63. **Toda Lattice Hierarchy. II**

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0. Introduction. This note is a sequel to the preceding paper [1]. Here we shall discuss the $l$-reduced family, (TL)$_l$, of the Toda lattice (TL) hierarchy, and shall construct the $N$-soliton solutions by making use of the Riemann-Hilbert decomposition, which is an infinite dimensional generalization of the classical Riemann-Hilbert problem.

The TL hierarchy was introduced in [1] as follows: Let $x=(x_1, x_2, \ldots), y=(y_1, y_2, \ldots)$ be two time flows. Let $L, M, B_n, C_n$ ($n=1, 2, \ldots$) be matrices of size $\mathbb{Z} \times \mathbb{Z}$ of the form

\begin{equation}
L = \sum_{-\infty < j < \infty} \text{diag} [b_j(s; x, y) A^j], \quad b_j(s; x, y) = 1,
\end{equation}

\begin{equation}
M = \sum_{-\infty < j < \infty} \text{diag} [c_j(s; x, y) A^j], \quad c_j(s; x, y) \neq 0
\end{equation}

\begin{equation}
B_n = (L^n)_{+}, \quad C_n = (M^n)_{-}.
\end{equation}

(As for the notations, the readers should refer to [1].) Then the TL hierarchy is defined by the following Zakharov-Shabat equations;

\begin{equation}
\partial_s B_n - \partial_{x_n} B_n + [B_n, B_n] = 0, \quad \partial_{y_n} C_n - \partial_{y_n} C_n + [C_n, C_n] = 0
\end{equation}

\begin{equation}
\partial_{y_n} B_n - \partial_{x_n} C_n + [B_n, C_n] = 0, \quad m, n = 1, 2, \ldots.
\end{equation}

The hierarchy is linearized by the equations

\begin{equation}
LW^{(\omega)} = W^{(\omega)} A, \quad MW^{(\theta)} = W^{(\theta)} A^{-1},
\end{equation}

\begin{equation}
\partial_x W = B_x W, \quad \partial_y W = C_x W, \quad W = W^{(\omega)}, W^{(\theta)}, \quad n = 1, 2, \ldots.
\end{equation}

As fundamental solution matrices to these equations, one may choose matrices of the form

\begin{equation}
W^{(\omega)}(x, y) = \hat{W}^{(\omega)}(x, y) \exp \xi(x, A),
\end{equation}

\begin{equation}
W^{(\theta)}(x, y) = \hat{W}^{(\theta)}(x, y) \exp \xi(y, A^{-1}), \quad \xi(x, A^{-1}) = \sum_{n=1}^{\infty} x_n A^{-n},
\end{equation}

\begin{equation}
\hat{W}^{(\omega)}(x, y) = \sum_{j=0}^{\infty} \text{diag} [a_j^{(\omega)}(s; x, y)] A^j.
\end{equation}

Such fundamental solution matrices will be referred to as wave matrices.

The linearization (4) equivalently leads to the bilinear relation

\begin{equation}
W^{(\omega)}(x, y) W^{(\omega)}(x', y')^{-1} = W^{(\theta)}(x', y) W^{(\theta)}(x', y')^{-1}
\end{equation}

for any $x, x', y, y'$.
From this relation, we deduce the existence of $\tau$ functions, $\tau(s; x, y)$, defined by

$$
\tau(s; x, y) = p_j(-\partial_x) \tau(s; x, y) / \tau(s; x, y),
$$

where $p_j(x)$ is introduced through $\sum_{j=0}^{\infty} p_j(x) x^j = \exp \xi(s, x)$. Substituting (7) to (6), the TL hierarchy is converted to Hirota's bilinear equations.

1. L-periodic reduction. Let $l$ be a positive integer. Let us consider a subfamily of the TL hierarchy with the additional constraints

$$
L' = A', \quad M' = A'^{-1}.
$$

This subfamily will be referred to as the $l$-periodic Toda lattice ((TL)$_l$) hierarchy. One easily sees that the $l$-periodic condition (8) yields

$$
[L, A] = [M, A'] = 0, \quad [W^{(s)}, A'] = [W^{(s)}, A] = 0,
$$

where $x_n, y_n$ are independent of $x_n, y_n$ for $n \equiv 0 \mod l$. Let us introduce $\tau$ functions, $\tau(s; x, y)$, by $\tau(s; x, y) = \exp \left( \sum_{n=0}^{\infty} n x_n y_n \right) \tau(s; x, y)$ (cf. [1]). Then we obtain

**Proposition 1.** Under a suitable choice of elementary multipliers of $\tau$ functions (cf. [1], Theorem 4), one finds that the $l$-periodic condition (8) equivalently reads

$$
\partial_x \tau(s; x, y) = \partial_y \tau(s; x, y) = 0,
$$

for any $s$, $n \equiv 0 \mod l$.

In order to investigate the linearization scheme for the (TL)$_l$ hierarchy (cf. [4]), let us recall some basic facts about Lie algebras. Let $g(((\infty)))$ be the formal Lie algebra of matrices of size $Z \times Z$, and $g(((\infty)))$, be a subalgebra of $g(((\infty)))$, defined by

$$
g(((\infty))) = \{ A \in g(((\infty))) | [A, A'] = 0 \text{ for } n \equiv 0 \mod l \}
$$

$g((l, C[[\zeta, \zeta^{-1}]])$ is isomorphic to $g(((\infty)))$, under the correspondence

$$
\begin{align*}
A(\zeta) = \sum_{j \in Z} \text{diag}[a_j(0), \ldots, a_j(l-1)] A(\zeta) & \quad \longrightarrow \quad A = \sum_{j \in Z} \text{diag}[a_j(0), \ldots, a_j(l-1)] A',
\end{align*}
$$

where

$$
A(\zeta) = \begin{bmatrix}
0 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 1 \\
\zeta & \cdots & 0 & 0 & 0
\end{bmatrix},
$$

and $\text{diag}[a_j(0), \ldots, a_j(l-1)]$ denotes a diagonal matrix $\text{diag}(\ldots, a_j(0), \ldots, a_j(l-1), a_j(0), \ldots, a_j(l-1), \ldots)$. Under this isomorphism, $L, M$, $W^{(s)}(x, y)$, $W^{(s)}(x, y)$ are identified with $L(\zeta), M(\zeta), W^{(s)}(x, y; \zeta), W^{(s)}(x, y; \zeta) \in g((l, C[[\zeta, \zeta^{-1}])))$, which take the form
\[ L(\zeta) = W^{(0)}(x, y; \zeta) A(\zeta) W^{(0)}(x, y; \zeta)^{-1}, \]
\[ M(\zeta) = W^{(0)}(x, y; \zeta) A(\zeta)^{-1} W^{(0)}(x, y; \zeta)^{-1}, \]
\[ W^{(0)}(x, y; \zeta) = W^{(0)}(x, y; \zeta) \exp \xi(x, A(\zeta)), \]
\[ W^{(0)}(x, y; \zeta) = W^{(0)}(x, y; \zeta) \exp \xi(y, A(\zeta)^{-1}), \]
\[ \hat{\varphi}^{(0)}(x, y; \zeta) = \sum_{j=0}^\infty \text{diag}[\hat{w}_j^{(0)}(0; x, y), \ldots, \hat{w}_j^{(0)}(l-1; x, y)] A(\zeta)^{ij}. \]

Proposition 2. \( W^{(0)}(x, y; \zeta) \) solve the linear equations
\[
(12) \quad \partial_{x^n} W(\zeta) = B_n(\zeta) W(\zeta), \quad \partial_{x^n} W(\zeta) = C_n(\zeta) W(\zeta), \quad n = 1, 2, \ldots,
\]
where \( B_n(\zeta) = [L(\zeta)]^n, \quad C_n(\zeta) = [M(\zeta)]^n. \) The symbols \( [A(\zeta)]_+ \) stand for the part of \( A(\zeta) \) of non-negative order, and of strictly negative order with respect to \( A(\zeta) \), respectively. The compatibility condition for (12) gives the (TL) hierarchy, and \( B_n(\zeta), C_n(\zeta) \) belong to \( \mathfrak{sl}(l, C[\zeta, \zeta^{-1}]). \)

2. The Riemann-Hilbert decomposition and its application.

We define \( \hat{\mathcal{V}}^{(0)}(x, y) \) for wave matrices \( W^{(0)}(x, y) \) by \( W^{(0)}(x, y) = \hat{\mathcal{V}}^{(0)}(x, y) \exp(\xi(x, A) + \xi(y, A^{-1})) \) They take the form \( \hat{\mathcal{V}}^{(0)}(x, y) = \sum_{j=0}^\infty \text{diag}[\hat{v}_j^{(0)}(s; x, y)] A^{ij} \), and \( \hat{v}_0^{(0)}(s; x, y) = 1 \). Then the bilinear relation (6) implies that there exists an invertible matrix \( A \) of size \( Z \times Z \) such that
\[
(13) \quad \hat{\mathcal{V}}^{(0)}(x, y) = \hat{\mathcal{V}}^{(0)}(x', y) H(x, y)
\]
where \( H(x, y) = \exp(\xi(x, A) + \xi(y, A^{-1})) \). This implies decomposition of \( H(x, y) \) to the upper triangular matrix \( \hat{\mathcal{V}}^{(0)} \) and the lower triangular matrix \( \hat{\mathcal{V}}^{(0)} \), and is regarded as an infinite dimensional generalization of the classical Riemann-Hilbert (RH) problem. In fact, if \( A \in \mathfrak{sl}(l) \), then (13) reduces to the classical RH problem. Therefore (13) will be called the RH decomposition.

Proposition 3. Let \( A = I + \sum_{j=1}^N a_j X_{p_j}, \) in (13), where
\[
X_{p_j} = \sum_{n \in \mathbb{Z}} p^n q^{-n} E_{m \times n} \quad (E_{m \times n} = (\delta_{n,0})_{n \in \mathbb{Z}})
\]
is the so-called vertex operator \([3]\). Suppose \( a_j > 0 \), and \( 0 < q_1 < \ldots < q_i < p_1 < \ldots < p_N \). Then the RH decomposition has a unique pair of solutions \( \hat{\mathcal{V}}^{(0)}(x, y) \), and their entries are expressed in the form
\[
\hat{v}^{(0)}(s; x, y) = p_i(- \hat{\delta}_s) \tau^{(s; x, y)}, \quad \hat{v}^{(0)}(s; x, y) = p_i(- \hat{\delta}_s) \tau^{(s; x, y)},
\]
Here the \( \tau \) functions, \( \tau^{(s; x, y)} \), is given by
\[
(14) \quad \tau^{(s; x, y)} = \sum_{\ell=0}^N \sum_{1 \leq i_1 < \cdots < i_l \leq \ell} \sum_{0 \leq j \leq s} c_{i_1, \ldots, i_l}(s) \cdot a_{s_i}(s) \exp \left( \sum_{p=1}^l \xi(p, q_p) - \eta(q_p) \right)
\]
where
\[
a_{i}(s) = a_{i}(p, q), \quad a'(s) = \sum_{i, j \leq s} c_{i, j} q_{i-j}, \quad a_{i+j} = (p_j - q_{i+j}), \quad \gamma(p) = \xi(x, p) + \xi(y, p^{-1}).
\]
The $\tau$ function (14) coincides with the $\tau$ function introduced in [5], [6].

3. Remarks. (1) Setting $A=I+\sum_{j=1}^{k} a_j E_{x_jy_j}$ and $y=0$ in (13), one can obtain rational solutions of the KP hierarchy (cf. Theorem 5 in [1]).

(2) The TL hierarchies of orthogonal or symplectic type, and the multi-component TL hierarchy can be also considered.

These topics are investigated in detail in [2].

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References