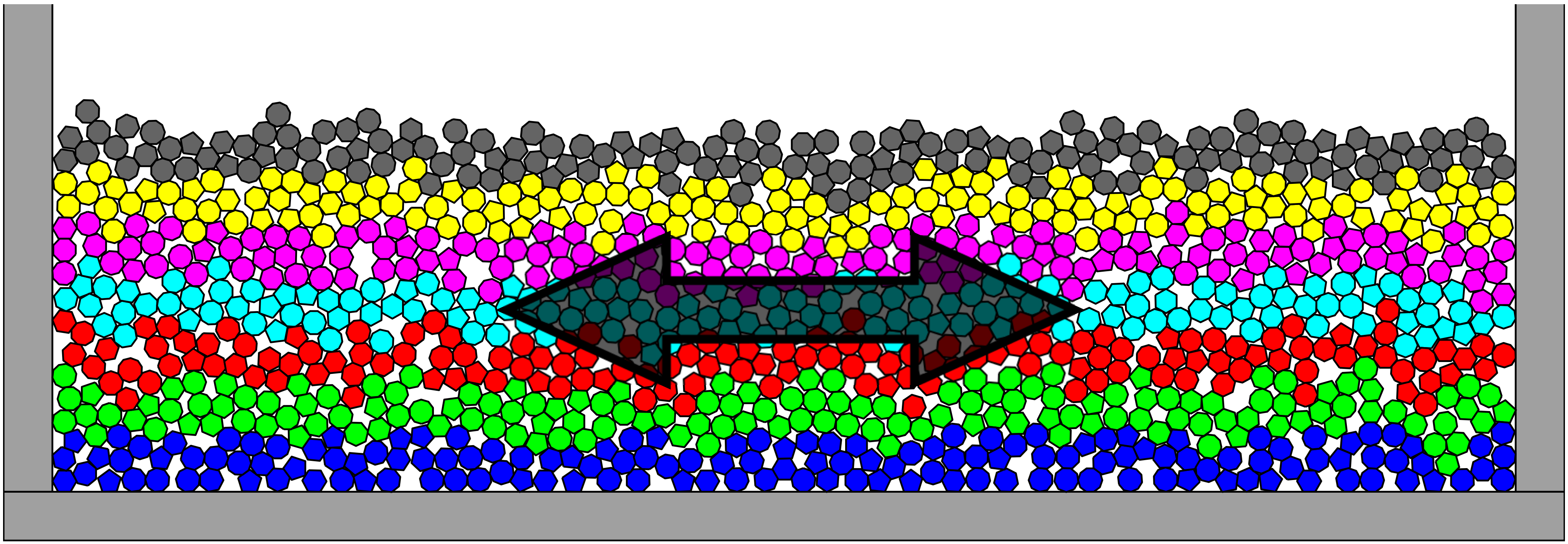



Example: drum rotating with $\omega_{\text{rot}}$.
Particle in bulk rotates with $\omega_0 = \omega_{\text{rot}}$.
Particle on surface avalance independent of wall motion, $\omega_0 = 0$

# 11. Static friction in a vibrated granular system

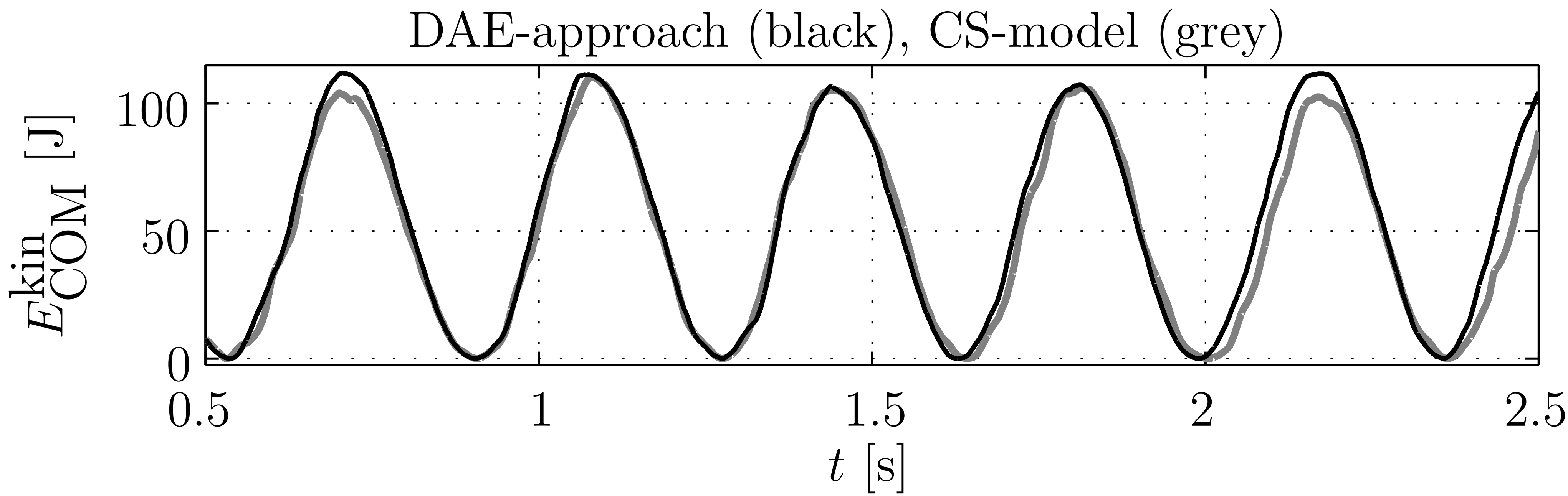
Setup:
Horizontally vibrated box, dimensionless acceleration $\Gamma = 0.75$,
1008 particles, 5-10 corners, regular polygons

☞ As long as the accelerations are not too large, interparticle forces are static coulomb friction



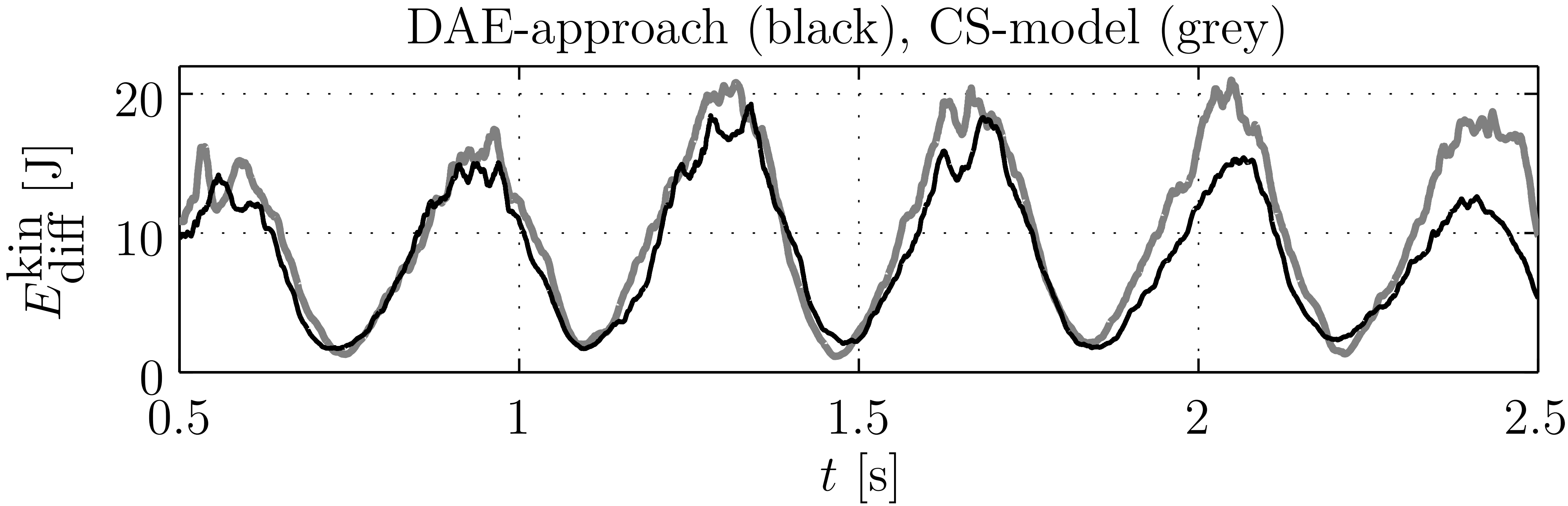
☞ Compare our DAE-approach with Cundall-Strack (CS) friction model (incrementation of tangential forces)

Total kinetic energy $E^{\text{kin}}_{\text{COM}} = \frac{1}{2}\left(\sum m_i\right) v^2_{\text{COM}}$: shows how fast friction grips and how large the portion of influenced particles is




DAE-approach – Granular assembly moves consistently as bulk
– Static friction dominates due to faster grip of friction
– Better reaction to external forces
– Bulk more coherent than for CS
(slower grip, under-/overcompensation)

Kinetic energy for velocity deviation from the centre of mass $E^{\text{kin}}_{\text{diff}} = \dfrac{1}{2}\sum m_i \left(v_{\text{COM}} - v_i\right)^2$: shows particle separation from bulk motion

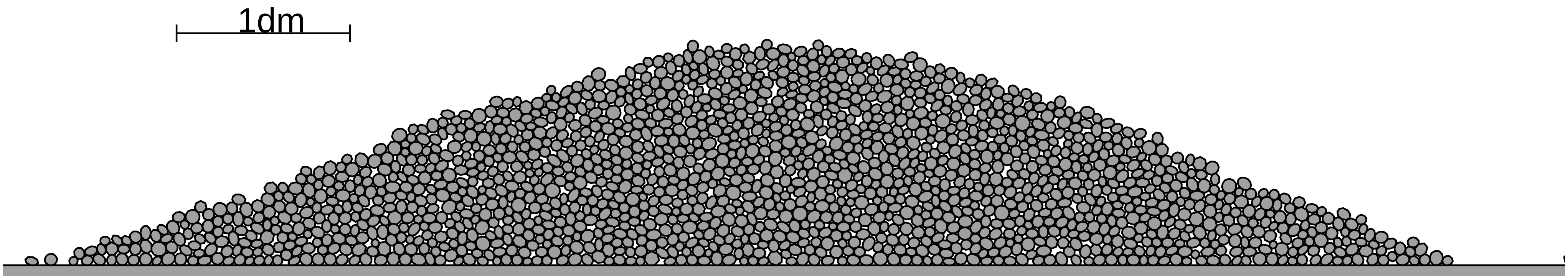



DAE-approach – Less independent motion for individual particles
– Particles more likely to stick with bulk

→ DAE-approach better suited for dynamical processes

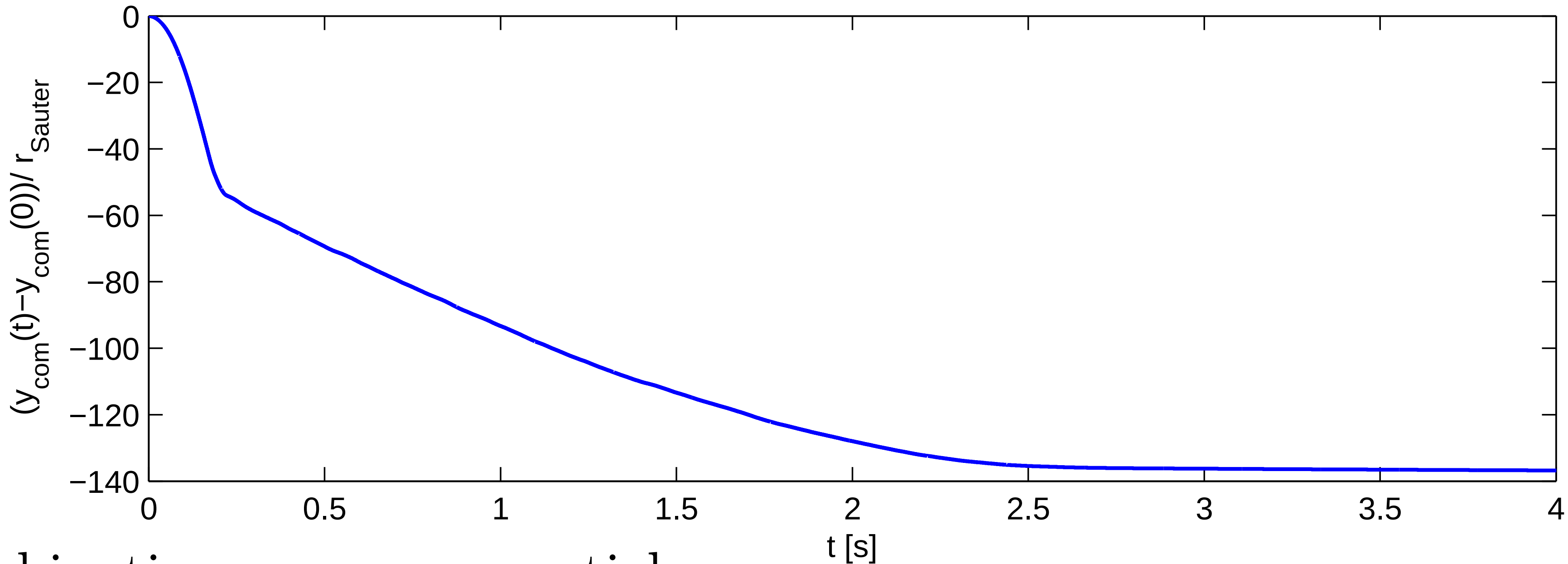
# 12. Static friction in a heap of particles

Setup: 1622 polygonal particles dropped from a hopper
Particles with 5-10 corners, 20% variation in size
Smooth floor, but finite friction ($\mu = 0.6$) between particles and particles-floor

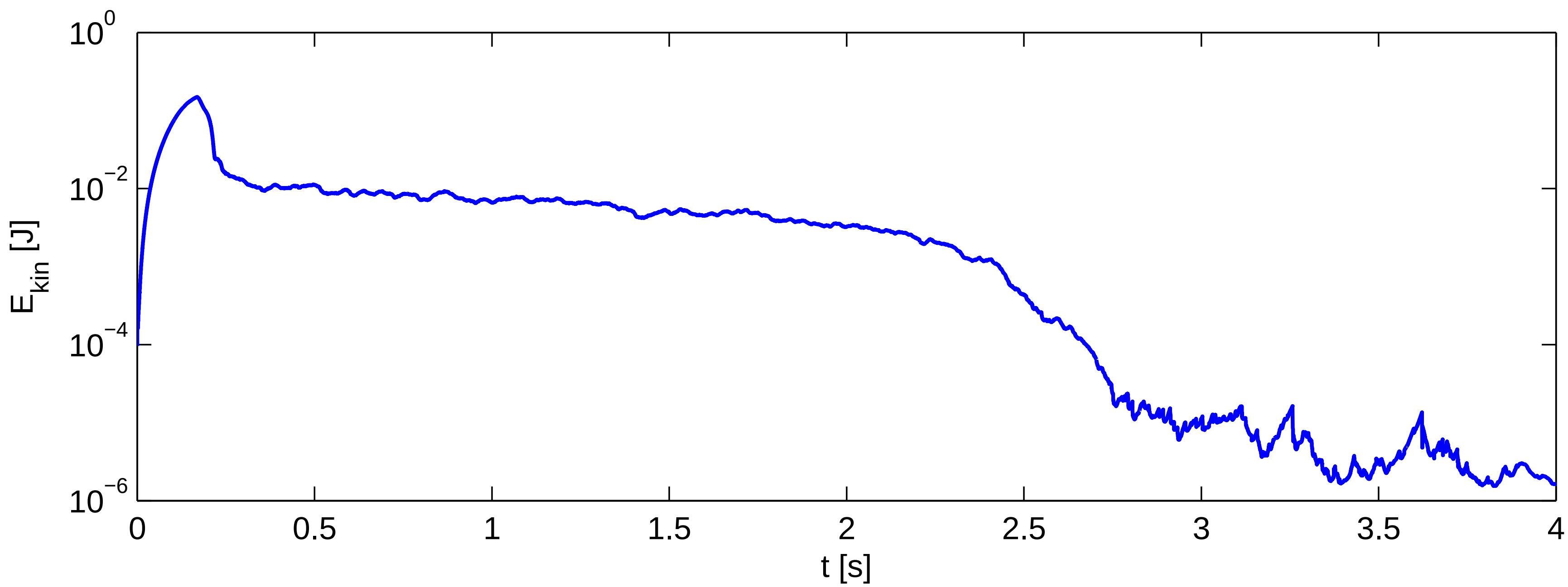



No sidewalls or artificial forces required to keep heap stable, correct angle of slopes
Slopes straight, except region where particles have been dropped down

Relative position for centre of mass with respect to $r_{\text{Sauter}} = \sqrt{\frac{A_{\text{mean}}}{\pi}}$

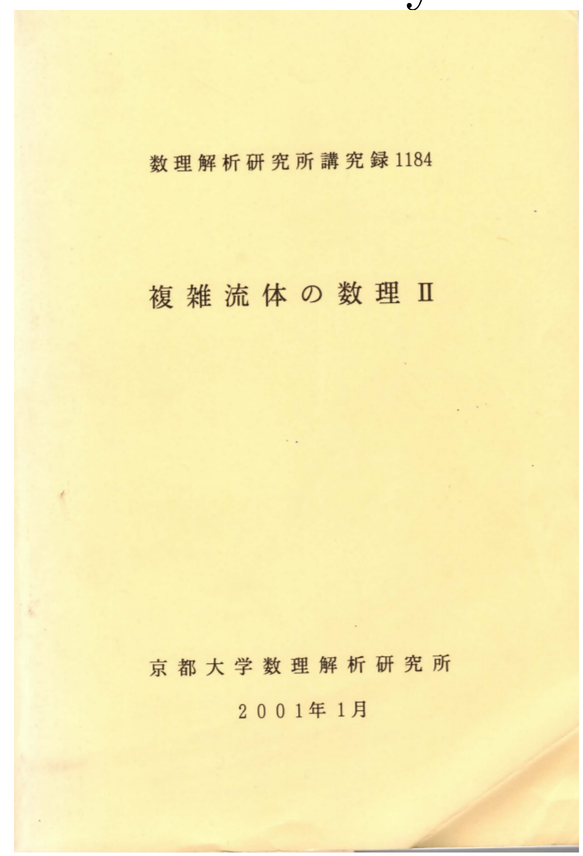



Mean kinetic energy per particle



# 13. Conclusion

– Computation of static friction based on constraints / Differential Algebraic Equations works
– Even though dry Coulomb friction at $v = 0$ is highly non-linear, no need to solve any non-linear equations
– No additional physical parameters required
– More accurate modeling of bulk motion in dynamic situations, especially under vibration
– Finite drift in static configurations problem of numerical stability
– Same treatment possible for other kinds of friction, but for rolling friction (coefficient too small) and pivoting friction (in 3D, surface roughness dependence)



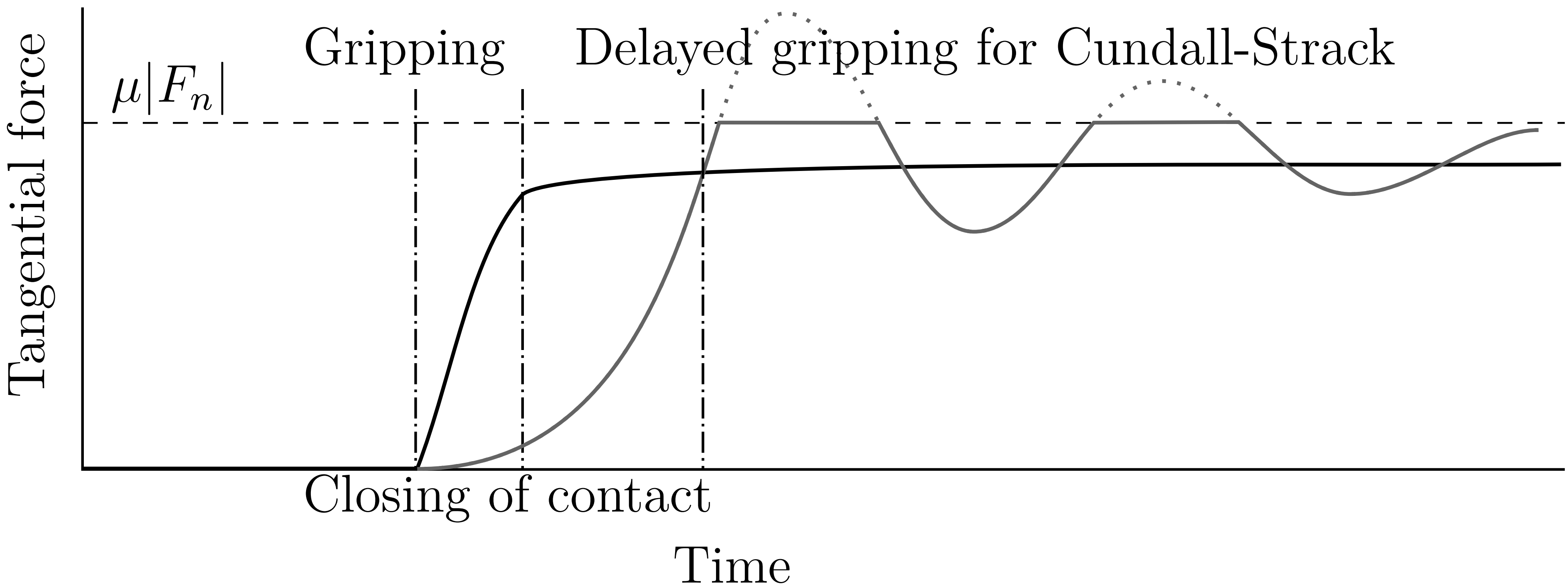

In 2001, 17 years ago, my supervisor used the Cundall-Strack model with $\mu$,$Y_t$, and $\gamma_t$ as parameters.
Prof. Hayakawa was very unhappy and asked: "Can you work with less parameters?"
☞ Now we can ☜

# A. Cundall-Strack model

– Commonly used model: “breaking tangential springs”

$$F^{\tan}(t) = F^{\tan}(t-\tau) - k^{\tan} v^{\tan} \cdot \tau,$$

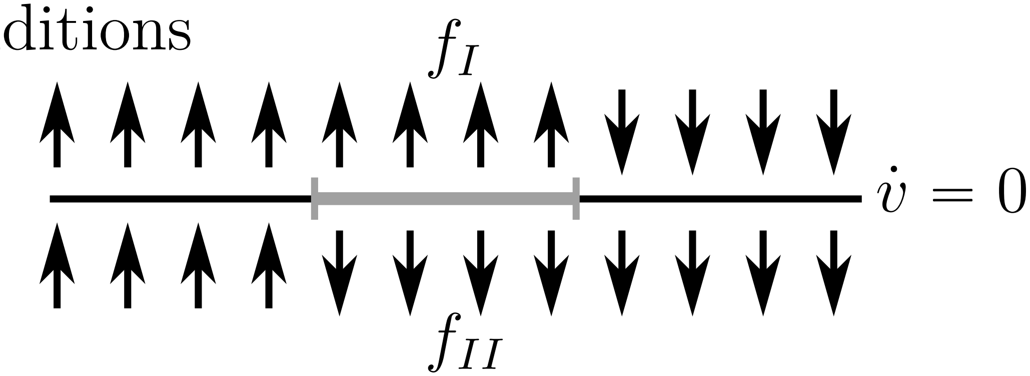
☺ Finite $F^{\tan}$ possible for $v = 0$
☹ Oscillatory behaviour
☹ Degree of freedom can store and release energy
☹ slower “grip” than real friction
☹ requires tracking of contact point



# B. Painlevé Paradoxon

– Rigid body problem apparently without any solution:
Friction force accelerates body, irrespective of choice of direction of friction
– Resulting from problematic assumption: Static friction must have magnitude $\mu|F_n|$ at all times
$\rightarrow$ Case of inconsistent initial conditions

Phaseflow is pulling on the constraint from both sides
Solution is not unique




– For $v = 0$ static Coulomb friction must exactly compensate external forces

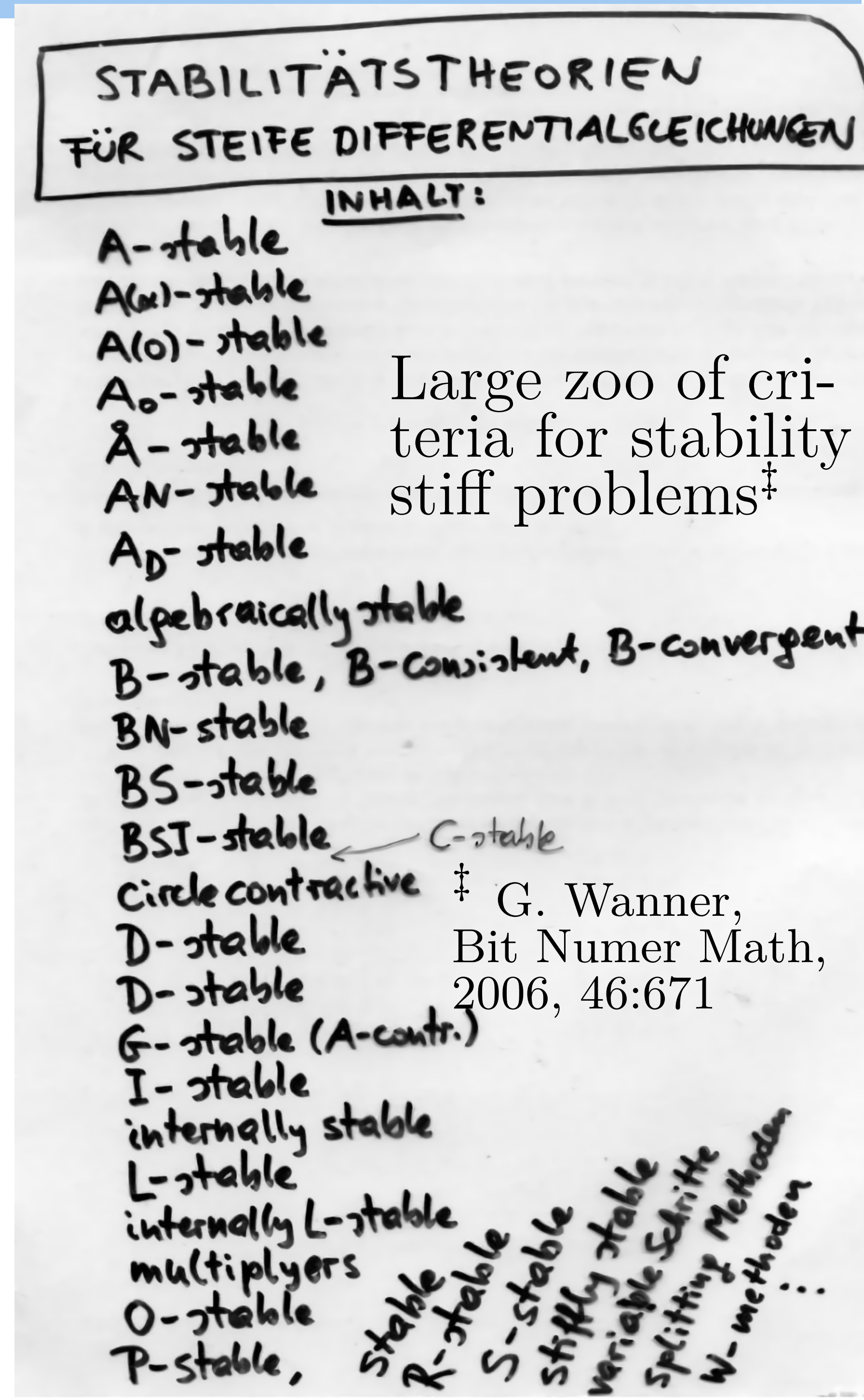
# C. Stiff Ordinary Differential Equations

Stiffness, tautological definition: "stiff solvers work than non-stiff solvers"
Equations per se not stiff, equations become stiff for specific initial values and parameter regions
Possible causes for stiffness:

☞ stability more important than accuracy
☞ multiple timescales in the problem
☞ large variations of the solution in small intervalls
☞ large variation in the eigenvalues of the Jacobian

More pragmatically: "Problems for which explicit methods don't work."† Less computational cost, no additional noise with stiff solvers→ equation is stiff
Details unimportant, in our case stability obtained by modeling

† E. Hairer, G. Wanner, Solving Ordinary Differential Equations II: Stiff and differential algebraic problems, Springer 1996, 2nd ed., p.2



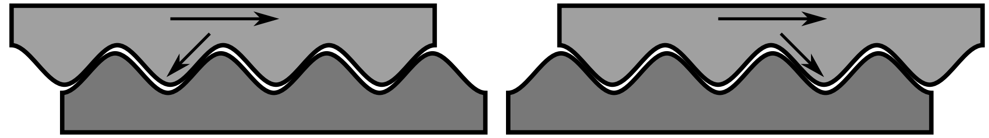
STABILITÄTSTHEORIEN
FÜR STEIFE DIFFERENTIALGLEICHUNGEN
INHALT:
A-stable
A(α)-stable
A(0)-stable
$A_0$-stable
Å-stable
AN-stable
$A_D$-stable
algebraically stable
B-stable, B-consistent, B-convergent
BN-stable
BS-stable
BSI-stable ← C-stable
Circle contractive
D-stable
D-stable
G-stable (A-contr.)
I-stable
internally stable
L-stable
internally L-stable
multiplyers
O-stable
P-stable, stable, R-stable, S-stable, stiffly stable, variable Schritte, Splitting Methoden, W-methoden ...

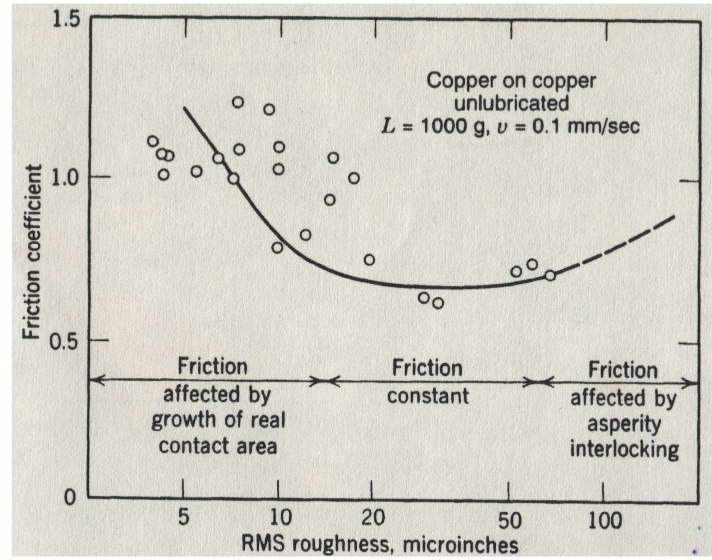
Large zoo of criteria for stability stiff problems‡

‡ G. Wanner, Bit Numer Math, 2006, 46:671

# D. Friction is not Roughness

– Friction dependent on absolute contact area, not surface roughness
– Roughnes *can* increase contact area






– Atomically smooth surfaces (split mica) → high friction forces
(E. Rabinowicz, Friction and Wear of Materials, Wiley, New York, 1965)

– Possible to build heap on polished mirror surfaces