88. *A Class of Solutions to the Self-Dual Yang-Mills Equations*

By Kanehisa TAKASAKI
Department of Mathematics, University of Tokyo

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This note presents a method for generating a class of special solutions to the self-dual Yang-Mills equations. They are shown to be parametrized by some matrices which serve as "frames" representing the corresponding points of the (infinite dimensional) Grassmann manifolds. Thus a remarkable similarity to the results of Sato [7] and Date *et al.* [5] for the "soliton equations" is revealed. Also it should be added that the method presented here is closely related with those of Cherednik [3], Date [4] and Krichever [6].

§1. **The self-dual Yang-Mills equations and the linearization.** Hereafter, in contrast to the usual formulation in the real domain (see, for example, [1] and the references therein), we shall work with the complex analytic theory of the self-dual Yang-Mills fields with structure group $GL(r, \boldsymbol{C})$ ($r \geq 2$). All the functions which will appear in what follows are supposed to be holomorphic in some complex domains. Thus, the self-dual Yang-Mills equations which we shall consider are, by definition, given by

$$(1)\qquad \begin{aligned} &[\partial_y + A_y, \partial_{\bar z} + A_{\bar z}] = 0, \qquad [\partial_z + A_z, \partial_{\bar y} + A_{\bar y}] = 0, \\ &[\partial_y + A_y, \partial_{\bar y} + A_{\bar y}] = [\partial_z + A_z, \partial_{\bar z} + A_{\bar z}], \end{aligned}$$

where $x = (y, \bar y, z, \bar z)$ are complex independent variables in $\boldsymbol{C}^4$ ($\bar y$ and $\bar z$ do not indicate the complex conjugates of y and z), $\partial_y = \partial/\partial y, \cdots, \partial_{\bar z} = \partial/\partial \bar z$, and $A_y, \cdots, A_{\bar z}$ are unknown matrices of size $r \times r$ of holomorphic functions of x.

Introducing another independent complex variable λ, we can rewrite (1) into

$$(2)\qquad [-\lambda(\partial_y + A_y) + (\partial_z + A_z), \lambda(\partial_{\bar z} + A_{\bar z}) + (\partial_{\bar y} + A_{\bar y})] = 0,$$

so that, as pointed out first by Belavin-Zakharov [2] and Ward [8], the linear system

$$(3)\qquad \begin{aligned} &(-\lambda(\partial_y + A_y) + (\partial_z + A_z))\Psi(x, \lambda) = 0, \\ &(\lambda(\partial_{\bar z} + A_{\bar z}) + (\partial_{\bar y} + A_{\bar y}))\Psi(x, \lambda) = 0 \end{aligned}$$

presents a linearization of (1). Note that if (3) is fulfilled for an invertible matrix $\Psi(x, \lambda)$ of size $r \times r$, (2) immediately follows.

§2. **Special solutions.** As the data for the special solution stated below, inspired by [4], [6], let us consider $\{\lambda_j(x), m_j, c_j(x, \lambda), j = 1, \cdots, N\}$, where $\lambda_j(x)$, $j = 1, \cdots, N$, are holomorphic functions with

(4) $$[-\lambda_j(x)\partial_y+\partial_z,\ \lambda_j(x)\partial_{\bar{z}}+\partial_{\bar{y}}]=0,$$
m_j, $j=1,\cdots,N$, are positive integers with

(5) $$\sum_{j=1}^{N} m_j=rm, \qquad m \text{ is a positive integer,}$$

and $c_j(x,\lambda)$, $j=1,\cdots,N$, are r-column vectors of holomorphic functions, defined near $(x,\lambda_j(x))$ respectively, of the form

(6) $$c_j(x,\lambda)=\hat{c}_j(\lambda,\ y+\lambda z,\ -\bar{z}+\lambda\bar{y}),$$

where $\hat{c}_j(\lambda,p,q)$, $j=1,\cdots,N$, are r-column vectors of holomorphic functions of three variables (λ,p,q). Let us define $c_{j,k,l}(x)$, $1\le j\le N$, $0\le k,l$, by the expansion

(7) $$\lambda^k c_j(x,\lambda)=\sum_{l=0}^{\infty} c_{j,k,l}(x)(\lambda-\lambda_j(x))^l.$$

A special solution to the self-dual Yang-Mills equations is constructed from the above data as follows:

Theorem 1. *Suppose*

(8) $$\det\,[(c_{1,k,l}(x))_{\substack{k=0,\cdots,m-1\\ l=0,\cdots,m_1-1}}\,|\cdots|\,(c_{N,k,l}(x))_{\substack{k=0,\cdots,m-1\\ l=0,\cdots,m_N-1}}]\not\equiv 0.$$

Then, for any invertible matrix $W_0(x)$ *of size* $r\times r$ *of holomorphic functions of* x, *a matrix* $\Psi(x,\lambda)=\sum_{k=0}^{m}W_k(x)\lambda^{m-k}$ *of size* $r\times r$ *is uniquely determined by the conditions*

(9) $$\Psi(x,\lambda)c_j(x,\lambda)=O\,((\lambda-\lambda_j(x))^{m_j})\quad(\lambda\to\lambda_j(x)),\quad j=1,\cdots,N.$$

Furthermore, the matrices $A_y(x),\cdots,A_{\bar{z}}(x)$ *defined by*

(10) $$\begin{aligned} &A_y=-\partial_y W_0\cdot W_0^{-1}, \qquad A_{\bar{z}}=-\partial_{\bar{z}}W_0\cdot W_0^{-1},\\ &A_z=-(\partial_z W_0-\partial_y W_1-A_y W_1)\cdot W_0^{-1},\\ &A_{\bar{y}}=-(\partial_{\bar{y}}W_0+\partial_{\bar{z}}W_1+A_{\bar{z}}W_1)\cdot W_0^{-1}\end{aligned}$$

and $\Psi(x,\lambda)$ *solve equations* (1)–(3).

Remark. A holomorphic function $\lambda_j(x)$ with condition (4) can be generated, for example, by solving (locally) an equation of the form $f_j(\lambda,\ y+\lambda z,\ -\bar{z}+\lambda\bar{y})=0$ with respect to λ, where $f_j(\lambda,p,q)$ is a holomorphic function of three variables (λ,p,q).

The structure of higher evolutions ("hierarchy") similar to those of the soliton equations [7], [5] can be also specified: Let us introduce a series of independent variables $t=(t^{(\alpha)}_{\nu,\rho,\sigma})_{1\le\alpha\le r,0\le\nu\le\rho,\sigma}$ and a diagonal matrix

(11) $$T(t,\lambda)=\sum_{\nu,\rho,\sigma=0}^{\infty}\operatorname{diag}[t^{(1)}_{\nu,\rho,\sigma},\cdots,t^{(r)}_{\nu,\rho,\sigma}]\lambda^\nu(y+\lambda z)^\rho(-\bar{z}+\lambda\bar{y})^\sigma.$$

Theorem 2. *Suppose* (8). *Then, for any invertible matrix* $W_0(x,t)$, *a matrix* $\Psi(x,t,\lambda)=\sum_{k=0}^{m}W_k(x,t)\lambda^{m-k}e^{T(t,\lambda)}$ *of size* $r\times r$ *is uniquely determined by the conditions*

(12) $$\Psi(x,t,\lambda)c_j(x,\lambda)=O\,((\lambda-\lambda_j(x))^{m_j})\quad(\lambda\to\lambda_j(x)),\quad j=1,\cdots,N.$$

$\Psi(x,\lambda)$ *satisfies* (3) *for the matrices* $A_y(x,t),\cdots,A_{\bar{z}}(x,t)$ *defined by* (10) *with* $W_0=W_0(x,t)$, $W_1=W_1(x,t)$, *and also the linear systems*

(13) $$\partial\Psi(x,t,\lambda)/\partial t^{(\alpha)}_{\nu,\rho,\sigma}=B^{(\alpha)}_{\nu,\rho,\sigma}(x,t,\lambda)\Psi(x,t,\lambda),\quad 1\le\alpha\le r,\quad 0\le\nu,\rho,\sigma,$$

where $B^{(\alpha)}_{\nu,\rho,\sigma}(x,t,\lambda)$ *is a matrix of size* $r\times r$ *whose components are* *polynomials of* λ.

§3. **Parametrization by "frames".** We shall, first, investigate a general framework of the parametrization; later, we shall go back to the solutions mentioned in §2.

Let us consider the correspondence between two matrices, $\Psi(x,\lambda)=\sum_{k=0}^{m} W_k(x)\lambda^{m-k}$ (of size $r\times r$) and $\xi(x)=(\xi_k(x))_{k=0,1,\cdots,(\downarrow)}$ (of size $\infty\times rm$ with $\xi_k(x)$, $k=0,1,\cdots$, being $r\times rm$-blocks), defined by

(14) $$(W_m(x), W_{m-1}(x), \cdots, W_0(x), 0, 0, \cdots)\xi(x)=0,$$

where $\Psi(x,\lambda)$ and $\xi(x)$ are supposed to satisfy the conditions

(15) $$\det W_0(x)\not\equiv 0, \qquad \det(\xi_k(x))_{k=0,\cdots,m-1}\not\equiv 0,$$

(16) $$\Lambda\xi(x)=(\xi_{k+1}(x))_{k=0,1,\cdots}=\xi(x)C(x)$$
for some matrix $C(x)$ of size $rm\times rm$.

Here Λ denotes the block-wise shifting matrix $(\delta_{i,j-1}1_r)_{i,j=0,1,\cdots}$ with 1_r the unit matrix of size $r\times r$.

Theorem 3. *Each of two matrices* $\Psi(x,\lambda)$ *and* $\xi(x)$ *with properties* (15) *and* (16) *is determined by the other via* (14), *uniquely up to the arbitrariness* $\Psi(x,\lambda)\to G(x)\Psi(x,\lambda)$, $\xi(x)\to\xi(x)H(x)$, *where* $G(x)$ *and* $H(x)$ *are invertible matrices of size* $r\times r$ *and* $rm\times rm$ *respectively. Furthermore,* (1)–(3) *are fulfilled for the matrices* $A_y(x),\cdots,A_{\bar z}(x)$ *defined by* (10) *if and only if*

(17) $$(-\Lambda\partial_y+\partial_z)\xi(x)=\xi(x)A(x), \qquad (\Lambda\partial_{\bar z}+\partial_{\bar y})\xi(x)=\xi(x)B(x)$$

are satisfied for some matrices $A(x)$ *and* $B(x)$ *of size* $rm\times rm$.

It should be noted that $\Psi(x,\lambda)\to G(x)\Psi(x,\lambda)$ corresponds to the gauge transformation, while $\xi(x)\to\xi(x)H(x)$ defines the equivalence of "frames" representing a common point of the Grassmann manifold as appeared in [7], i.e. the equivalence of $\infty\times rm$-matrices whose column vectors span a common linear subspace of dimension rm in the vector space of column vectors of size ∞.

The structure of higher evolutions is described as follows:

Theorem 4. *Suppose* (15)–(17). *Then, for any invertible matrix* $W_0(x,t)$, *a matrix* $\Psi(x,t,\lambda)=\sum_{k=0}^{m} W_k(x,t)\lambda^{m-k}e^{T(t,\lambda)}$ *of size* $r\times r$ *is uniquely determined by*

(18) $$(W_m(x,t), W_{m-1}(x,t), \cdots, W_0(x,t), 0, 0, \cdots)e^{T(t,\Lambda)}\xi(x)=0.$$

$\Psi(x,t,\lambda)$ *satisfies* (3) *for the matrices* $A_y(x,t),\cdots,A_{\bar z}(x,t)$ *defined by* (10) *with* $W_0=W_0(x,t)$, $W_1=W_1(x,t)$, *and also linear systems* (13).

The solutions presented in §2 are recovered if we set

(19) $$\xi(x)=[(c_{1,k,l}(x))_{\substack{k=0,1,\cdots\\ l=0,\cdots,m_1-1}} \mid\cdots\mid (c_{N,k,l}(x))_{\substack{k=0,1,\cdots\\ l=0,\cdots,m_N-1}}].$$

$A(x)$, $B(x)$ and $C(x)$ are given by

(20) $$\begin{aligned} A(x)&=-\oplus_{j=1}^{N}\operatorname{diag}[\partial_y\lambda_j(x), 2\partial_y\lambda_j(x), \cdots, m_j\partial_y\lambda_j(x)],\\ B(x)&=\oplus_{j=1}^{N}\operatorname{diag}[\partial_{\bar z}\lambda_j(x), 2\partial_{\bar z}\lambda_j(x), \cdots, m_j\partial_{\bar z}\lambda_j(x)],\\ C(x)&=\oplus_{j=1}^{N} J(\lambda_j(x), m_j),\end{aligned}$$

where $J(\lambda,m)$ denotes the Jordan cell of size $m\times m$ with eigenvalue λ.

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