First-passage properties of persistent random walks/ run-and-tumble particles

Grégory Schehr
Laboratoire de Physique Théorique et Hautes Energies,
CNRS - Sorbonne Université

Japan-France joint seminar
Kyoto, 13-16 Dec. 2023

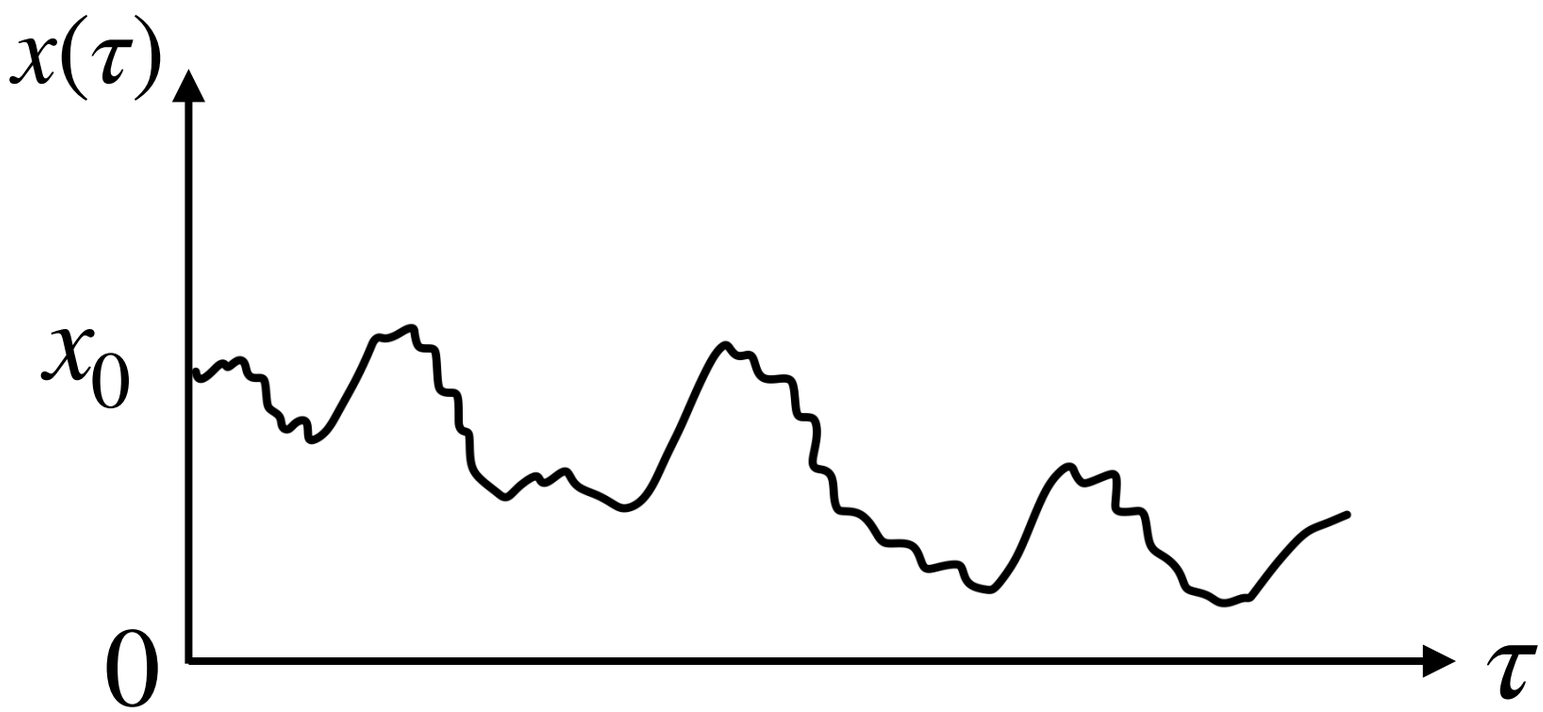
in collaboration with

- Pierre Le Doussal (LPENS, Paris)
- Satya N. Majumdar (LPTMS, Orsay)
- Francesco Mori (Univ. Oxford)

Phys. Rev. Lett. **124**, 090603 (2020), Phys. Rev. E **102**, 042133 (2020)

Persistence/Survival probability

- One-dimensional continuous-time stochastic process $x(\tau)$



- Persistence or survival probability

$$S(x_0, t) = \text{Prob}\,(x(\tau) > 0\,,\, \forall \tau \in [0,t]\,|\,x(0) = x_0 > 0)$$

A classical (and difficult!) question in the theory of stochastic processes

Persistence/Survival probability $S(X_0, t)$

- It is easy to compute for continuous time Markov processes

For 1d-Brownian motion with diffusion constant D

$$S(X_0, t) = \mathrm{erf}\left(\frac{X_0}{\sqrt{4Dt}}\right) \quad \text{where} \quad \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\,dx$$

$$\underset{t\to\infty}{\sim} \frac{X_0}{\sqrt{\pi D t}}$$

- Much harder for non-Markov processes: it has generated enormous activities in maths and in stat. mech. over the last decades

A. J. Bray, S. N. Majumdar, G. S., Adv. Phys. **62**, 225 (2013)

F. Aurzada, T. Simon, Lévy matters **V**, 185, (Springer, 2015)

- This talk: exact results for the persistence in a class of non Markov processes, namely d-dimensional persistent random walks/ run-and-tumble processes

$\Longrightarrow$ Exact and suprisingly universal results !

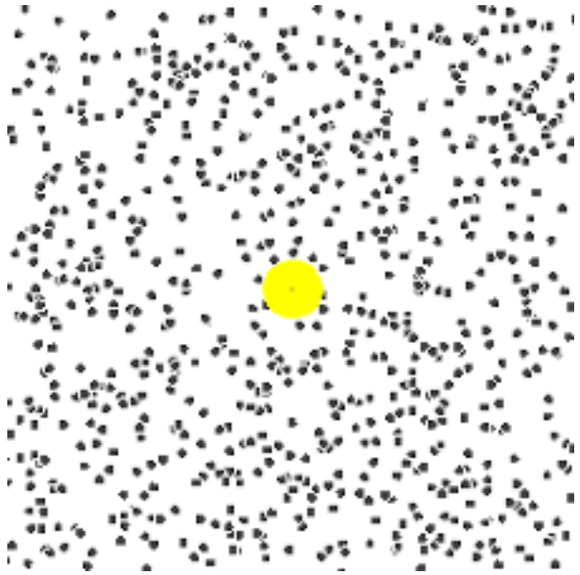
Outline

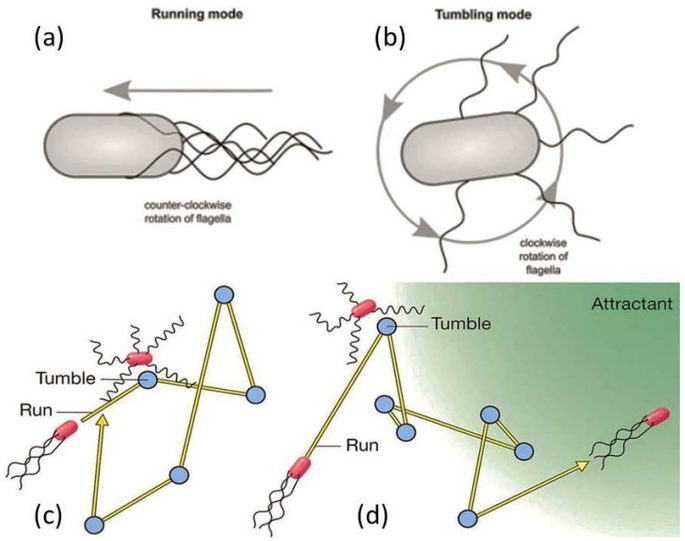
- Run-and-tumble particle (RTP): a model of active matter
- A first stage with the Sparre Andersen theorem
- From the Sparre Andersen theo. to the survival proba. of and RTP
- Conclusion

Outline

- Run-and-tumble particle (RTP): a model of active matter
- A first stage with the Sparre Andersen theorem
- From the Sparre Andersen theo. to the survival proba. of and RTP
- Conclusion

Passive vs active particles





- Passive BM: random motion due to collisions with other molecules
- Active particle: the particle absorbs energy directly from the environment $\Longrightarrow$ a ballistic motion (Run) with a constant velocity $\vec{\mathbf{v}}$ during an exponentially distributed random time with mean γ^{-1} (persistence time), followed by a local reorientation of the velocity (Tumble)... another run...

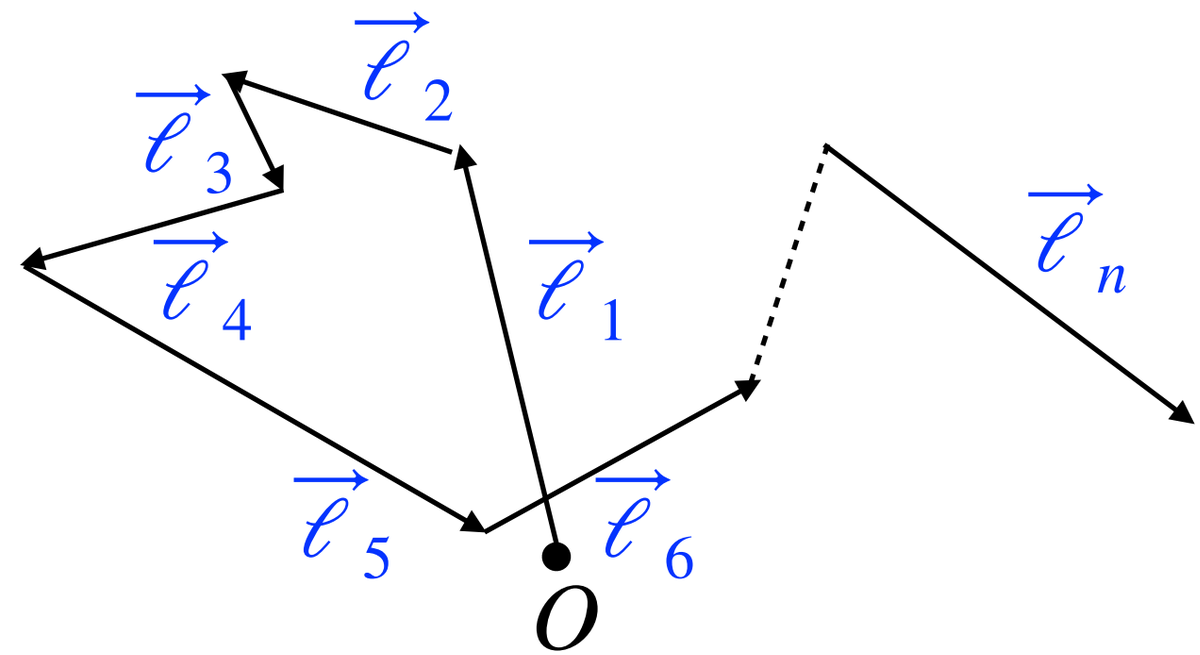
Ex: widely used to model dynamics of living matter, like E. Coli

Berg (2004), Tailleur and Cates (2008), ...

Run and tumble particle in d dimensions: the model

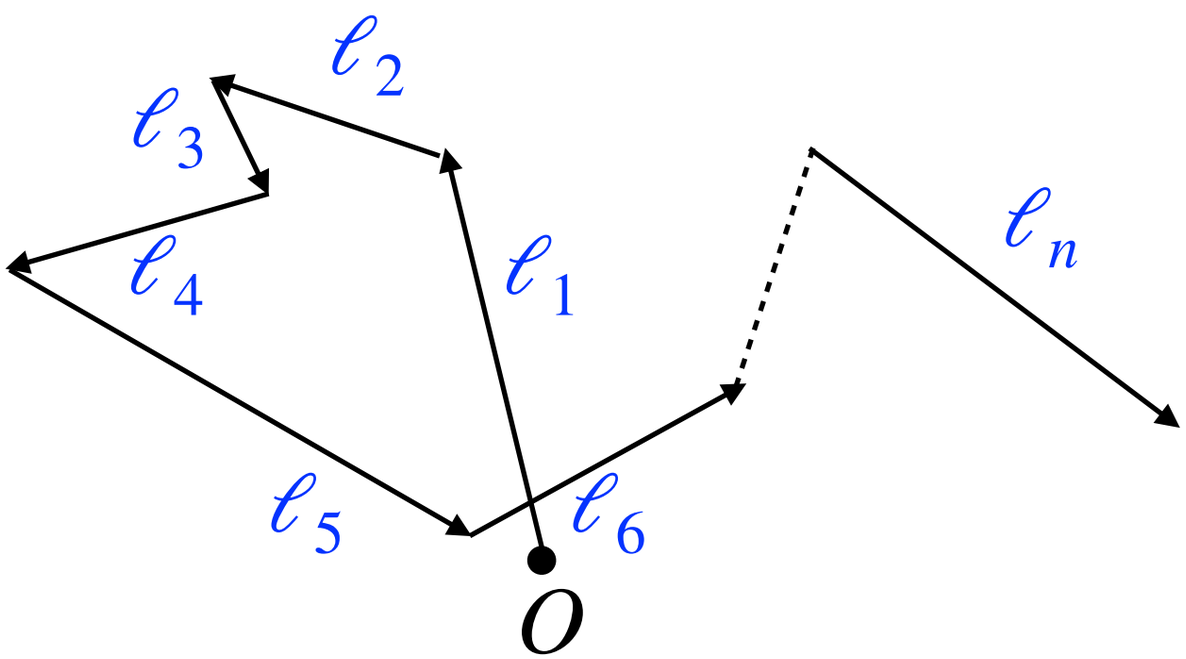
persistence time: γ^{-1}

run lengths: $\ell_i = |\vec{\mathbf{v}}_\mathbf{i}|\,\tau_i$



- The particle, starting from the origin, chooses a random velocity $\vec{\mathbf{v}}_1$ from a distribution $W(\vec{\mathbf{v}})$ and runs ballistically during a random run-time τ_1 drawn (independently) from an exponential distribution $\tau_1 \sim \mathrm{Exp}(\gamma)$
- At the end of the run, the particle tumbles instantaneously, chooses a new velocity $\vec{\mathbf{v}}_2$ from the same distribution $W(\vec{\mathbf{v}})$ (independently of $\vec{\mathbf{v}}_1$) and runs ballistically during a random run-time $\tau_2 \sim \mathrm{Exp}(\gamma)$ also independently of τ_1
- ...

Run and tumble particle in d dimensions: the model



- The time scale is set by γ^{-1}
- Two ``parameters": d and $W(\vec{\mathbf{v}})$
- The special choice:

$$W(\vec{\mathbf{v}}) = \frac{1}{S_d v_0^{d-1}} \delta(|\vec{\mathbf{v}}| - v_0) \quad , \quad v_0 > 0$$

is the standard RTP or persistent random walk

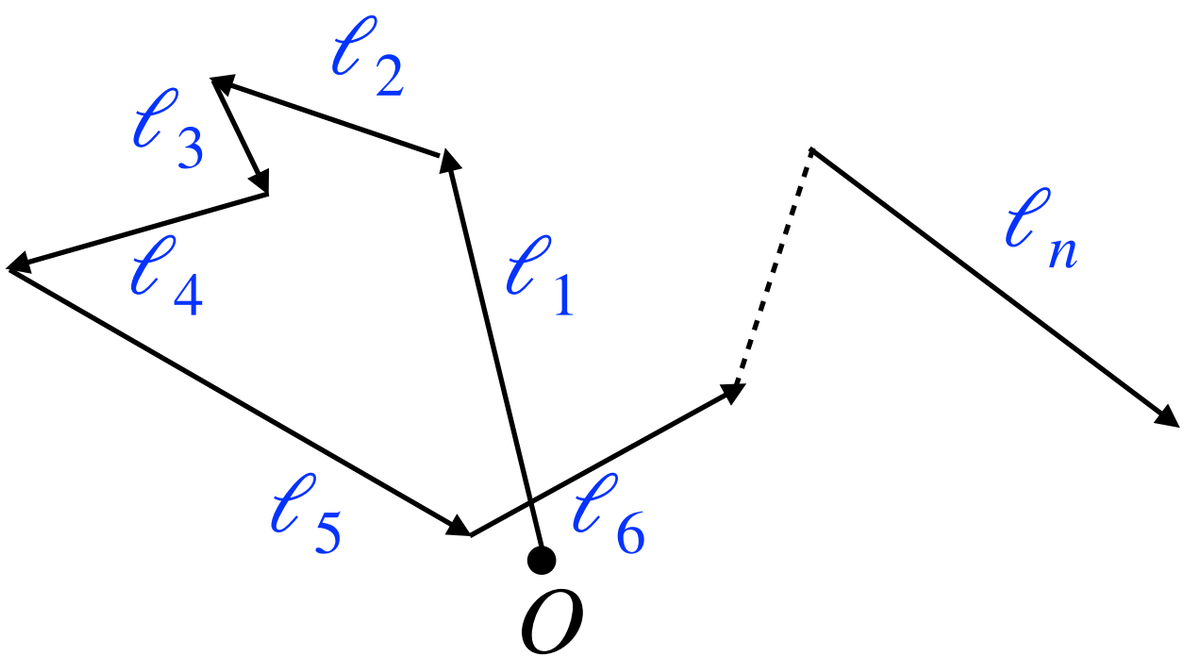
- The persistent random walk has already a long story

▸ R. Fürth (1920) ``The Brownian motion when considering persistence of the direction of movement. With applications to the movement of living infusoria"

▸ M. Kac (1974), ``A stochastic model related to the telegrapher's equation"

▸ see also R. P. Feynman (1965), ``Relativistic chessboard model"

Run and tumble particle in d dimensions: the model



- The time scale is set by γ^{-1}
- Two ``parameters": d and $W(\vec{\mathbf{v}})$
- The special choice:

$$W(\vec{\mathbf{v}}) = \frac{1}{S_d v_0^{d-1}} \delta(|\vec{\mathbf{v}}| - v_0) \quad , \quad v_0 > 0$$

is the standard RTP or persistent random walk

- Several properties, like the proba. distribution at time t, are well known

e. g., K. Martens, L. Angelani, R. Di Leonardo, L. Bocquet '12

- However, the survival probability was only known for

$$d = 1 \quad \text{and} \quad W(v) = \frac{1}{2}\delta(v - v_0) + \frac{1}{2}\delta(v + v_0)$$

Orsingher '95, Weiss '02,..., Angelani et al. '14, Artuso et al. '14, Malakar et al. '18, Evans, Majumdar '18, Le Doussal, Majumdar, G. S. '19

Survival probability in $d = 1$ and constant speed v_0

$$\frac{dX}{dt} = v_0\,\sigma(t) \quad , \quad \begin{cases} X(0) = X_0 \\ \sigma(0) = \pm 1 \quad \text{w. proba} \quad 1/2 \end{cases}$$

- Exact solution via coupled backward Fokker-Planck equations

- The survival probability $S(t) = S(X_0 = 0, t)$ starting from the origin reads

$$\boxed{S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0\left(\frac{\gamma t}{2}\right) + I_1\left(\frac{\gamma t}{2}\right) \right)}$$

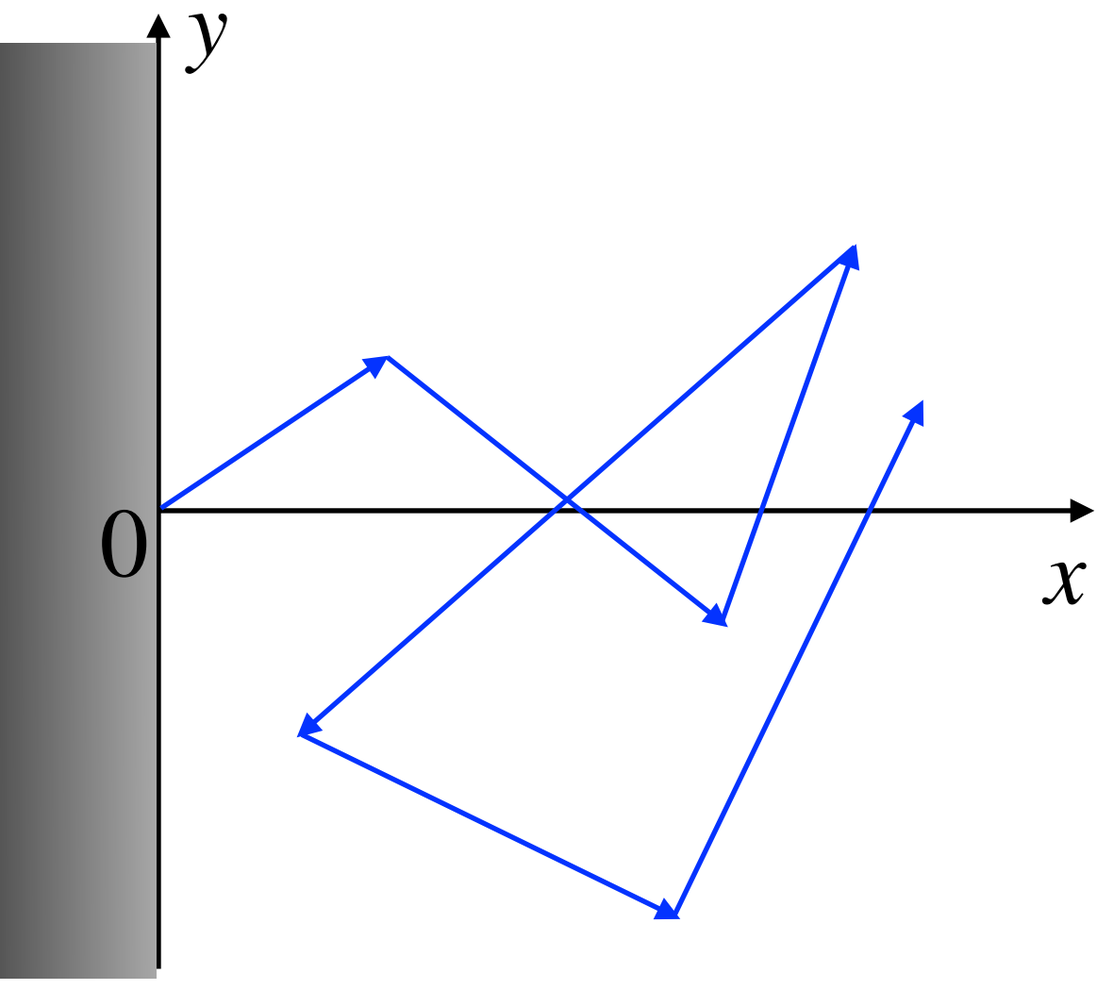
modified Bessel functions

Orsingher '95, Weiss '02,..., Angelani et al. '14, Artuso et al. '14, Malakar et al. '18, Evans, Majumdar '18, Le Doussal, Majumdar, G. S. '19

- Algebraic decay for $t \gg \gamma^{-1}$, $S(t) \sim 1/\sqrt{\pi \gamma t}$

- How to compute $S(t)$ for $d > 1$? Much more difficult because the different components of $\vec{\mathbf{X}}(t)$ get coupled (unlike Brownian motion)...

A simple question for the d-dimensional RTP model

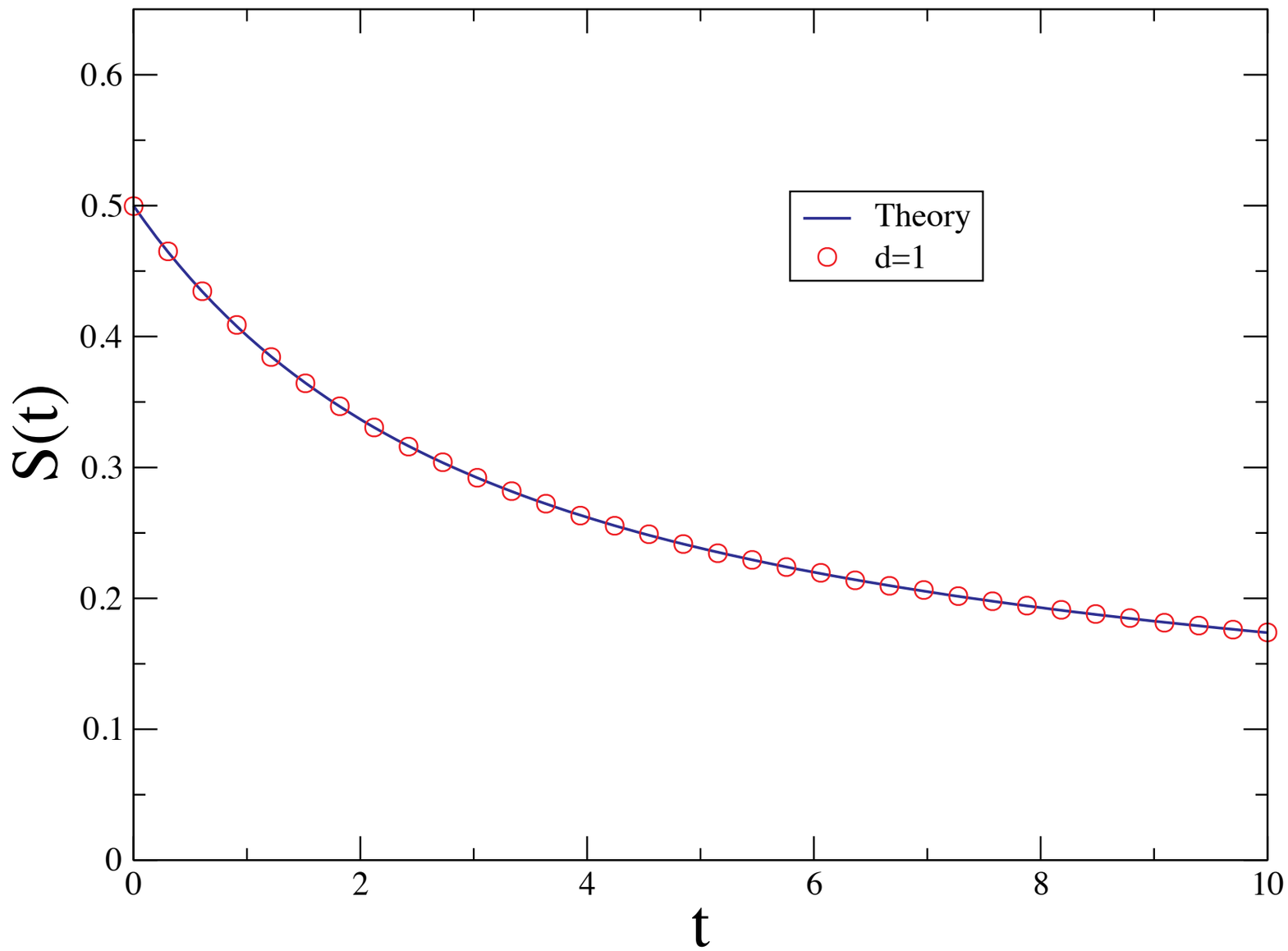


- The RTP starts from the origin at $t = 0$
- Two parameters: d and $W(\vec{\mathbf{v}}) = W(-\vec{\mathbf{v}})$

$S(t) =$ proba. that the $x-$component of the RTP's position does not become negative up to time t, i.e., the proba. that the RTP does not cross the hyperplane $x = 0$ up to t

Q: how does $S(t)$ depend on the dimension d and $W(\vec{\mathbf{v}})$?

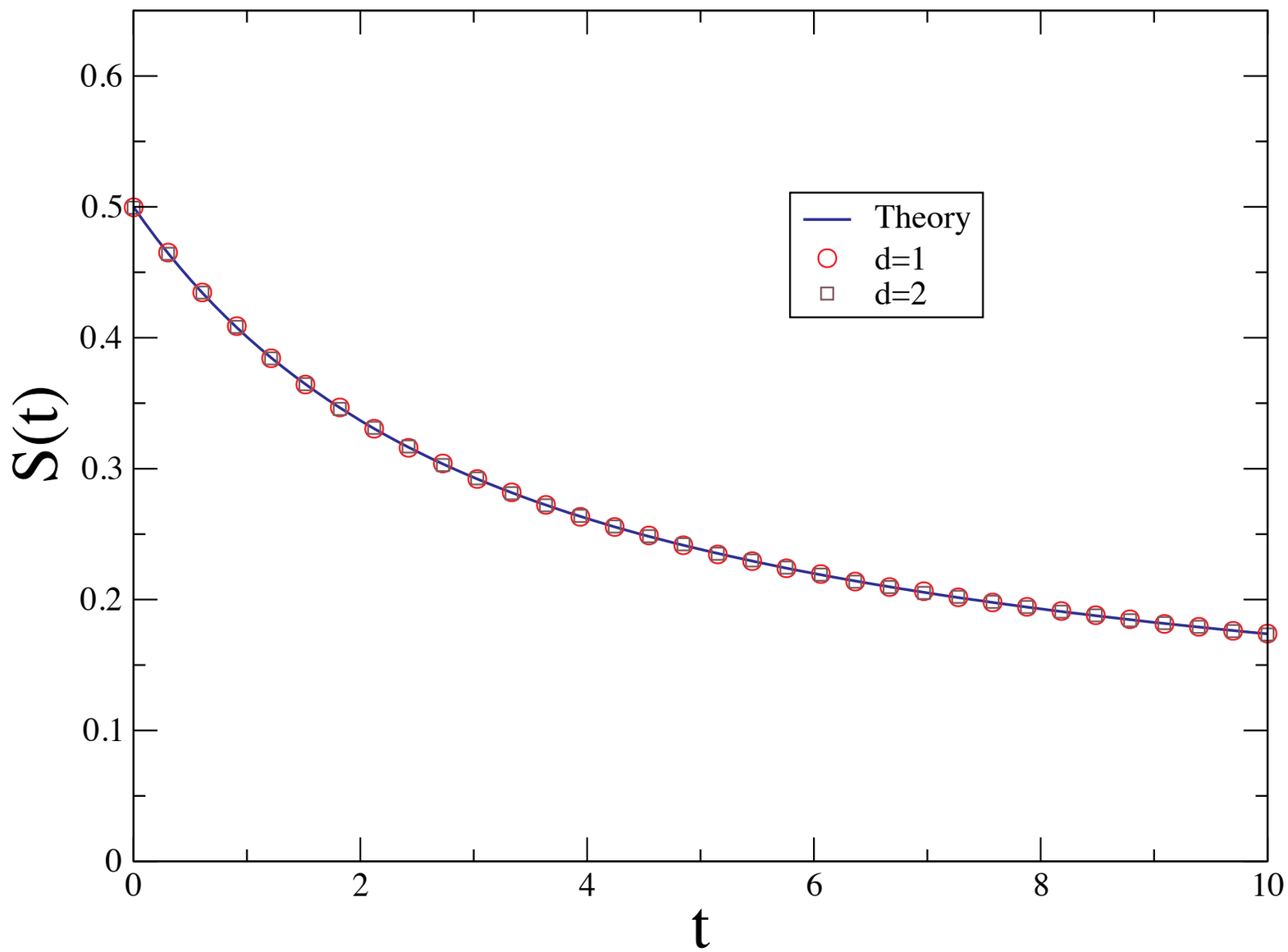
Survival probability $S(t)$ vs t in $d > 1$: start with numerics



Theory: $S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0 \left(\frac{\gamma t}{2} \right) + I_1 \left(\frac{\gamma t}{2} \right) \right)$

Velocity distribution: $W(v) = \frac{1}{2} \delta(v - v_0) + \frac{1}{2} \delta(v + v_0)$

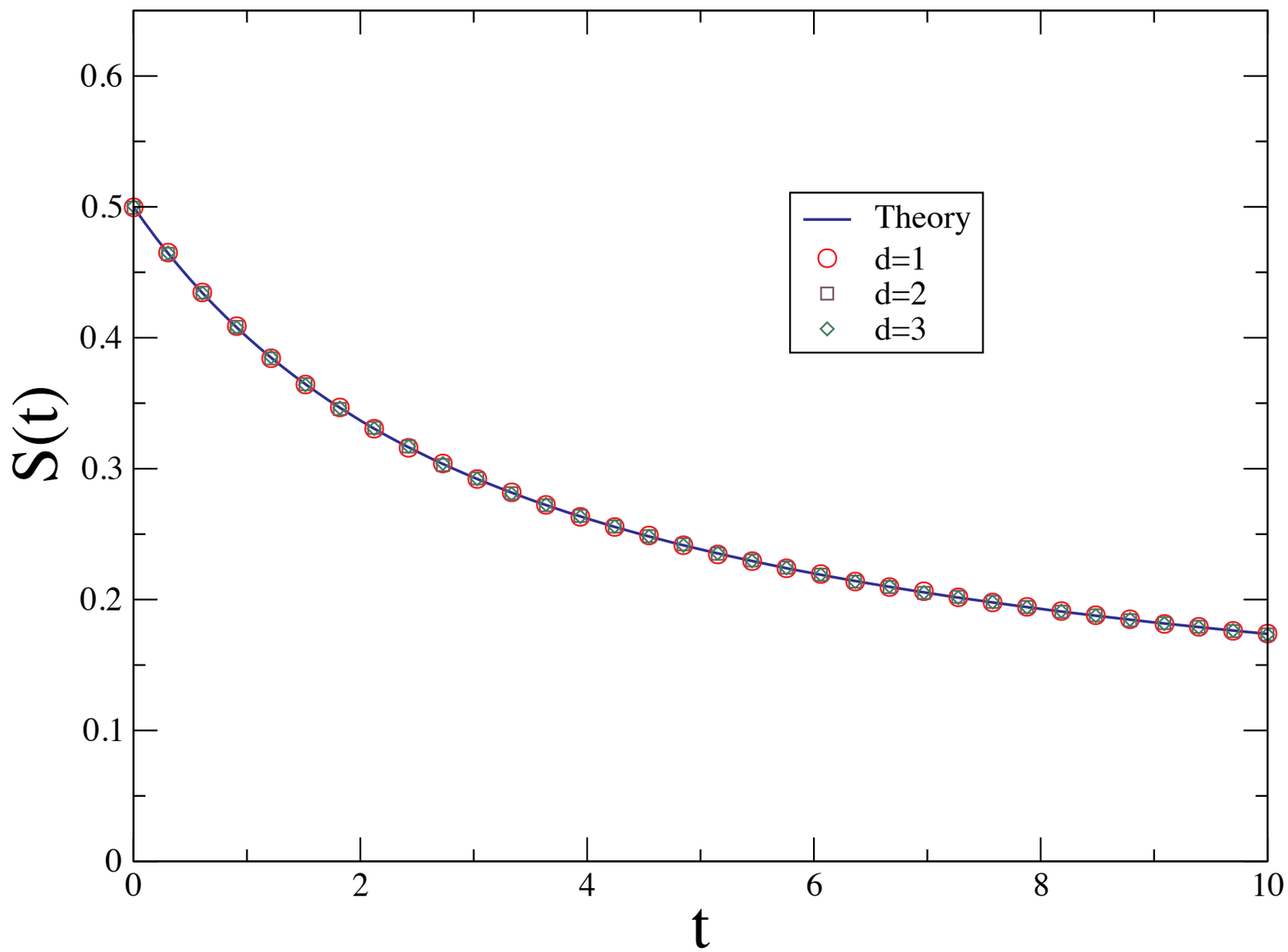
Survival probability $S(t)$ vs t in $d > 1$: start with numerics



Theory: $S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0 \left(\frac{\gamma t}{2} \right) + I_1 \left(\frac{\gamma t}{2} \right) \right)$

Isotropic velocity distribution: $W(\vec{\mathbf{v}}) \propto \delta(|\vec{\mathbf{v}}| - v_0)$ in $d = 2$

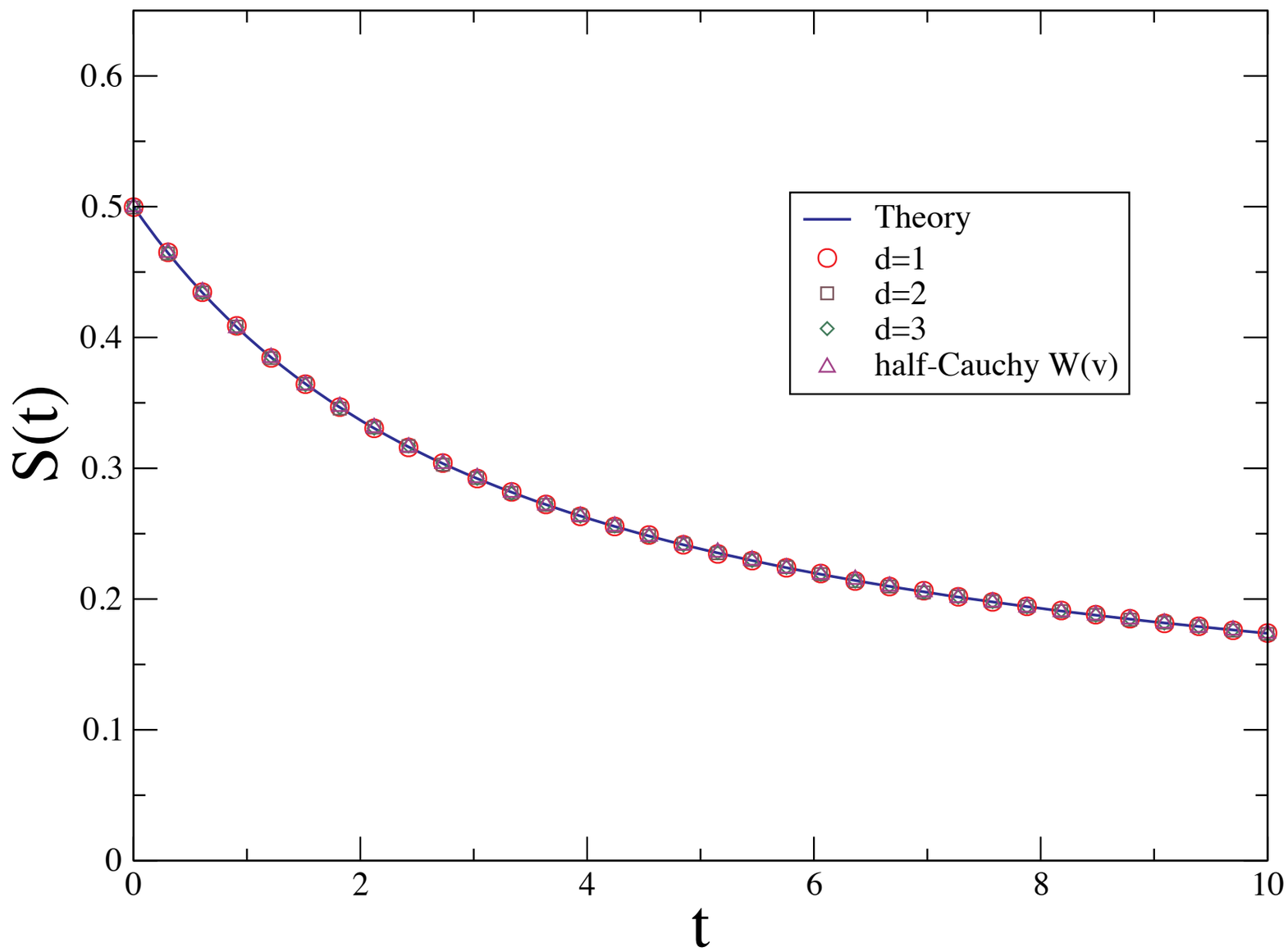
Survival probability $S(t)$ vs t in $d > 1$: start with numerics



Theory: $S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0 \left(\frac{\gamma t}{2} \right) + I_1 \left(\frac{\gamma t}{2} \right) \right)$

Isotropic velocity distribution: $W(\vec{\mathbf{v}}) \propto \delta(|\vec{\mathbf{v}}| - v_0)$ in $d = 3$

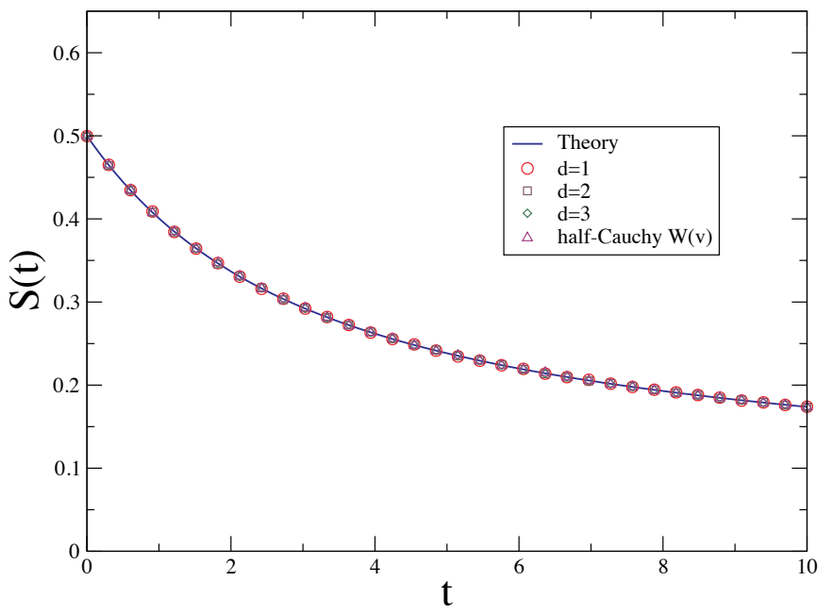
Survival probability $S(t)$ vs t in $d > 1$: start with numerics



Theory: $S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0 \left(\frac{\gamma t}{2} \right) + I_1 \left(\frac{\gamma t}{2} \right) \right)$

Isotropic velocity distribution: $W(\vec{\mathbf{v}}) \propto \frac{\theta(|\vec{\mathbf{v}}|)}{1 + \mathbf{v}^2}$ in $d = 2$

Survival probability $S(t)$ vs t: a universal behavior



$$S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0\left(\frac{\gamma t}{2}\right) + I_1\left(\frac{\gamma t}{2}\right) \right)$$

→ This suggests that this result for is universal for all time t (and not just for large t)

→ $S(t)$ is independent of the dimension d and the symmetric velocity distribution $W(\vec{\mathbf{v}})$

F. Mori, P. Le Doussal, S. N. Majumdar, G. S. PRL (2020)

→ This is a consequence of the Sparre Andersen theorem

Outline

- Run-and-tumble particle (RTP): a model of active matter
- A first stage with the Sparre Andersen theorem
- From the Sparre Andersen theo. to the survival proba. of an RTP
- Conclusion

Sparre Andersen theorem for 1d random walk

- Random walk in dimension d=1

  initial position : $X_0 = 0$

  Markov dynamics : $X_k = X_{k-1} + \eta_k \quad , \quad k \geq 1$

  i.i.d. random variables with a continuous and symmetric distribution $p(\eta)$

- Note that $p(\eta)$ is arbitrary and includes Lévy flights, i.e.,

$$p(\eta) \underset{\eta \to \pm\infty}{\propto} |\eta|^{-\mu-1} \, , \, 0 < \mu < 2$$

Sparre Andersen theorem for 1d random walk

- Survival probability, starting from the origin $X_0 = 0$

$$q(n) = \text{Prob}\,(X_1 \geq 0, X_2 \geq 0, \cdots, X_n \geq 0 \,|\, X_0 = 0)$$

- Sparre Andersen theorem (1954)

$$\boxed{q(n) = \frac{1}{2^{2n}} \binom{2n}{n}}$$

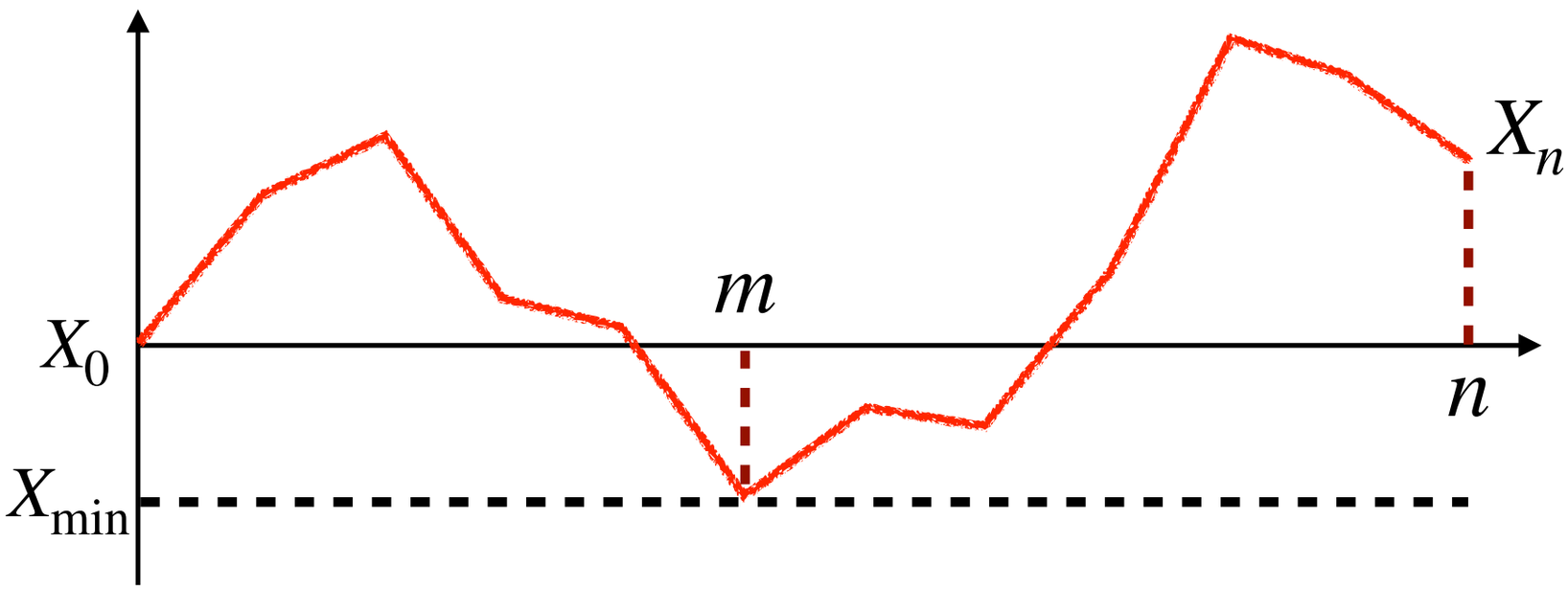
holds for any continuous and symmetric jump distribution $p(\eta)$

- Its generating function is thus given by

$$\boxed{\tilde{q}(z) = \sum_{n \geq 0} q(n)\, z^n = \frac{1}{\sqrt{1-z}}}$$

A simple proof of the Sparre Andersen theorem

Ph. Mounaix, S. N. Majumdar, G. S., J. Phys. A (2020)



$$X_0 = 0$$

$$X_k = X_{k-1} + \eta_k$$

- Consider the time of the minimum $t_{\min}$

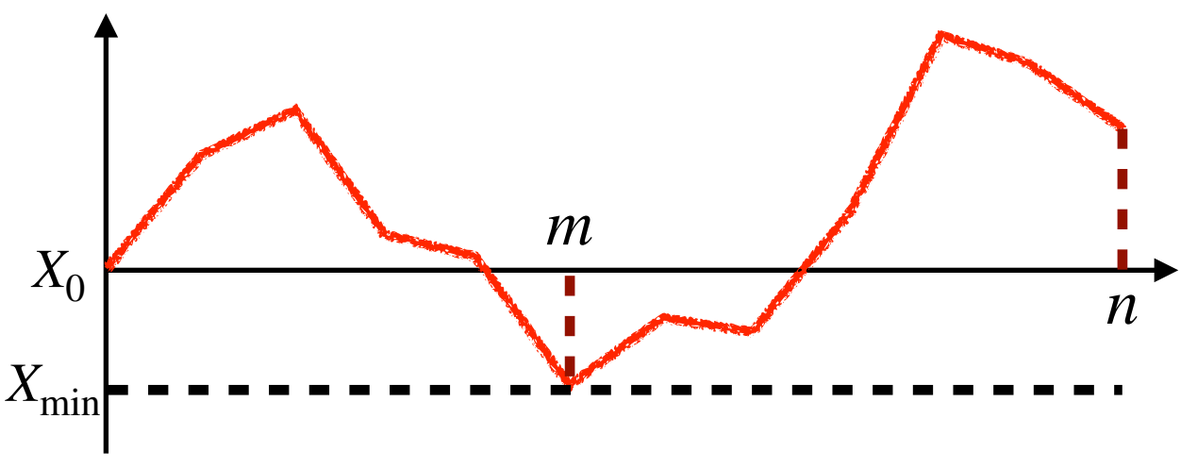
$$\text{``}t_{\min} = m\text{''} \quad \Longleftrightarrow \quad \text{``}X_m = X_{\min} = \min\{X_0, X_1, \cdots, X_n\}\text{''}$$

- Probability distribution of the minimum $t_{\min}$

$$P_n(m) = \text{Prob}(t_{\min} = m) = q(m)\underbrace{q(n-m)}_{\text{survival proba. up to step } n-m} \quad , \quad 0 \leq m \leq n$$

A simple proof of the Sparre Andersen theorem

Ph. Mounaix, S. N. Majumdar, G. S., J. Phys. A (2020)



$$X_0 = 0$$

$$X_k = X_{k-1} + \eta_k$$

- Probability distribution of the minimum $t_{\min}$

$$P_n(m) = \text{Prob}(t_{\min} = m) = q(m)q(n-m) \quad , \quad 0 \leq m \leq n$$

- Normalization condition imposes

$$\sum_{m=0}^{n} P_n(m) = 1 \Longleftrightarrow \sum_{m=0}^{n} q(m)q(n-m) = 1$$

- Taking the generating function w.r.t. n

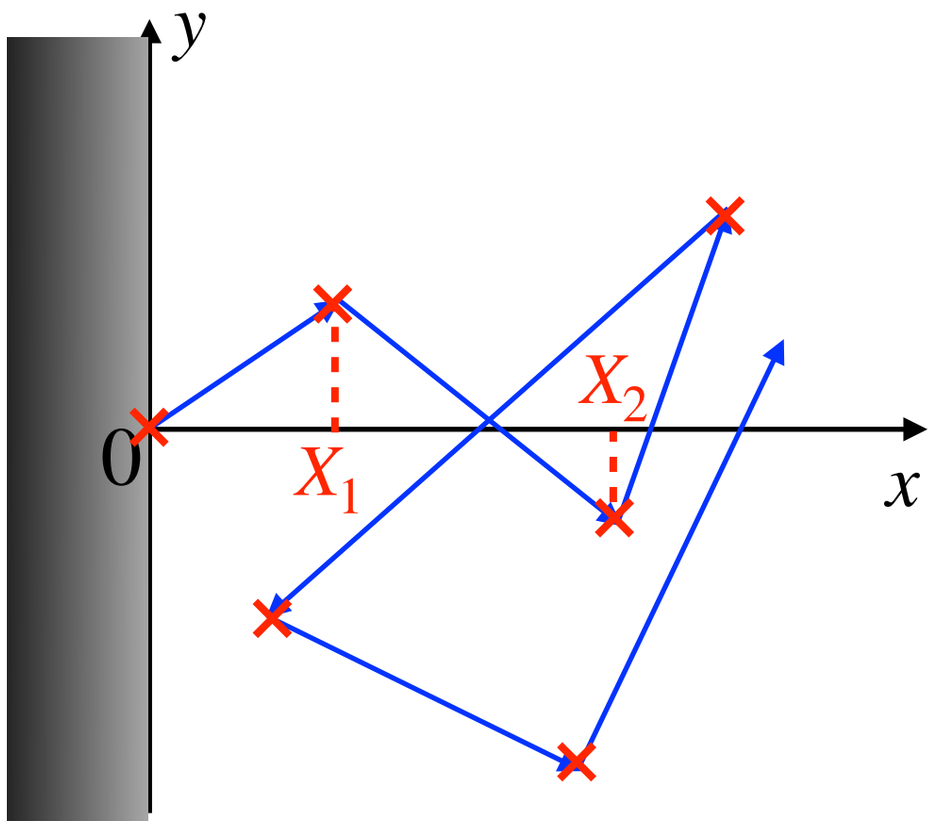
$$\tilde{q}(z) = \sum_{m \geq 0} z^n q(n)$$

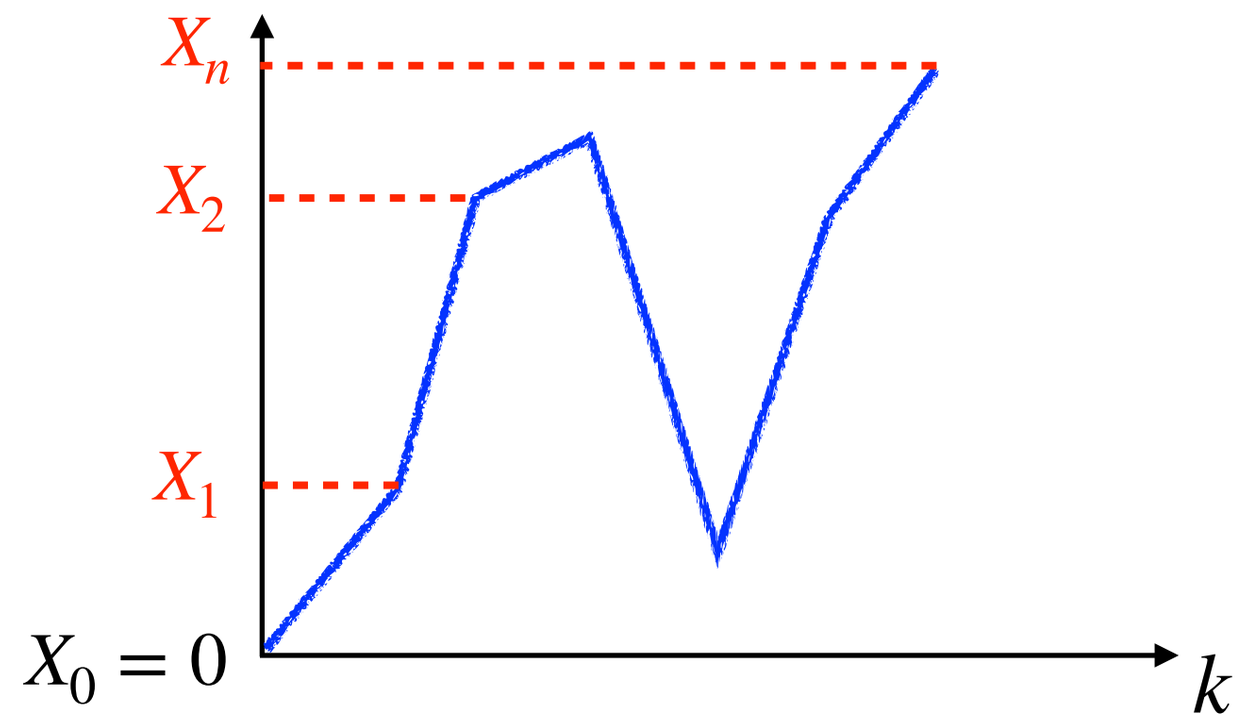
$$\tilde{q}(z)^2 = \frac{1}{1-z} \Longrightarrow \tilde{q}(z) = \frac{1}{\sqrt{1-z}}$$

Outline

- Run-and-tumble particle (RTP): a model of active matter
- A first stage with the Sparre Andersen theorem
- From the Sparre Andersen theo. to the survival proba. of and RTP
- Conclusion

Step 1: dynamics of the x-component

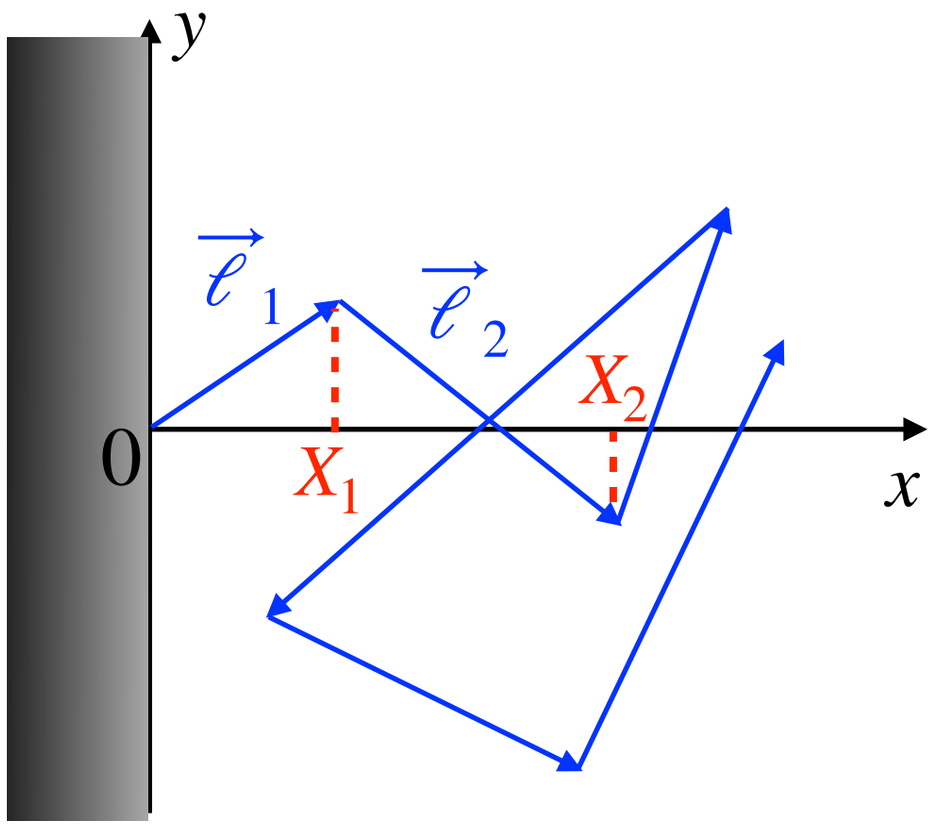


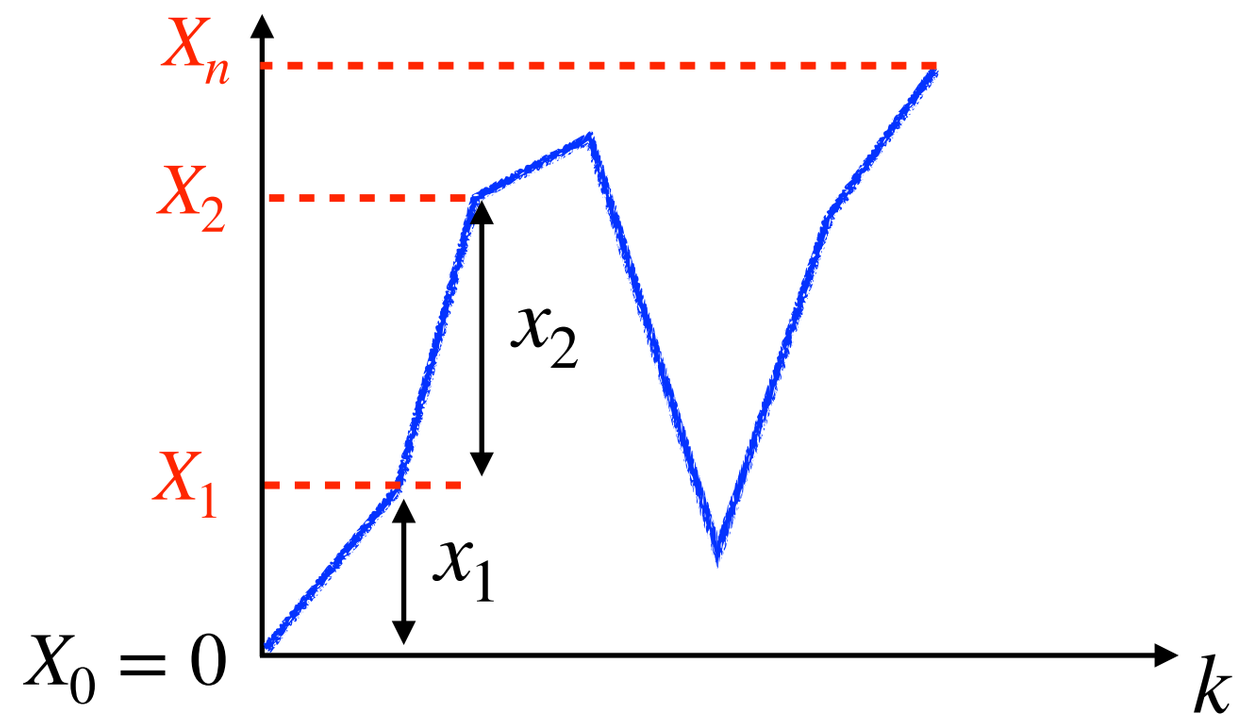


- Let $X(\tau)$ denote the x-component of the RTP at time τ
- X_k: the x-component of the RTP at the instant of the $(k+1)^{\text{th}}$ tumbling
- The nber of tumblings $N_T(t)$ on a fixed time interval $[0, t]$ is a random variable
- Survival proba. $S(t) = \text{Prob}[X(\tau) \geq 0\ , \forall \tau \in [0,t] \mid X(0) = 0]$

$$S(t) = \sum_{n=1}^{\infty} \boxed{\text{Prob}[X_1 \geq 0,\ X_2 \geq 0,\ \cdots, X_n \geq 0, N_T(t) = n \,|\, X_0 = 0]}$$

Step 1: dynamics of the x-component





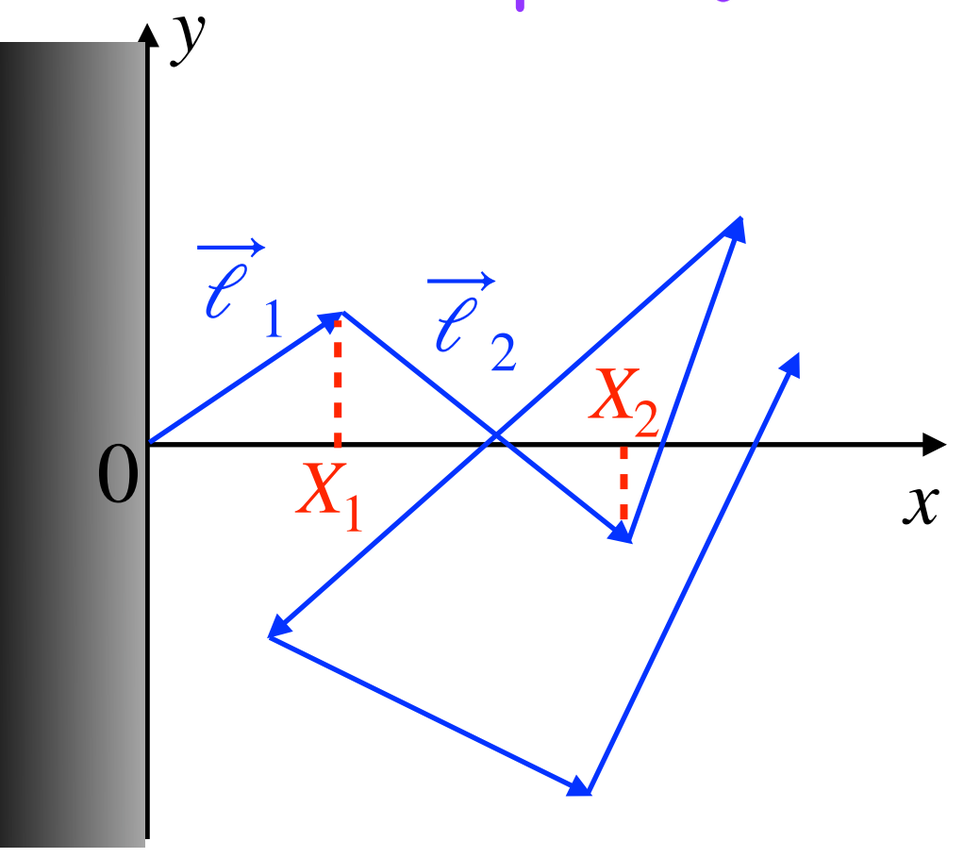
- Recall that the run lengths are given by $\ell_i = |\vec{\mathbf{v}_\mathbf{i}}|\tau_i$

independent random variables

- To compute $\text{Prob}[X_1 \geq 0, X_2 \geq 0, \cdots, X_n \geq 0, N_T(t) = n \,|\, X_0 = 0]$ we need the joint distribution of $\{\vec{\mathbf{v}_\mathbf{i}}\}_{1 \leq i \leq n}$, $\{\tau_i\}_{1 \leq i \leq n}$ & $N_T(t)$

$$\prod_{i=1}^{n} W(\vec{\mathbf{v}_\mathbf{i}})$$

Step 2: joint distribution of $\{\tau_i\}_{1\leq i\leq n}$ & $N_T(t)$



- Duration of the ith run: τ_i
- Let $\{\tau_1, \tau_2, \cdots, \tau_n\}$ be a realisation with $N_T(t) = n$ runs
- Note that the last run τ_n is unfinished and is thus different from the other run times

- $P(\{\tau_i\}_{1\leq i\leq n}, n\,|\,t)$: proba weight of a « configuration » $\{\tau_1, \tau_2, \cdots, \tau_n\}$ & $N_T(t) = n$

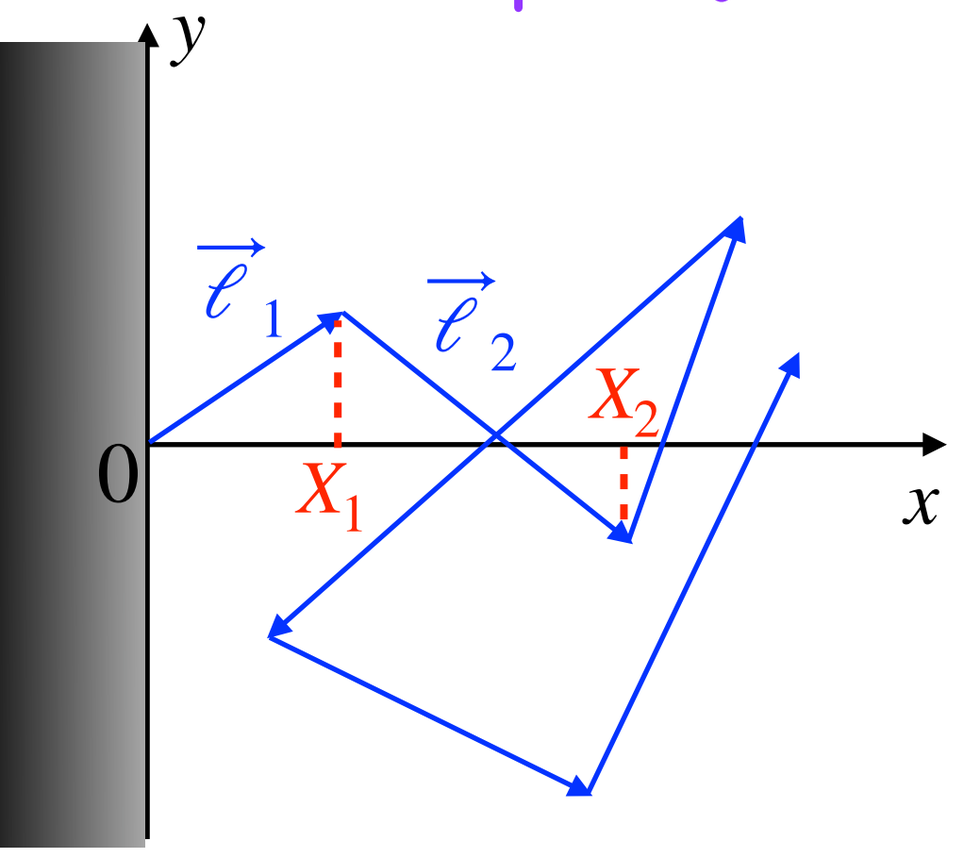
$$P(\{\tau_i\}_{1\leq i\leq n}, n\,|\,t) = \left[\prod_{i=1}^{n-1} p(\tau_i)\right] \int_{\tau_n}^{\infty} p(\tau)\, d\tau\, \delta\left(\sum_{i=1}^{n} \tau_i - t\right), \qquad p(\tau) = \gamma e^{-\gamma\tau}$$

$$= \frac{1}{\gamma}\left[\prod_{i=1}^{n} \gamma\, e^{-\gamma\tau_i}\right] \delta\left(\sum_{i=1}^{n} \tau_i - t\right)$$

only true for exp. run times !

→ Puts all n run times on equal footing (up to a factor γ)

Step 3: joint distribution of $\{x_i\}_{1\le i\le n}$ & $N_T(t)$



- Let x_i be the x-component of $\vec{\ell}_i = \tau_i \vec{\mathbf{v}}_\mathbf{i}$, i.e. $x_i = \vec{\ell}_\mathbf{i} \cdot \vec{\mathbf{e}}_x$ in d dimensions

$$P(\{x_i\}_{1\le i\le n}, n\,|\,t) = \frac{1}{\gamma}\left[\prod_{i=1}^{n}\int_0^\infty d\tau_i\,\gamma e^{-\gamma\tau_i}\int d^d\vec{\mathbf{v}}_\mathbf{i}\;W(\vec{\mathbf{v}}_\mathbf{i})\delta\left(x_i - \tau_i\,\vec{\mathbf{v}}_i\cdot\vec{\mathbf{e}}_\mathbf{x}\right)\right]$$
$$\times\,\delta\left(\sum_{i=1}^{n}\tau_i - t\right)$$

→ Use Laplace transform with respect to t

Step 4: go to Laplace space (« grand-canonical » ensemble)

- Taking Laplace transform with respect to t and re-organizing

$$\boxed{\int_0^\infty e^{-st}\, P\left(\{x_i\}_{1\le i\le n}, n; t\right)\, dt = \frac{1}{\gamma}\left(\frac{\gamma}{\gamma+s}\right)^n \prod_{i=1}^{n} \tilde{p}_s(x_i)}$$

$$\tilde{p}_s(x) = (\gamma+s)\int_0^\infty d\tau\, e^{-(\gamma+s)\tau} \underbrace{\int d^d\vec{\mathbf{v}}\; W(\vec{\mathbf{v}})\, \delta\left(x - \tau\vec{\mathbf{v}}\cdot\vec{\mathbf{e}}_x\right)}_{\text{contains all the dependence on } d \;\&\; W(\vec{\mathbf{v}})}$$

- The crucial point is that $\tilde{p}_s(x)$ can be interpreted as a proba. density
  - Easy to see that $\tilde{p}_s(x) \ge 0\,, \forall\, x \in \mathbb{R}$
  - One can check that it is normalized $\displaystyle\int_{-\infty}^{\infty} \tilde{p}_s(x)\, dx = 1$
  - It is symmetric, $\tilde{p}_s(x) = \tilde{p}_s(-x)$ and continuous

Step 4: go to Laplace space (« grand-canonical ensemble »)

- Taking Laplace transform with respect to t and re-organizing

$$\boxed{\int_0^\infty e^{-st}\, P\left(\{x_i\}_{1\le i\le n}, n; t\right)\, dt = \frac{1}{\gamma}\left(\frac{\gamma}{\gamma+s}\right)^n \prod_{i=1}^n \tilde{p}_s(x_i)}$$

$$\tilde{p}_s(x) = (\gamma+s)\int_0^\infty d\tau\, e^{-(\gamma+s)\tau}\int d^d\vec{\mathbf{v}}\; W(\vec{\mathbf{v}})\, \delta\left(x - \tau\vec{\mathbf{v}}\cdot\vec{\mathbf{e}}_x\right)$$

- Inverting the Laplace transform yields

$$\boxed{P\left(\{x_i\}_{1\le i\le n}, n; t\right) = \int_\Gamma \frac{ds}{2\pi i} e^{st}\, \frac{1}{\gamma}\left(\frac{\gamma}{\gamma+s}\right)^n \prod_{i=1}^n \tilde{p}_s(x_i)}$$

Step 5: back to Sparre Andersen

- Survival proba. $S(t)$ = proba. that the $x-$component of the RTP's position does not become negative up to time t
- Let 's relate it to the survival proba. of the effective $1d-$random walk

$$P\left(\{x_i\}_{1\leq i\leq n}, n; t\right) = \int_\Gamma \frac{ds}{2\pi i} e^{s\,t} \frac{1}{\gamma}\left(\frac{\gamma}{\gamma+s}\right)^n \prod_{i=1}^n \tilde{p}_s(x_i)$$

$$\rightarrow \quad S(t) = \sum_{n=1}^\infty \int_{-\infty}^\infty dx_1 \cdots \int_{-\infty}^\infty dx_1 P\left(\{x_i\}_{1\leq i\leq n}, n; t\right) \theta(x_1)\theta(x_1+x_2)\cdots\theta(x_1+x_2+\cdots x_n)$$

$$\boxed{S(t) = \int_\Gamma \frac{ds}{2\pi i} e^{s\,t} \frac{1}{\gamma} \sum_{n=1}^\infty \left(\frac{\gamma}{\gamma+s}\right)^n q_n}$$

$$\theta(x) = \begin{cases} 1\,, x \geq 0 \\ 0\,, x < 0 \end{cases}$$

$$q_n = \int_{-\infty}^\infty dx_1 \cdots \int_{-\infty}^\infty dx_1 \prod_{i=1}^n \tilde{p}_s(x_i)\, \theta(x_1)\, \theta(x_1+x_2)\cdots\theta(x_1+x_2+\cdots x_n)$$

$$q_n = \frac{1}{2^{2n}}\binom{2n}{n}$$

universal, thanks to Sparre Andersen thm !

Step 5: back to Sparre Andersen

$$\boxed{S(t) = \int_\Gamma \frac{ds}{2\pi i} e^{s\,t} \frac{1}{\gamma} \sum_{n=1}^{\infty} \left(\frac{\gamma}{\gamma + s} \right)^n q_n}$$

with

$$q_n = \frac{1}{2^{2n}} \binom{2n}{n}$$

- Survival probability

$$S(t) = \int_\Gamma \frac{ds}{2\pi\, i} e^{s\,t} \frac{1}{\gamma} \left[\sqrt{\frac{\gamma + s}{s}} - 1 \right]$$

- Inverting the Laplace transform yields

$$\boxed{S(t) = \frac{1}{2} e^{-\gamma t/2} \left(I_0 \left(\frac{\gamma t}{2} \right) + I_1 \left(\frac{\gamma t}{2} \right) \right)}$$

universal, i.e., independent of d and

$$W(\vec{\mathbf{v}}) = W(-\vec{\mathbf{v}})\ !$$

Outline

- Run-and-tumble particle (RTP): a model of active matter
- A first stage with the Sparre Andersen theorem
- From the Sparre Andersen theo. to the survival proba. of and RTP
- Conclusion

Summary and Conclusion

- Universal behaviour of the survival proba. $S(t)$ for a wide class of run-and-tumble model

  - Independent of dimension d and velocity distribution $W(\vec{\mathbf{v}})$
  - Consequence of the Sparre Andersen theorem
  - Universality of other related observables: dist. of the time of the maximum, record statistics, occupation time (more to discover ?)

- Universality is lost for power law distribution of the run-times (Lévy walks) — universality is recovered only at late times

- Similar universality found in a discrete-time version of the RTP

  B. Lacroix-A-Chez-Toine, F. Mori, J. Phys. A (2020)

- Beyond universality using Spitzer 's formula for $S(X_0 > 0\,, t)$

  B. De Bruyne, S. N. Majumdar, G. S. (2021)

Survival probability starting from $X_0 > 0$

- Exact result for the double Laplace transform in $d = 1$ and arbitrary velocity distribution $W(v)$ — not necessarily symmetric

$$\int_0^\infty dX_0 \int_0^\infty dt\, S(X_0, t)\, e^{-\lambda X_0 - s t} = \frac{\gamma + s}{\gamma \lambda s} \exp\left(-\frac{i}{2\pi} \int_{i\mathbb{R}} dz \ln\left(\frac{z+\lambda}{z}\right) \frac{\int_{-\infty}^{\infty} dv \, \frac{v\, W(v)}{(\gamma + s + z\, v)^2}}{\frac{1}{\gamma} - \int_{-\infty}^{\infty} dv \, \frac{W(v)}{(\gamma + s + z\, v)}} \right) - \frac{1}{\gamma \lambda}$$

B. De Bruyne, S. N. Majumdar, G. S., J. Stat. Mech. (2021)

- Simplest example: « standard » RTP with a uniform drift μ

$$W(v) = \frac{1}{2}\, \delta(v - \mu - v_0) + \frac{1}{2}\, \delta(v - \mu + v_0)$$

  - Explicit result for $S(X_0, t)$ in terms of Bessel functions
  - Rich behaviour in the (μ, v_0) plane