63. *Toda Lattice Hierarchy. II*

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0. Introduction. This note is a sequel to the preceding paper [1]. Here we shall discuss the l-reduced family, $(\mathrm{TL})_l$, of the Toda lattice (TL) hierarchy, and shall construct the N-soliton solutions by making use of the Riemann-Hilbert decomposition, which is an infinite dimensional generalization of the classical Riemann-Hilbert problem.

The TL hierarchy was introduced in [1] as follows: Let $x=(x_1, x_2, \cdots)$, $y=(y_1, y_2, \cdots)$ be two time flows. Let L, M, B_n, C_n $(n=1, 2, \cdots)$ be matrices of size $Z\times Z$ of the form

$$(1)\quad \begin{aligned} L&=\sum_{-\infty<j\leq 1}\mathrm{diag}[b_j(s\,;x,y)]\Lambda^j, & b_1(s\,;x,y)&=1,\\ M&=\sum_{-1\leq j<+\infty}\mathrm{diag}[c_j(s\,;x,y)]\Lambda^j, & c_{-1}(s\,;x,y)&\neq 0,\end{aligned}$$

$$(2)\quad B_n=(L^n)_+,\qquad C_n=(M^n)_-.$$

(As for the notations, the readers should refer to [1].) Then the TL hierarchy is defined by the following Zakharov-Shabat equations;

$$(3)\quad \begin{aligned} &\partial_{x_n}B_m-\partial_{x_m}B_n+[B_m, B_n]=0, && \partial_{y_n}C_m-\partial_{y_m}C_n+[C_m, C_n]=0,\\ &\partial_{y_n}B_m-\partial_{x_m}C_n+[B_m, C_n]=0, && m, n=1, 2, \cdots.\end{aligned}$$

The hierarchy is linearized by the equations

$$(4)\quad \begin{aligned} &LW^{(\infty)}=W^{(\infty)}\Lambda, \qquad MW^{(0)}=W^{(0)}\Lambda^{-1},\\ &\partial_{x_n}W=B_nW,\quad \partial_{y_n}W=C_nW,\quad W=W^{(\infty)}, W^{(0)},\quad n=1, 2, \cdots.\end{aligned}$$

As fundamental solution matrices to these equations, one may choose matrices of the form

$$(5)\quad \begin{aligned} &W^{(\infty)}(x,y)=\hat{W}^{(\infty)}(x,y)\exp\xi(x,\Lambda),\\ &W^{(0)}(x,y)=\hat{W}^{(0)}(x,y)\exp\xi(y,\Lambda^{-1}), \qquad \xi(x,\Lambda^{\pm 1})=\sum_{n=1}^{\infty}x_n\Lambda^{\pm n},\\ &\hat{W}^{\binom{0}{\infty}}(x,y)=\sum_{j=0}^{\infty}\mathrm{diag}[\hat{w}_j^{\binom{0}{\infty}}(s\,;x,y)]\Lambda^{\pm j}.\end{aligned}$$

Such fundamental solution matrices will be referred to as wave matrices.

The linearization (4) equivalently leads to the bilinear relation

$$(6)\quad W^{(\infty)}(x,y)W^{(\infty)}(x',y')^{-1}=W^{(0)}(x,y)W^{(0)}(x',y')^{-1}\quad \text{for any } x, x', y, y'.$$

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From this relation, we deduce the existence of τ functions, $\tau(s; x, y)$, defined by

$$
(7) \quad \begin{aligned} \hat{w}_j^{(\infty)}(s; x, y) &= p_j(-\tilde{\partial}_x)\tau(s; x, y)/\tau(s; x, y), \\ \hat{w}_j^{(0)}(s; x, y) &= p_j(-\tilde{\partial}_y)\tau(s+1; x, y)/\tau(s; x, y) \end{aligned}
$$

where $p_j(x)$ is introduced through $\sum_{j=0}^{\infty} p_j(x)\lambda^j = \exp \xi(x, \lambda)$. Substituting (7) to (6), the TL hierarchy is converted to Hirota's bilinear equations.

1. l-periodic reduction. Let l be a positive integer. Let us consider a subfamily of the TL hierarchy with the additional constraints

$$
(8) \quad L^l = \Lambda^l, \qquad M^{-l} = \Lambda^{-l}.
$$

This subfamily will be referred to as the l-periodic Toda lattice ($(\mathrm{TL})_l$) hierarchy. One easily sees that the l-periodic condition (8) yields

$$
(9) \quad \begin{aligned} &[L, \Lambda^n] = [M, \Lambda^n] = 0, \qquad [W^{(\infty)}, \Lambda^n] = [W^{(0)}, \Lambda^n] = 0, \\ &\partial_{x_n} L = \partial_{y_n} L = 0, \quad \partial_{x_n} M = \partial_{y_n} M = 0 \qquad \text{for } n \equiv 0 \bmod l. \end{aligned}
$$

That is, solutions of the $(\mathrm{TL})_l$ hierarchy are independent of x_n, y_n ($n \equiv 0 \bmod l$). Let us introduce τ functions, $\tau'(s; x, y)$, by $\tau'(s; x, y) = \exp\left(\sum_{n=1}^{\infty} n x_n y_n\right)\tau(s; x, y)$ (cf. [1]). Then we obtain

Proposition 1. *Under a suitable choice of elementary multipliers of τ functions* (cf. [1], Theorem 4), *one finds that the l-periodic condition* (8) *equivalently reads*

$$
(10) \quad \begin{aligned} &\partial_{x_n}\tau'(s; x, y) = \partial_{y_n}\tau'(s; x, y) = 0, \\ &\tau'(s; x, y) = \tau'(s+l; x, y), \qquad \textit{for any } s, \ n \equiv 0 \bmod l. \end{aligned}
$$

In order to investigate the linearization scheme for the $(\mathrm{TL})_l$ hierarchy (cf. [4]), let us recall some basic facts about Lie algebras. Let $\mathfrak{gl}((\infty))$ be the formal Lie algebra of matrices of size $Z \times Z$, and $\mathfrak{gl}((\infty))_l$ be a subalgebra of $\mathfrak{gl}((\infty))_l$ defined by

$$
\mathfrak{gl}((\infty))_l = \{A \in \mathfrak{gl}((\infty)) \mid [A, \Lambda^n] = 0 \qquad \text{for } n \equiv 0 \bmod l\}
$$

$\mathfrak{gl}(l, \boldsymbol{C}[[\zeta, \zeta^{-1}]])$ is isomorphic to $\mathfrak{gl}((\infty))_l$ under the correspondence

$$
(11) \quad \begin{aligned} &\mathfrak{gl}(l, \boldsymbol{C}[[\zeta, \zeta^{-1}]]) \longrightarrow \mathfrak{gl}((\infty))_l \\ &A(\zeta) = \sum_{j \in Z} \mathrm{diag}[a_j(0), \cdots, a_j(l-1)]\Lambda_l(\zeta)^j \\ &\qquad \longmapsto A = \sum_{j \in Z} \mathrm{diag}[a_j(0), \cdots, a_j(l-1)]_l \Lambda^j, \end{aligned}
$$

where

$$
\Lambda_l(\zeta) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \zeta & & & 0 \end{bmatrix},
$$

and $\mathrm{diag}[a_j(0), \cdots, a_j(l-1)]_l$ denotes a diagonal matrix $\mathrm{diag}(\cdots, a_j(0), \cdots, a_j(l-1), a_j(0), \cdots, a_j(l-1), \cdots]$. Under this isomorphism, L, M, $W^{(\infty)}(x, y)$, $W^{(0)}(x, y)$ are identified with $L(\zeta)$, $M(\zeta)$, $W^{(\infty)}(x, y; \zeta)$, $W^{(0)}(x, y; \zeta) \in \mathfrak{gl}(l, \boldsymbol{C}[[\zeta, \zeta^{-1}]])$, which take the form

$$L(\zeta)=W^{(\infty)}(x,y;\zeta)\Lambda_l(\zeta)W^{(\infty)}(x,y;\zeta)^{-1},$$
$$M(\zeta)=W^{(0)}(x,y;\zeta)\Lambda_l(\zeta)^{-1}W^{(0)}(x,y;\zeta)^{-1},$$
$$W^{(\infty)}(x,y;\zeta)=\hat{W}^{(\infty)}(x,y;\zeta)\exp\xi(x,\Lambda_l(\zeta)),$$
$$W^{(0)}(x,y;\zeta)=\hat{W}^{(0)}(x,y;\zeta)\exp\xi(y,\Lambda_l(\zeta)^{-1}),$$
$$\hat{W}^{\binom{0}{\infty}}(x,y;\zeta)=\sum_{j=0}^{\infty}\mathrm{diag}[\hat{w}_j^{\binom{0}{\infty}}(0;x,y),\cdots,\hat{w}_j^{\binom{0}{\infty}}(l-1;x,y)]\Lambda_l(\zeta)^{\pm j}.$$

Proposition 2. $W^{\binom{0}{\infty}}(x,y;\zeta)$ *solve the linear equations*

(12) $\quad \partial_{x_n}W(\zeta)=B_n(\zeta)W(\zeta),\quad \partial_{y_n}W(\zeta)=C_n(\zeta)W(\zeta),\quad n=1,2,\cdots,$

where $B_n(\zeta)=[L(\zeta)^n]_+$, $C_n(\zeta)=[M(\zeta)^n]_-$. *The symbols* $[A(\zeta)]_\pm$ *stand for the part of* $A(\zeta)$ *of non-negative order, and of strictly negative order with respect to* $\Lambda_l(\zeta)$, *respectively. The compatibility condition for* (12) *gives the (TL) hierarchy, and* $B_n(\zeta)$, $C_n(\zeta)$ *belong to* $\mathfrak{sl}(l,\mathbf{C}[\zeta,\zeta^{-1}])$.

2. The Riemann-Hilbert decomposition and its application. We define $\hat{V}^{\binom{0}{\infty}}(x,y)$ for wave matrices $W^{\binom{0}{\infty}}(x,y)$ by $W^{\binom{0}{\infty}}(x,y)=\hat{V}^{\binom{0}{\infty}}(x,y)\exp(\xi(x,\Lambda)+\xi(y,\Lambda^{-1}))$. They take the form $\hat{V}^{\binom{0}{\infty}}(x,y)=\sum_{j=0}^{\infty}\mathrm{diag}[\hat{v}_j^{\binom{0}{\infty}}(s:x,y)]\Lambda^{\pm j}$, and $\hat{v}_0^{(\infty)}(s;x,y)=1$. Then the bilinear relation (6) implies that there exists an invertible matrix A of size $Z\times Z$ such that

(13) $$\hat{V}^{(0)}(x,y)=\hat{V}^{(\infty)}(x,y)H(x,y)$$

where $H(x,y)=\exp(\xi(x,\Lambda)+\xi(y,\Lambda^{-1}))\ A\ \exp(\xi(-x,\Lambda)+\xi(-y,\Lambda^{-1}))$. This implies decomposition of $H(x,y)$ to the upper triangular matrix $\hat{V}^{(0)}$ and the lower triangular matrix $\hat{V}^{(\infty)}$, and is regarded as an infinite dimensional generalization of the classical Riemann-Hilbert (RH) problem. In fact, if $A\in\mathfrak{gl}((\infty))_l$ then (13) reduces to the classical RH problem. Therefore (13) will be called the RH decomposition.

Proposition 3. *Let* $A=I+\sum_{j=1}^{N}a_jX_{p_jq_j}$ *in* (13), *where*

$$X_{pq}=\sum_{m,n\in Z}p^mq^{-n}E_{mn}\ (E_{mn}=(\delta_{mi}\delta_{nj})_{i,j\in Z})$$

is the so-called vertex operator [3]. *Suppose* $a_j>0$, *and* $0<q_N<\cdots<q_1<p_1<\cdots<p_N$. *Then the RH decomposition has a unique pair of solutions* $\hat{V}^{\binom{0}{\infty}}(x,y)$, *and their entries are expressed in the form*

$$\hat{v}_j^{(\infty)}(s;x,y)=p_j(-\tilde{\partial}_x)\tau'(s;x,y)/\tau'(s;x,y),$$
$$\hat{v}_j^{(0)}(s;x,y)=p_j(-\tilde{\partial}_y)\tau'(s+1;x,y)/\tau'(s;x,y).$$

Here the τ *functions,* $\tau'(s;x,y)$, *is given by*

(14) $$\tau'(s;x,y)=\sum_{l=0}^{N}\sum_{i_1<\cdots<i_l}c_{i_1\cdots i_l}a_{i_1}(s)\cdots a_{i_l}(s)\exp\Bigl(\sum_{\mu=1}^{l}\eta(p_{i_\mu})-\eta(q_{i_\mu})\Bigr)$$

where

$$a_i(s)=a_i(p_i/q_i)^sq_i/(p_i-q_i),$$
$$c_{i_1\cdots i_l}=\sum_{1\le\mu<\nu\le l}c_{i_\mu i_\nu},\quad c_{ij}=(p_i-p_j)(q_i-q_j)/(p_i-q_j)(q_i-p_j),$$
$$\eta(p)=\xi(x,p)+\xi(y,p^{-1}).$$

The τ function (14) coincides with the τ function introduced in [5], [6].

3. Remarks. (1) Setting $A=I+\sum_{j=1}^{r} a_j E_{m_j n_j}$ and $y=0$ in (13), one can obtain rational solutions of the KP hierarchy (cf. Theorem 5 in [1]).

(2) The TL hierarchies of orthogonal or symplectic type, and the multi-component TL hierarchy can be also considered.

These topics are investigated in detail in [2].

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References

[1] K. Ueno and K. Takasaki: Toda lattice hierarchy I. RIMS preprint (1983).
[2] ——: Toda lattice hierarchy. RIMS preprint (1983) (to appear in Advanced Study in Pure. Math.).
[3] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa: Transformation groups for soliton equations II. Proc. Japan Acad., **57**A, 387–392 (1981); ditto. III. J. Phys. Soc. Japan, **40**, 3806–3812 (1981).
[4] A. V. Mikhailov, M. A. Olshanetsky, and A. M. Perelomov: Two-dimensional generalized Toda lattice. Commun. Math. Phys., **79**, 473–488 (1981).
[5] R. Hirota: Discrete analogue of the generalized Toda equation. J. Phys. Soc. Japan, **50**, 3875 (1981); Tech. Rep., A-6, A-9, Hiroshima Univ. (1981).
[6] T. Miwa: On Hirota's difference equation. Proc. Japan Acad., **58**A, 8–11 (1982).