91. **Singular Cauchy Problems for a Class of Weakly Hyperbolic Differential Operators**

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In these notes singular Cauchy problems of Hamada's type are studied in the category of holomorphic functions and hyperfunctions for a class of hyperbolic differential operators with non-involutive multiple characteristics. Integral representations of their solutions are given.

1. **Introduction.** Let \( P(t, x, D_t, D_x) \) be a differential operator of order \( m \) of the form

\[
P(t, x, D_t, D_x) = D_t^m + \sum_{\ell=1}^{m} A_\ell(t, x, D_x)D_t^{m-\ell},
\]

where \( D_t = (1/\sqrt{-1})(\partial/\partial t), \ D_x = (1/\sqrt{-1})(\partial/\partial x) \) and \( A_\ell(t, x, D_x) \) is a differential operator at most of order \( \ell \), not containing \( D_t \), whose coefficients are holomorphic functions defined in a neighborhood of \((t, x) = (0, 0)\) in \( C \times C^n \).

We assume the following conditions:

(A-1) (Degeneracy of characteristic roots). There exists a non-negative integer \( q \) such that the principal symbol \( P_n(t, x, \tau, \xi) \) of \( P(t, x, D_t, D_x) \) is expressed in the form

\[
P_n(t, x, \tau, \xi) = \prod_{j=1}^{m} (\tau - t^\ell \lambda_j(\xi)),
\]

where \( \lambda_j(\xi) (1 \leq j \leq m) \) are holomorphic functions defined in a conic open neighborhood \( \Omega_0 \) of \( \xi_0 = (1, 0, \cdots, 0) \) in \( C^n \) and homogeneous of degree 1 such that

\[
\lambda_j(\xi) \neq \lambda_k(\xi), \quad \text{if } j \neq k \text{ and } \xi \in \Omega_0.
\]

(A-2) (Hyperbolicity). \( \lambda_j(\xi) (1 \leq j \leq m) \) are real if \( \xi \) is real.

(A-3) (Levi condition). Let \( A_{\ell,j}(t, x, \xi) \) be the homogeneous part of \( A_{\ell}(t, x, \xi) \) of degree \( j \) with respect to \( \xi \) and let

\[
A_{\ell,j}(t, x, \xi) = \sum_{k=0}^{\infty} t^k A_{\ell,j,k}(x, \xi)
\]

be the Taylor expansion of \( A_{\ell,j}(t, x, \xi) \) with respect to \( t \). Then

\[
A_{\ell,j,k}(x, \xi) = 0, \quad \text{if } k < (q+1)j - i.
\]

Alinhac [1], Amano [2], Amano-Nakamura [13], Nakamura-Uryu [6], Nakane [7], Taniguchi-Tozaki [10] and Yoshikawa [12] studied the Cauchy problem for weakly hyperbolic operators of the above type, and constructed parametrices, using a type of ordinary differential operators with polynomial coefficients which determine the principal
parts of parametric. (Nakamura-Uryu [6] and Amano-Nakamura [13] studied a more general case.) All of these authors, except Nakane [7], treated these subjects in the category of $C^\infty$ functions.

We deal with the singular Cauchy problem of Hamada's type

\[
\begin{align*}
& \text{(CP)}_l \quad \begin{cases} P(t, x, D_t, D_x)u_l(t, x, y, \xi) = 0, \\ D^j_u|_{y = \xi} = 0, \end{cases} \\
& \text{for } 0 \leq j \leq m - 1,
\end{align*}
\]
and its version in the complex domains

\[
\begin{align*}
& \text{(CP)}_c \quad \begin{cases} P(t, x, D_t, D_x)u_c(t, x, y, \xi) = 0, \\ D^j_u|_{y = \xi} = \delta_{j,1} (x - y, \xi) \xi^{-n} \end{cases} \\
& \text{for } 0 \leq j \leq m - 1,
\end{align*}
\]

for $0 \leq i \leq m - 1$, where $\delta_{j,1}$ is Kronecker's delta, and show that the solution $u_i$ is obtained as an infinite series of Radon integrals (see (1.2)). Parametric is obtained as integrals

\[
\int_{|\xi| = 1} u_i(t, x, y, \xi) \omega(\xi) \quad (0 \leq i \leq m - 1),
\]

where

\[
\omega(\xi) = \sum_{j = 1}^n (-1)^{j-1} \xi^j d\xi^1 \cdots d\xi^{j-1} \cdot d\xi_{j-1} \cdot d\xi_{j+1} \cdot \cdots d\xi_n.
\]

Our construction of solutions is similar to those of Yoshikawa [12] and Nakamura-Uryu [6], but we need much more delicate estimations. Our Radon integrals are modifications of those studied by Kataoka [4] and Aoki [3].

Before stating our main theorems we introduce the notations

\[
\begin{align*}
\varphi_j(t, \xi) &= (q + 1)^{-1} \psi_j(t \xi^{-1}), \\
\psi_j(t, x, y, \xi) &= (x - y, \xi) + \psi_j(t, \xi), \\
r_j(t, \xi) &= \max_{1 \leq k \leq m} |\varphi_j(t, \xi - 1) - \varphi_j(t, \xi - 1)|, \\
d(t, \xi) &= ((t^{(q + 1)} + |\xi|^{-1})^{(q + 1)}, \\
X &= \{x \in \mathbb{C}^n \mid |x| < a\}, \\
\Omega &= \{\xi = (\xi_1, \xi_2) \in \mathbb{C}^n \mid 0; |\xi| < b, |\arg(\xi)| < b\}, \\
S_+ &= \{t \in \mathbb{C}^n \mid |\arg(a t^2) < (2\nu + 2)^{-1} \pi - \epsilon\}, \\
Z &= \mathbb{C} \times X \times \Omega, \\
D(r) &= \{p \in \mathbb{C} \mid \text{Im } (p > r)\}, \\
D_{l}(d, r, R) &= \bigcup_{-b \leq \xi \leq b} \{p \in \mathbb{C} \mid \text{Im } (pe^{-r}) > r, \text{ dist}(p, R < 1)\},
\end{align*}
\]

for positive constants $a, b, d, r, R$ and $\sigma = \pm 1, 1 \leq j \leq m$.

Under Assumptions (A-1)-(A-3) we have

**Theorem 1.** For any sufficiently small positive constant $\varepsilon$ there exist positive constants $a, b, h$ and holomorphic functions $u_{\sigma_0}^{(i)}(a = \pm 1, 0 \leq i \leq m - 1, 1 \leq j \leq m, \nu \geq 0)$ defined in $Z$ such that the following hold:

(i) A solution $u_i$ of (CP)$_c$ is obtained in the form

\[
\begin{align*}
& u_i(t, x, y, \xi) = \sum_{j = 1}^n u_{\sigma, i, j, k}(t, x, y, \xi; \varphi_j(t, x, y, \xi)) + h_{\sigma, i, k}(t, x, y, \xi), \\
& \text{for } \sigma = \pm 1, \text{ where } u_{\sigma, i, j, k} \text{ is given by}
\end{align*}
\]

(1.2) $u_{\sigma, i, j, k}(t, x, \xi; p)$
\[ = \sum_{i=1}^{m} \int_{|\varrho|<\varrho_{i}} \exp \left( -\frac{1}{\varrho_{i}} \varrho \bar{\varrho} \right) \varrho^{-n} u_{i,j}(t, x, \rho \bar{\xi}^{-1}, \bar{\xi}) \varrho^{-1} \, \mathrm{d}\varrho, \]

for any positive constant \( R > h^{s+1} \), and \( h_{s,i,n} \) is a holomorphic function defined in a neighborhood of \( (t, x, y, \xi) = (0, 0, 0, \xi) \) and homogeneous of degree \((-n)\) with respect to \( \xi \).

(iii) The series (1.2) converges uniformly in every compact subset of the domain

\[ \{(t, x, \xi, p) \in \mathbb{Z} \times C \mid d(t, R)h < 1, \xi^{-1}p \in D_{\varrho}(0)\} \]

\[ \cup \bigcap_{j=1}^{m} \{(t, x, \xi, p) \in \mathbb{Z} \times C \mid d(t, R)h < 1, \xi^{-1}p \in D_{\varrho}(r_{j}(t, \xi))\}, \]

and hence it is a holomorphic function which is defined in this domain and homogeneous of degree \((-n)\) with respect to \( (\xi, p) \).

(iv) By deforming the integration path of the \( \nu \)-th term of (1.2) into

\[ C_{s,R} = \{(\nu+1)R \exp \left( -1\varrho\theta \right) ; 0 \leq \varrho \leq 1 \} \cup \{(\nu+1)R \exp \left( -1\varrho\theta \right) ; 1 \leq \varrho \} \]

\( (|\theta| < \varrho) \), \( u_{s,i,j,n} \) is continued to a holomorphic function defined in the domain

\[ \{(t, x, \xi, p) \in \mathbb{Z} \times C \mid \xi^{-1}p \in D_{\varrho}(d(t, R)h, 0, R)\} \]

\[ \cup \bigcap_{j=1}^{m} \{(t, x, \xi, p) \in \mathbb{Z} \times C \mid \xi^{-1}p \in D_{\varrho}(d(t, R)h, r_{j}(t, \xi), R)\}. \]

**Theorem 2.** The solution of (CP), is given by the “boundary value hyperfunction” of (1.1), namely,

\[ u_{i}(t, x, y, \xi) = \sum_{j=1}^{m} u_{s,i,j,n}(t, x, \xi) + \varphi_{i}(t, x, y, \xi) + \sqrt{-1} \varrho \cdot D_{\varrho}(t, x, y, \xi) \infty; \varrho_{i} = 0. \]

The singularity support and the singularity spectrum of \( u_{i} \), if we regard \( \xi \) as a parameter, are estimated as follows:

\[ \text{sing. supp.} (u_{i}) \subset \bigcup_{j=1}^{m} \{ \varphi_{j} = 0 \}, \]

\[ \text{S.S.} (u_{i}) \subset \bigcup_{j=1}^{m} \{(t, x, y, \xi) ; \sqrt{-1} \varrho \cdot D_{\varrho}(t, x, y, \xi) \infty; \varrho_{j} = 0 \}. \]

(As for the terminologies of hyperfunctions and singularity spectra, we refer to Sato-Kawai-Kashiwara [9].)

**Remark 1.** As Amano [2] and Amano-Nakamura [13] pointed out, our method for the construction of solutions will be effective in the analysis of the “branching of singularities” at multiple characteristic points. Alinhac [1], Nakane [7] and Taniguchi-Tozaki [10] carried out the analysis in the case \( m = 2 \), using special functions of the hypergeometric or confluent hypergeometric type.

**Remark 2.** (1.4) implies that the singularities of solutions are concentrated on the union of bicharacteristic strips associated with \( \tau - t \lambda_{j} \) \((1 \leq j \leq m)\) which pass \((t, x, y, \xi) = (0, 0, 0, \xi)\). This result is entirely different from those in the case of involutive multiple charac-
teristics. (See, for example, Kawai-Nakamura [5].)

2. Outline of the proof. We choose \( u^{(\nu)}_{i,j}(t, x, \xi) \) to be "semi-homogeneous" of degree \((-i-\nu)/(q+1), \) namely, for \( c \in C-0, \)
\[
(2.1) \quad u^{(\nu)}_{i,j}(c^{-i}t, x, c^{i+1}\xi) = c^{-i-\nu}u^{(\nu)}_{i,j}(t, x, \xi)
\]
holds. Then, at least formally, we can reduce the problem to the "transport equation"
\[
(2.2) \quad \left( D_t^n + \sum_{k=1}^{m} A_i^{(\nu)}(t, x, \xi) D_t^{n-k} \right) \left( \exp \left( \sqrt{-1} \psi_j(t, \xi) \right) u_{i,j}^{(\nu)} \right)
= -\sum_{k = 1}^{m} \sum_{\nu \geq 0} \frac{\alpha!}{\nu!} \left( \partial_{\nu}^{k} A_i^{(\nu)} \right) D_t^{n-k}
\times \left( \exp \left( \sqrt{-1} \psi_j(t, \xi) \right) u_{i,j}^{(\nu)} \right),
\]
with the "initial condition"
\[
(2.3) \quad \sum_{j=1}^{d} D_t^{j}(\exp \left( \sqrt{-1} \psi_j \right) u_{i,j}^{(\nu)})(t = 0) = \delta_{i,j} \delta_{\nu;0} (\sqrt{-1})^{-\nu} / (n-1)!,
\]
for \( 0 \leq k \leq m-1, 1 \leq j \leq m, \nu \geq 0, \) where \( \partial_{\nu}^{k} = (\partial / \partial \xi)^{k}, \) and \( A_i^{(\nu)}(t, x, \xi) \) is defined by
\[
A_i^{(\nu)}(t, x, \xi) = \sum_{k \geq 0 \nu \geq 1 \leq \nu = 0} t^k A_{i,i,k}(x, \xi).
\]

Actually we can show that the proof is reduced to the construction of solutions of (2.2) and (2.3) which satisfy the "growth condition"
\[
(2.4) \quad |D_t^k u_{i,j}^{(\nu)}| \leq C \nu^{(\xi)^{\nu+1}} \nu^{(\xi)^{\nu+1}} d(t, \xi)^{\nu,0} d(t, \nu+1)^{\nu,0}
\times \left\{ \begin{array}{ll}
d(t \xi^{(q+1)}, 1)^{(\nu,0)} \xi^{(\nu,0)} & \text{if } (t, x, \xi) \in Z, \\
d(t \xi^{(q+1)}, 1)^{(\nu,0)} \xi^{(\nu,0)} \xi^{(\nu,0)} & \text{if } (t, x, \xi) \in Z,
\end{array} \right.
\]
for \( 0 \leq k \leq m \) and \( \nu \geq 0, \) where we set
\[
\pi_j(x, \xi) = \sum_{i=1}^{m} [(q/2)(m-i+1)(m-i) A_{i,i-1,i-1,i-1}(x, \xi)
\times + \sqrt{-1} A_{i,i-1,i-1,i-1}(x, \xi)]
\times \lambda_j(\xi)^{\nu-1} \prod_{\nu \leq m} (\lambda_j(\xi) - \lambda_i(\xi))^{-1},
\]
\[
\mu_j(x, \xi) = \text{Re} \left( \pi_j(x, \xi) \right), \quad M(x, \xi) = \max_{1 \leq j \leq m} \mu_j(x, \xi).
\]

Since \( \psi_j(t, \xi) \) and \( A_i^{(\nu)}(t, x, \xi) \) are semi-homogeneous of degree 0 and \((i-\nu)/(q+1), \) respectively, (2.2)–(2.4) are compatible with the semi-homogeneity. Namely, if \( u_{i,j}^{(\nu)}(t, x, \xi) \) \( (\sigma = \pm 1, 1 \leq j \leq m, 0 \leq i \leq m-1, \nu \geq 0) \) satisfy (2.2)–(2.4) with the constraint \( \xi_1 = 1 \) (which we abbreviate to (2.2)|_{\xi_1 = 1}, \quad (2.4)|_{\xi_1 = 1}, \) and if we set
\[
u_{i,j}^{(\nu)}(t, x, \xi) = \xi_1^{(i-\nu)/(q+1)} u_{i,j}^{(\nu)}(t \xi_1^{(q+1)}, x, \xi_1^{i+1} \xi_1),
\]
then they are solutions of (2.2)–(2.4). Thus essentially we have only to consider the case \( \xi_1 = 1. \)

We construct \( u_{i,j}^{(\nu)} \) by the induction on \( \nu. \)

For \( \nu = 0, \) (2.2)|_{\xi_1 = 1} \) is a homogeneous ordinary differential equation with polynomial coefficients with respect to \( t, \) and has an irregular singular point of Poincaré's rank \((q+1) \) at \( t = \infty. \) It has formal solutions of the form
\( v_j = \hat{v}_j t^{q_j(x, \xi)} \exp (\sqrt{-1} \psi_j(t, 1, \xi')) \) for \( 1 \leq j \leq m \),

where \( \hat{v}_j \) is a formal power series in \( t^{-q+1} \) whose coefficients are holomorphic functions in \( (x, \xi') \). By a version of the "asymptotic existence theorem" (Wasow [11, Theorem 12.3]), there exists, for each \( j \), a holomorphic solution (not formal) of (2.2)\(_{t=1}\) whose asymptotic expansion in the sector \( S_\nu \) coincides with \( v_j \). Using these solutions we get solutions of (2.2)--(2.4) for \( \nu = 0 \).

For \( \nu \neq 0 \), we construct solutions by the "method of the variation of constants", using the same integration paths as those which appeared in Nishimoto [8]. Through some delicate estimations we can show that these solutions actually satisfy the former half of (2.4).

To show the latter half of (2.4), we choose a suitable family of finitely many sectors which cover the whole plane \( C \). Then we repeat similar arguments to express \( u^{(\nu)}_{x,i,k} \), for each sector \( S \) of this family, in such a manner that

\[
u^{(\nu)}_{x,i,k} - \sum_{j=1}^{m} \exp (\sqrt{-1} (\psi_j - \psi_x)) u^{(\nu)}_{x,i,k,\nu,j}
\]

holds, where \( u^{(\nu)}_{x,i,k,\nu,j} \) satisfies the transport equations and the former half of (2.4) in \( S \), instead of \( S_\nu \). This implies the latter half of (2.4).

The last argument is indispensable, because "Stokes' phenomena" (Wasow [11, §15]) may occur and the factor \( \exp (\sqrt{-1} (\psi_j - \psi_x))(j \neq k) \) may break the validity of the former half of (2.4) in the sector \( S (\neq S_\nu) \). This is the reason why we need the latter half of (2.4). This relates to the "branching of singularities". (See Remark 1.)

The detailed proof will appear elsewhere.

References