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Metrics and Related Equations

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Reprinted from

PUBLICATIONS OF THE RESEARCH INSTITUTE FOR

MATHEMATICAL SCIENCES

KYOTO UNIVERSITY

Vol. 22, No. 5, 1986

Aspects of Integrability in Self-Dual Einstein Metrics and Related Equations

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Abstract

The nonlinear system describing self-dual Einstein metrics and its generalizations are discussed from the point of view of integrability. It is shown that these nonlinear systems share a variety of remarkable features (such as the existence of a linear scattering problem, a group-theoretical solution technique similar to the Riemann-Hilbert problem, and a geometric interpretation as dynamical motion in an infinite dimensional Grassmann manifold) with nonlinear integrable systems known until now. Differences of the relevant group-theoretical structures between these two classes of nonlinear systems are also pointed out. These results lead to the conclusion that the nonlinear systems in question do form a new class of nonlinear integrable systems.

Introduction

Along with the developments in the last decade the notion of “complete integrability,” or “integrability” for brevity, has become considerably familiar to us. Some of its mathematical issues, nevertheless, seem to remain to be fully solved. A basic question in this context is, for example, how far the notion of nonlinear integrable systems can be really extended. In the beginning of this field just a few examples, including the celebrated Korteweg-de Vries equation, were known as nonlinear integrable systems, but after the discovery and progress of a number of techniques we now have an enormous list of “soliton equations” [1–4]. There are also attempts of quantization of these equations [5]. Further, some of these techniques later turned out to be also applicable to the self-dual Yang-Mills equations [6–10]. Recently their higher dimensional [11] and supersymmetric [12–14] analogues were investigated along the same lines. The frontier of what should be called nonlinear integrable

Received May 13, 1986.

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systems thus has been (and, perhaps, is still) expanding continuously.

The purpose of this paper is to show that the system of nonlinear equations defining self-dual Einstein metrics, which we call the *self-dual Einstein equations*, share various remarkable features with nonlinear integrable systems known until now. This means that the self-dual Einstein equations should be considered a sort of nonlinear integrable system. Such a point of view is in fact not very new. Rather, in view of the fact that the integrability of the self-dual Yang-Mills equations is a consequence of their twistor-theoretical interpretation [15], it would be quite natural to expect a similar situation for the self-dual Einstein equations, because the latter also admit a twistorial description due to Penrose [16]. The recent work of Boyer and Plebanski [17, 18] seems to be indeed based, at least partly, on such an idea. This paper is intended to present a more detailed analysis on this issue.

Of course the notion of integrability includes in general a variety of contents, but this paper deals with the self-dual Einstein equations from the following three aspects (which are in fact closely connected with each other):

- i) The existence of a linear “scattering” problem whose integrability conditions in the sense of Frobenius agree with the nonlinear system in question.
- ii) A group-theoretical description of solutions with the aid of the Riemann-Hilbert problem or of its appropriate analogue.
- iii) A geometric interpretation of the nonlinear system as some dynamical motion in a symmetric space (in particular, in an infinite dimensional Grassmann manifold).

Detailed discussions on these aspects will be presented in the subsequent sections (see §2 for (i), §4 for (ii) and §5 for (iii)). It would be however worth mentioning here in advance that group-theoretical structures relevant to the self-dual Einstein equations are somewhat distinct from (though seemingly fairly similar to) the case of more classical examples of nonlinear integrable systems. For example, as already pointed by Boyer and Plebanski [18], an infinite dimensional Lie algebra characteristic of the present case takes the form of a tensor product $\mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ where λ is a parameter and \mathfrak{g} a Lie algebra of vector fields which is infinite dimensional in itself; on the other hand for soliton equations [2–4] and the self-dual or supersymmetric Yang-Mills equations [10, 13, 14] the same role is played by Kac-Moody-type Lie algebras, which are typically written $\mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ with \mathfrak{g} finite dimensional Lie algebras. For this and some other reasons to be made clear in the subsequent sections, it

seems very natural to recognize the self-dual Einstein equations as an essentially new type of nonlinear integrable system. Besides, this is by no means an isolated example; it will be indeed shown in §3 that there is a broad class of nonlinear systems that should be considered “integrable” in the same sense.

§1. Potentials for Self-Dual Einstein Metrics

1.1. Self-Dual Einstein Metrics

Let $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ denote a four-dimensional Riemannian metric with coordinates $x = (x^1, x^2, x^3, x^4)$, and $R_{\alpha\beta\mu\nu}$ the components of the Riemann curvature forms. The metric ds^2 is said to be a *self-dual Einstein metric* if the Riemann curvature forms are self-dual with respect to the Hodge *-operator, i.e.

$$(1.1) \quad *R_{\alpha\beta\mu\nu}dx^\mu \wedge dx^\nu = R_{\alpha\beta\mu\nu}dx^\mu \wedge dx^\nu.$$

The usual (vacuum) Einstein equations $R_{\mu\nu} = 0$ automatically follow from Eq. (1.1) (see, for example, [19, 20]), thus self-dual Einstein metrics form part of Einstein metrics. We call Eqs. (1.1) the self-dual Einstein equations.

Since the self-dual Einstein equations, as well as the Einstein equations, include only rational functions of derivatives of the metric components $g_{\mu\nu}(x)$, one may readily replace the real coordinates $x = (x^1, x^2, x^3, x^4)$ by complex ones $z = (z^1, z^2, z^3, z^4)$ and consider these equations in complex domains. In this paper we mainly deal with these “complexified” equations, calling them also the self-dual Einstein equations.

1.2. Null-Tetrad Formalism

Plebanski [20] pointed out that null-tetrad formalism is very useful for the study of self-dual Einstein metrics. In this formalism a metric is written

$$(1.2) \quad ds^2 = 2e^1e^2 + 2e^3e^4 = 2 \det \begin{pmatrix} e^3 & e^1 \\ -e^2 & e^4 \end{pmatrix},$$

where e^1, \dots, e^4 denote a tetrad (vierbein), i.e. linearly independent 1-forms $e^a = e^a_\mu dz^\mu$. Writing the metric as above is not unique; there remains the following arbitrariness:

$$(1.3) \quad \begin{pmatrix} e^3 & e^1 \\ -e^2 & e^4 \end{pmatrix} \longrightarrow l \begin{pmatrix} e^3 & e^1 \\ -e^2 & e^4 \end{pmatrix} l'$$

where $l=l(z)$ and $l'=l'(z)$ are $SL(2, \mathbf{C})$ -valued arbitrary functions. We call these transformations induced by l and l' , respectively, *left and right $SL(2, \mathbf{C})$ gauge transformations of the tetrad*.

Main results of Plebanski [20] can be summarized as follows. First, by choosing an appropriate right $SL(2, \mathbf{C})$ gauge transformation the self-dual Einstein equations for metric (1.2) can be reduced to the exterior differential equations

$$(1.4) \quad d(e^2 \wedge e^3)=0, \quad d(e^3 \wedge e^4 - e^1 \wedge e^2)=0, \quad d(e^1 \wedge e^4)=0.$$

By introducing a new parameter λ , they can be written more compactly as:

$$(1.5) \quad d((e^3 + \lambda e^1) \wedge (-e^2 + \lambda e^4))=0 \quad (\lambda \in \mathbf{C}).$$

Second, performing further a left $SL(2, \mathbf{C})$ gauge transformation one can bring the tetrad into simple canonical forms. Actually Plebanski presented two sorts of such canonical forms, one of which takes the following form:

$$(1.6) \quad \begin{aligned} e^3 &= dx - \Theta_{xy} dp - \Theta_{yy} dq, & e^1 &= dp, \\ -e^2 &= dy + \Theta_{xx} dp + \Theta_{xy} dq, & e^4 &= dq, \end{aligned}$$

where (p, q, x, y) are appropriate coordinates and $\Theta = \Theta(p, q, x, y)$ is a solution of the equation (second heavenly equation)

$$(1.7) \quad \Theta_{xq} - \Theta_{yp} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0,$$

whereas the indices to Θ as usual denote the differentiation with respect to the variables indicated therein. It should be also noted that Eqs. (1.4) are form-invariant under left $SL(2, \mathbf{C})$ gauge transformations.

1.3. Definition of u -Potentials

What play central roles throughout this paper are potentials $u_n^\alpha = u_n^\alpha(z)$ and $\hat{u}_n^\alpha = \hat{u}_n^\alpha(z)$ ($n=1, 0, -1, \dots, \alpha=1, 2$) and their generating functions (which we call, for brevity, *u -potentials*)

$$u^\alpha = \sum_{n=-\infty}^1 u_n^\alpha \lambda^n, \quad \hat{u}^\alpha = \sum_{n=0}^{\infty} \hat{u}_n^\alpha \lambda^n \quad (\alpha=1, 2)$$

to be defined respectively as a solution of the exterior differential equations

$$(1.8) \quad du^1 \wedge du^2 = (e^3 + \lambda e^1) \wedge (-e^2 + \lambda e^4),$$

$$(1.9) \quad d\hat{u}^1 \wedge d\hat{u}^2 = (e^3 + \lambda e^1) \wedge (-e^2 + \lambda e^4),$$

where (and also throughout this paper) λ denotes a parameter moving in the Riemann sphere \mathbf{P}^1 , and d the total differentiation *with respect to space-time coordinates alone* (i.e. $d\lambda=0$). In general these potentials u^α and \hat{u}^α are required to be just formal Laurent series of λ (though in applications to various actual solutions they can be chosen convergent ones; see §4). The existence of such potentials evidently implies that the tetrad satisfies Eqs. (1.4), because the left side of Eqs. (1.8) and (1.9) are always closed forms. What is important is that the converse is also true:

Proposition (1.10). *For a given tetrad such potentials u^α and \hat{u}^α do exist if and only if Eqs. (1.4) are satisfied.*

In other words Eqs. (1.4) are exactly the integrability conditions for Eqs. (1.8) and (1.9) to have a solution (*not unique*).

There are various ways to verify the above basic fact. Newman et al. argued a construction of such potentials in [21], which seems to be an earliest attempt that introduced equations like (1.8) and (1.9) into the study of the self-dual Einstein equations. Boyer and Plebanski [17, 18] presented another construction exploiting a geometric formalism. (In fact, what Boyer and Plebanski called “potentials” are different from ours; they used this word to mean another sort of potential functions including the function Θ .) According to their argument, our u -potentials can be identified with coordinate components of a section of a certain infinite dimensional vector bundle (on the z -space) endowed with a sort of symplectic structure; the problem of solving Eqs. (1.8) and (1.9) then becomes equivalent to finding an isotropic section of this vector bundle. A common characteristic of these two constructions is to reduce the problem into solving an infinite system of equations which can be obtained by expanding the both sides of Eqs. (1.8) and (1.9) into Laurent series of λ . There, however, is another method that enables one to construct the u -potentials more directly. This method is an application of Darboux’s theorem [22] on the canonical forms of exterior differential 2-forms. This theorem shows that if a 2-form ω on a finite dimensional manifold is closed (i.e. $d\omega=0$) and for some integer r ($r \geq 1$) $\omega \wedge \dots \wedge \omega$ (r -fold) $\neq 0$ and $\omega \wedge \dots \wedge \omega$ ($r+1$ -fold) $= 0$, then at any point there are a neighborhood of this point and functions $P_1, \dots, P_r, Q_1, \dots, Q_r$ defined therein such that $dP_1 \wedge \dots \wedge dP_r \wedge dQ_1 \wedge \dots \wedge dQ_r \neq 0$ and $\omega = \sum_{i=1}^r dP_i \wedge dQ_i$. If the manifold is even, say $2n$, dimensional and $r=n$, then this is a

well known result in symplectic geometry; therefore following the terminology of symplectic geometry, let us call the functions $P_1, \dots, P_r, Q_1, \dots, Q_r$ *canonical variables*. Darboux's theorem can be extended to the case where the 2-form includes some additional parameters, and such a modified form of Darboux's theorem can be applied to the construction of the u -potentials. Indeed, the right side of Eqs. (1.8) and (1.9) is a closed 2-form with a parameter λ , provided the tetrad satisfies Eqs. (1.4). Furthermore, for this 2-form the integer r is equal to one. Therefore by virtue of Darboux's theorem one obtains a pair of canonical variables, which are exactly the u -potentials in question. Of course one should be careful here about the domains in the Riemann sphere in which the parameter λ moves. Namely, the two sorts of potentials u^α and \hat{u}^α both can be obtained as canonical variables in the canonical form of the 2-form $(e^1 + \lambda e^3) \wedge (-e^2 + \lambda e^4)$, but the domains in which the parameter λ is supposed to move are distinct; for the former ones the domains is a neighborhood of $\lambda = \infty$, whereas for the latter it is a neighborhood of $\lambda = 0$.

The notion of u -potentials forms a central theme of this paper; a variety of roles to be played by them will be described in detail in the following sections. It is however worth noting here that these potentials are also the most basic ingredients of the curved twistor construction of self-dual Einstein metrics due to Penrose [16]. For details, see §4; as we shall argue therein, these potentials can be regarded as representing a family of holomorphic curves in a curved twistor space.

1.4. u -Potentials and Canonical Form of Tetrad

We here show that the coordinates (p, q, x, y) can be identified with the leading and next-to-leading coefficients of the Laurent expansion of the potentials u^1 and u^2 ; this is a basic property of u -potentials, which sometimes enables us to save calculations to a great extent. To see the above fact, let us first note that Eq. (1.8) is equivalent to the system of equations

$$\begin{aligned} (1.10)_2 \quad & \omega^{(2)} = e^1 \wedge e^4, \\ (1.10)_1 \quad & \omega^{(1)} = e^3 \wedge e^4 - e^1 \wedge e^2, \\ (1.10)_0 \quad & \omega^{(0)} = e^2 \wedge e^3, \\ (1.10)_n \quad & \omega^{(n)} = 0 \quad (n < 0), \end{aligned}$$

where $\omega^{(n)}$ denotes the 2-form $\omega^{(n)} \equiv \sum_{m=n-1}^1 du_m^1 \wedge du_{n-m}^2$. From Eqs. (1.10)₂ and (1.10)₀ one finds that

$$(1.11) \quad du_1^1 \wedge du_1^2 \wedge du_0^1 \wedge du_0^2 = \omega^{(2)} \wedge \omega^{(0)} = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \neq 0,$$

therefore one may take $(u_1^1, u_1^2, u_0^1, u_0^2)$ as new space-time coordinates. We now prove:

Proposition (1.12). *Take $(p, q, x, y) \equiv (u_1^1, u_1^2, u_0^1, u_0^2)$ as space-time coordinates and regard the other Laurent coefficients as functions of (p, q, x, y) . Then there are an $SL(2, \mathbf{C})$ -valued function $l = (l_\alpha^\beta(p, q, x, y))$ and a scalar function $\Theta \equiv \Theta(p, q, x, y)$ satisfying the following equations:*

$$(1.12a) \quad \begin{pmatrix} e^3 & e^1 \\ -e^2 & e^4 \end{pmatrix} = l \begin{pmatrix} dx - \Theta_{xy} dp - \Theta_{yy} dq & dp \\ dy + \Theta_{xx} dp + \Theta_{xy} dq & dq \end{pmatrix},$$

$$(1.12b) \quad \Theta_{xq} - \Theta_{yp} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0,$$

$$(1.12c) \quad u_{-1}^1 = \Theta_y, \quad u_{-1}^2 = -\Theta_x.$$

Proof. From Eqs. (1.10)₁ and (1.10)₀

$$\omega^{(1)} \wedge \omega^{(0)} = (e^3 \wedge e^4 - e^1 \wedge e^2) \wedge e^2 \wedge e^3 = 0,$$

$$\omega^{(0)} \wedge \omega^{(0)} = e^2 \wedge e^3 \wedge e^2 \wedge e^3 = 0,$$

whereas in terms of the coordinates (p, q, x, y) one can calculate these 2-forms as:

$$\omega^{(1)} \wedge \omega^{(0)} = -dp \wedge dq \wedge dx \wedge dy \left(\frac{\partial u_{-1}^1}{\partial x} + \frac{\partial u_{-1}^2}{\partial y} \right),$$

$$\begin{aligned} \omega^{(0)} \wedge \omega^{(0)} = 2dp \wedge dq \wedge dx \wedge dy \left(\frac{\partial u_{-1}^1}{\partial p} + \frac{\partial u_{-1}^2}{\partial q} \right. \\ \left. + \frac{\partial u_{-1}^2}{\partial x} \frac{\partial u_{-1}^1}{\partial y} - \frac{\partial u_{-1}^1}{\partial y} \frac{\partial u_{-1}^2}{\partial x} \right), \end{aligned}$$

therefore one obtains the equations

$$(1.13a) \quad \frac{\partial u_{-1}^1}{\partial x} + \frac{\partial u_{-1}^2}{\partial y} = 0,$$

$$(1.13b) \quad \frac{\partial u_{-1}^1}{\partial p} + \frac{\partial u_{-1}^2}{\partial q} + \frac{\partial u_{-1}^2}{\partial x} \frac{\partial u_{-1}^1}{\partial y} - \frac{\partial u_{-1}^1}{\partial y} \frac{\partial u_{-1}^2}{\partial x} = 0.$$

Eq. (1.13a) ensures the existence of a function Θ that satisfies Eq. (1.12c), and inserting this expression of u_{-1}^1 and u_{-1}^2 into Eq. (1.13b) one obtains Eq. (1.12b). Thus what remains is to prove the existence of a left $SL(2, \mathbf{C})$ gauge transformation that connects the tetrad (e^1, \dots, e^4) with the following one:

$$(1.14) \quad \begin{aligned} \tilde{e}^3 &= dx - \Theta_{xy} dp - \Theta_{yy} dq, & \tilde{e}^1 &= dp, \\ -\tilde{e}^2 &= dy + \Theta_{xx} dp + \Theta_{xy} dq, & \tilde{e}^4 &= dq. \end{aligned}$$

To check this, let us note the equations

$$\begin{aligned} e^1 \wedge e^4 &= \omega^{(2)} = \tilde{e}^1 \wedge \tilde{e}^4, \\ e^3 \wedge e^4 - e^1 \wedge e^2 &= \omega^{(1)} = \tilde{e}^3 \wedge \tilde{e}^4 - \tilde{e}^1 \wedge \tilde{e}^2, \\ e^2 \wedge e^3 &= \omega^{(0)} = \tilde{e}^1 \wedge \tilde{e}^4, \end{aligned}$$

which follow from Eqs. (1.10) and (1.13). It is then a simple exercise of linear algebra to show from the last equations that there is certainly a left $SL(2, \mathbb{C})$ gauge transformation as such. This completes the proof of the proposition.

§2. Linear Scattering Problem for Self-Dual Einstein Metrics

2.1. Derivation of Linear Problem

In the preceding section we introduced the notion of u -potentials. We call these new dependent variables “potentials” (or “quasi-potentials”, according to the terminology of Estabrook and Wahlquist [23]) because they are defined as solutions of differential equations, i.e. (1.8) and (1.9), whose integrability conditions coincide with the equations in question, i.e. Eqs. (1.4). Eqs. (1.8) and (1.9) are however neither linear nor quasi-linear, thus the situation is seemingly very different from various (quasi-) potentials (in particular, “wave functions” in linear scattering problems) to be defined for nonlinear integrable systems known until now [1–14]. Thus naturally occurs a question, whether or not there is a linear problem which characterizes our u -potentials. The conclusion we here derive is that such a linear problem certainly exists if one admits a nonlinear constraint to be added to the relevant linear equations.

In order to give a precise statement, we introduce the notion of dual (or inverse) tetrad. The dual tetrad of a given tetrad (e^1, \dots, e^4) is by definition the tangent frame $(\partial_1, \dots, \partial_4)$ of vector fields to be determined by the equations

$$(2.1) \quad \langle e^a, \partial_b \rangle = \delta_b^a \quad \text{for } a, b = 1, \dots, 4,$$

where $\langle \ , \ \rangle$ denotes the natural inner product of 1-forms and vector fields, and δ_b^a the Kronecker delta. A basic property of the dual tetrad is that the following formula holds for any function $h = h(z)$:

$$(2.2) \quad dh = \sum_{a=1}^4 (\partial_a h) e^a.$$

The linear problem in question can be obtained by rewriting Eqs. (1.8) and

(1.9) using the dual tetrad of the tetrad included therein. Actually it has two equivalent representations, one of which takes the following form:

Proposition (2.3). *Let $(\partial_1, \dots, \partial_4)$ denote the dual tetrad of the tetrad (e^1, \dots, e^4) . Then Eq. (1.8) is equivalent to the system of equations*

$$(2.3a) \quad \begin{pmatrix} du^1 \\ du^2 \end{pmatrix} = \begin{pmatrix} \partial_3 u^1 & -\partial_2 u^1 \\ \partial_3 u^2 & -\partial_2 u^2 \end{pmatrix} \begin{pmatrix} e^3 + \lambda e^1 \\ -e^2 + \lambda e^4 \end{pmatrix},$$

$$(2.3b) \quad (\partial_2 u^1)(\partial_3 u^2) - (\partial_3 u^1)(\partial_2 u^2) = 1.$$

Similarly, Eq. (1.9) is equivalent to the same equations in which u^1 and u^2 are replaced by \hat{u}^1 and \hat{u}^2 .

Proof. It is obvious that Eq. (1.8) follows from Eqs. (2.3), so let us check the converse. In view of (2.2) one can expand the 2-form $du^1 \wedge du^2$ as:

$$(2.4) \quad \begin{aligned} du^1 \wedge du^2 &= \sum_{a < b} du^1 \wedge du^2(\partial_a, \partial_b), \quad \text{where} \\ du^1 \wedge du^2(\partial_a, \partial_b) &\equiv (\partial_a u^1)(\partial_b u^2) - (\partial_a u^2)(\partial_b u^1), \end{aligned}$$

therefore equating the coefficients of $e^a \wedge e^b$ in Eq. (1.8) one obtains:

$$(2.5) \quad \begin{aligned} du^1 \wedge du^2(\partial_1, \partial_2) &= -\lambda, & du^1 \wedge du^2(\partial_1, \partial_3) &= 0, \\ du^1 \wedge du^2(\partial_1, \partial_4) &= \lambda^2, & du^1 \wedge du^2(\partial_2, \partial_3) &= 1, \\ du^1 \wedge du^2(\partial_2, \partial_4) &= 0, & du^1 \wedge du^2(\partial_3, \partial_4) &= \lambda. \end{aligned}$$

The fourth equation in (2.5) is exactly Eq. (2.3b). This in particular implies:

$$\begin{pmatrix} \partial_3 u^1 & -\partial_2 u^1 \\ \partial_3 u^2 & -\partial_2 u^2 \end{pmatrix}^{-1} \begin{pmatrix} du^1 \\ du^2 \end{pmatrix} = \begin{pmatrix} -(\partial_2 u^2)du^1 + (\partial_2 u^1)du^2 \\ -(\partial_3 u^2)du^1 + (\partial_3 u^1)du^2 \end{pmatrix},$$

and calculating the right side further using Eqs. (2.5), one finds:

$$= \begin{pmatrix} e^3 + \lambda e^1 \\ -e^2 + \lambda e^4 \end{pmatrix},$$

thus it turns out that Eq. (2.3a), too, follows from Eq. (1.8). The latter half of the proposition can be checked the same way. This completes the proof of the proposition.

We now give another representation of the linear problem:

Proposition (2.6). *Let $(\partial_1, \dots, \partial_4)$ denote the dual tetrad of the tetrad (e^1, \dots, e^4) . Then Eq. (1.8) is equivalent to the system of equations*

$$(2.6a) \quad (-\lambda\partial_3 + \partial_1)u = 0, \quad (\lambda\partial_2 + \partial_4)u = 0 \quad (u = u^1, u^2),$$

$$(2.6b) \quad (\partial_2 u^1)(\partial_3 u^2) - (\partial_3 u^1)(\partial_2 u^2) = 1.$$

Similarly, Eq. (1.9) is equivalent to the same equations in which u^1 and u^2 are replaced by \hat{u}^1 and \hat{u}^2 .

Proof. One may rewrite Eq. (2.3a) into the following form:

$$du = (\lambda\partial_3u)e^1 + (\partial_2u)e^2 + (\partial_3u)e^3 - (\lambda\partial_2u)e^4 \quad (u = u^1, u^2).$$

On the other hand by virtue of (2.2) the 1-form du may be in general written

$$du = (\partial_1u)e^1 + (\partial_2u)e^2 + (\partial_3u)e^3 + (\partial_4u)e^4.$$

Therefore equating the coefficients of e^1 and e^4 one readily finds that Eqs. (2.6) are equivalent to Eqs. (2.3) and, consequently, to Eq. (1.8). The latter half of the proposition can be verified the same way. This completes the proof of the proposition.

The last system of equations, (2.6), is what we have sought for. This system indeed takes the form of a linear problem coupled with a nonlinear constraint, and the u -potentials are characterized by this system.

2.2. Symplectic Structure in Linear Problem

We here show that a sort of symplectic structure is underlying linear problem (2.6) and explains the meaning of nonlinear constraint (2.6b). This symplectic structure manifests itself when, as Proposition (1.12) ensures, the tetrad is cast into the second canonical form of Plebanski, (1.6). In order to see this, we use the following lemmas which can be readily checked:

Lemma (2.7). *Under a transformation of the tetrad as in (1.3) the corresponding dual tetrad changes as:*

$$\begin{pmatrix} \partial_3 & \partial_1 \\ -\partial_2 & \partial_4 \end{pmatrix} \longrightarrow {}^l l^{-1} \begin{pmatrix} \partial_3 & \partial_1 \\ -\partial_2 & \partial_4 \end{pmatrix} {}^{l'} l'^{-1}.$$

Lemma (2.8). *The dual tetrad of tetrad (1.6) is:*

$$\begin{pmatrix} \partial_3 & \partial_1 \\ -\partial_2 & \partial_4 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial p} - \Theta_{xx} \frac{\partial}{\partial y} + \Theta_{xy} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial q} - \Theta_{xy} \frac{\partial}{\partial y} + \Theta_{yy} \frac{\partial}{\partial x} \end{pmatrix}.$$

We call transformations induced by l and l' as above, respectively, *left and right* $SL(2, \mathbf{C})$ *gauge transformations of the dual tetrad*. Note that Eqs. (2.6) are form-invariant under left $SL(2, \mathbf{C})$ gauge transformations.

Let us now reduce Eqs. (2.6) into a more explicit form by using the above

lemmas. These lemmas, applied to the situation as shown in Proposition (1.12), show that the dual tetrad $(\partial_1, \dots, \partial_4)$ takes the following form:

$$(2.9) \quad \begin{pmatrix} \partial_3 & \partial_1 \\ -\partial_2 & \partial_4 \end{pmatrix} = {}^t l^{-1} \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial p} - \Theta_{xx} \frac{\partial}{\partial y} + \Theta_{xy} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial q} - \Theta_{xy} \frac{\partial}{\partial y} + \Theta_{yy} \frac{\partial}{\partial x} \end{pmatrix}.$$

In view of the invariance property of linear problem (2.6) mentioned above, it finally turns out that the linear problem can always be set into the following form:

$$(2.10a) \quad \begin{aligned} & \left(-\lambda \frac{\partial}{\partial x} + \frac{\partial}{\partial p} - \Theta_{xx} \frac{\partial}{\partial y} + \Theta_{xy} \frac{\partial}{\partial x} \right) u = 0, \\ & \left(-\lambda \frac{\partial}{\partial y} + \frac{\partial}{\partial q} - \Theta_{xy} \frac{\partial}{\partial y} + \Theta_{yy} \frac{\partial}{\partial x} \right) u = 0 \quad (u = u^1, u^2), \end{aligned}$$

$$(2.10b) \quad \frac{\partial(u^1, u^2)}{\partial(x, y)} \equiv \frac{\partial u^1}{\partial x} \frac{\partial u^2}{\partial y} - \frac{\partial u^1}{\partial y} \frac{\partial u^2}{\partial x} = 1,$$

where of course the following relations are assumed:

$$(2.11) \quad (u^1_1, u^2_1, u^1_0, u^2_0) = (p, q, x, y), \quad u^1_1 = \Theta_y, \quad u^2_1 = -\Theta_x.$$

It is now evident that the linear problem, written in the reduced form (2.10), is in close connection with the symplectic structure to be defined in the (x, y) -space with symplectic form $dx \wedge dy$. Eq. (2.10b) implies that the two dimensional map $(x, y) \rightarrow (u^1, u^2)$ is a canonical transformation with parameters (p, q, λ) ; this clearly explains the meaning of the nonlinear constraint in the linear problem. In addition, the linear part (2.10a) also reflects the structure of this symplectic form; indeed, the vector fields in Eqs. (2.10a) take the form of Hamiltonian vector fields. To be more precise, we introduce the following notation:

$$(2.12) \quad \begin{aligned} H_h &\equiv \frac{\partial h}{\partial x} \frac{\partial}{\partial y} - \frac{\partial h}{\partial y} \frac{\partial}{\partial x} \quad (\text{Hamiltonian vector field}), \\ \{h_1, h_2\} &\equiv H_{h_1} h_2 = \frac{\partial h_1}{\partial x} \frac{\partial h_2}{\partial y} - \frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial x} \quad (\text{Poisson bracket}). \end{aligned}$$

Then Eqs. (2.10) can be written

$$(2.13a) \quad \begin{aligned} & \left(-\lambda \frac{\partial}{\partial x} + \frac{\partial}{\partial p} - H_{\Theta_x} \right) u = 0, \quad \left(-\lambda \frac{\partial}{\partial y} + \frac{\partial}{\partial q} - H_{\Theta_y} \right) u = 0 \\ & \hspace{20em} (u = u^1, u^2), \end{aligned}$$

$$(2.13b) \quad \{u^1, u^2\} = 1,$$

thus giving a representation of the linear problem in which the relation to the

above-mentioned symplectic structure is made fully manifest.

2.3. Comparison with Other Nonlinear Integrable Systems

We now consider the implications of the results we have thus far derived. The motivation of our present discussion was the question of whether or not the u -potentials can be characterized as solutions of some linear problem. As a result we have found the system of equations (2.6) which is equivalent to Eq. (1.8) and, therefore, whose integrability conditions agree exactly with Eqs. (1.4). Does this system fulfill all requirements to be a linear "scattering" problem of the self-dual Einstein equations?

One might feel this system of equations still insufficient as such a linear problem, because it includes an essentially nonlinear equation, (2.6b); however this is *not correct*. In fact, linear scattering problems for various nonlinear integrable systems known by now are also frequently accompanied with similar nonlinear constraints; a main role of such constraints is to derive the equations in question from more general ones as their reductions. Nonlinear constraints used for that purpose usually take the form of the condition that the unknown function (wave function) of the relevant linear problem takes values in a group. Nonlinear constraint (2.6b) is evidently of the same nature; indeed, as we have remarked above, this constraint means that a certain two-dimensional map defined by the u -potentials takes values in the group (to be more precise, pseudo-group; see §4) of canonical transformations.

In order to make this analogy more explicit, let us attempt to compare the above linear problem for the self-dual Einstein equations with that of the self-dual Yang-Mills equations. The self-dual Yang-Mills equations in flat space-time have a "zero-curvature representation" [6-10] such as:

$$(2.14) \quad \left[-\lambda \left(\frac{\partial}{\partial x} + A_x \right) + \frac{\partial}{\partial p} + A_p, -\lambda \left(\frac{\partial}{\partial y} + A_y \right) + \frac{\partial}{\partial q} + A_q \right] = 0,$$

where (p, q, x, y) denote some complexified space-time coordinates and (A_p, A_q, A_x, A_y) Yang-Mills gauge potentials with values in a matrix Lie algebra \mathfrak{g} . This equation gives a representation of the integrability conditions of the linear problem

$$(2.15) \quad \begin{aligned} & \left(-\lambda \left(\frac{\partial}{\partial x} + A_x \right) + \frac{\partial}{\partial p} + A_p \right) W = 0, \\ & \left(-\lambda \left(\frac{\partial}{\partial y} + A_y \right) + \frac{\partial}{\partial q} + A_q \right) W = 0, \end{aligned}$$

where $W = W(p, q, x, y, \lambda)$ is a matrix-valued unknown function (wave function), usually required to take values in the Lie group G of the Lie algebra \mathfrak{g} . The last requirement, which ensures that the gauge potentials surely take values in \mathfrak{g} , becomes in general a nonlinear constraint to the wave function; imagine, for example, the case for $G = \text{SL}(r, \mathbf{C})$. It would be now quite clear from these observations that the linear problem for the self-dual Einstein equations has a structure very similar to that of the self-dual Yang-Mills equations.

Also significant, however, is the difference of the structure of groups relevant to the linear problems. Indeed in the case of the self-dual Yang-Mills equations, as well as other nonlinear integrable systems, the wave function takes values in a finite dimensional matrix group; whereas in the case of the self-dual Einstein equations the same role is played by the (pseudo-)group of two dimensional canonical transformations, an *infinite dimensional* object.

What we have viewed above is by no means superficial. In §4 we shall once again examine this issue from a somewhat different point of view.

§3. Generalizations of Self-Dual Einstein Equations from Point of View of Integrability

3.1. Simplest Example

As we have viewed, the self-dual Einstein equations (to be more precise, the reduced form of Plebanski (1.4)) may be interpreted as the integrability conditions of a certain linear problem. This is one of remarkable features that the self-dual Einstein equations share with various nonlinear integrable systems known until now. Bearing the above fact in mind, in this section we seek for other examples with the same features, and attempt to find some general aspects of this class of nonlinear equations.

One of the simplest ways to obtain such an example is to consider the integrability conditions of the linear system

$$(3.1) \quad (-\lambda\partial_3 + \partial_1)u = 0, \quad (\lambda\partial_2 + \partial_4)u = 0$$

without any constraint. Here “integrability” means, as one can imagine from our previous arguments, that this linear system has a pair of functionally independent solutions; a more precise formulation of this statement will be given later in 3.2. As we shall see therein, the integrability conditions of the above

linear problem in this sense take the form of a commutator relation such as:

$$(3.2) \quad [-\lambda\partial_3 + \partial_1, \lambda\partial_2 + \partial_4] \\ = (C_0 + \lambda C_1)(-\lambda\partial_3 + \partial_1) + (D_0 + \lambda D_1)(\lambda\partial_2 + \partial_4),$$

where C_0 , C_1 , D_0 and D_1 are some functions independent of λ to be determined by the frame of vector fields $(\partial_1, \dots, \partial_4)$.

From the above expression of the integrability conditions, however, the relation to the self-dual Einstein equations is still less clear. To improve this, let us recall that there are various equivalent formulations of Frobenius' theorem [22]. What we used above is one of them that employs vector fields. Another one formulates the problem as the integrability of a Pfaffian system. In the present setting the Pfaffian system corresponding to linear system (3.1) can be written

$$(3.3) \quad e^3 + \lambda e^1 = 0, \quad -e^2 + \lambda e^4 = 0,$$

where e^1, \dots, e^4 denote the 1-forms satisfying duality relation (2.1), and λ is considered a parameter, i.e. $d\lambda = 0$. The integrability conditions of this Pfaffian system take the form of exterior differential equations such as:

$$(3.4) \quad d(e^3 + \lambda e^1) \wedge (e^3 + \lambda e^1) \wedge (-e^2 + \lambda e^4) = 0, \\ d(-e^2 + \lambda e^4) \wedge (e^3 + \lambda e^1) \wedge (-e^2 + \lambda e^4) = 0 \quad (\lambda \in \mathbf{C}),$$

which therefore give another equivalent representation of Eq. (3.2). Once the integrability conditions are written as in (3.4), it is quite easy to see the relation to the self-dual Einstein equations. Indeed, Eqs. (1.4) imply Eqs. (3.4), but the converse is not true in general; thus one obtains a generalization of the self-dual Einstein equations in the sense mentioned at the beginning.

The above representation of the equations is also convenient to see their geometric meaning. Indeed with the aid of arguments exploited in refs. [17, 20, 24], one can show easily that Eqs. (3.4) describe a class of *conformally self-dual metrics*. This however does not exhaust all conformally self-dual metrics. For the description of general conformally self-dual metrics some modification of the present setting is required; this issue will be discussed elsewhere.

We finally note that the above equations, derived as a generalization of Eqs. (1.4), also have a sort of gauge invariance. Namely, Eqs. (3.2) and (3.4) are respectively form-invariant under transformations of the vector fields and 1-forms such as:

$$(3.5) \quad \begin{pmatrix} e^3 & e^1 \\ -e^2 & e^4 \end{pmatrix} \longrightarrow l \begin{pmatrix} \partial^3 & e^1 \\ -e^2 & e^4 \end{pmatrix}, \quad \begin{pmatrix} \partial_3 & \partial_1 \\ -\partial_2 & \partial_4 \end{pmatrix} \longrightarrow {}^t l^{-1} \begin{pmatrix} \partial_3 & \partial_1 \\ -\partial_2 & \partial_4 \end{pmatrix},$$

where 2×2 matrices $l = (l_\alpha^\beta(z))$ ($\alpha, \beta = 1, 2$) are simply assumed to be invertible; in other words the relevant structure group is $GL(2, \mathbb{C})$ unlike the $SL(2, \mathbb{C})$ for the case of Eqs. (1.4). We call the above transformations *left $GL(2, \mathbb{C})$ gauge transformations*. As we shall see later, the 1-forms e^1, \dots, e^4 and the vector fields $\partial_1, \dots, \partial_4$ can be reduced to a canonical form by an appropriate left $GL(2, \mathbb{C})$ transformation.

3.2. Solutions of Linear Problem and Application of Frobenius' Theorem

We now give a precise formulation of the integrability of linear problem (3.1). If the presence of the parameter λ may be ignored, this is just a simple application of Frobenius' theorem in a well known form, but in fact one has to take into consideration the presence of the parameter λ , which requires a more careful analysis. Bearing this in mind, we here prove:

Proposition (3.6). *Given a frame $(\partial_1, \dots, \partial_4)$ of linealy independent vector fields, linear problem (3.1) has two formal Laurent series solutions u^α ($\alpha = 1, 2$) with the expansion $u^\alpha = \sum_{n=-\infty}^1 u_n^\alpha \lambda^n$, where u_n^α 's are functions of space-time coordinates alone with $du_1^1 \wedge du_2^1 \wedge du_0^1 \wedge du_0^2 \neq 0$, if and only if Eq. (3.2) is satisfied for some functions C_0, C_1, D_0 , and D_1 .*

We first consider the "if" part. Writing $u = \sum_{n=-\infty}^1 u_n \lambda^n$ and equating the coefficients of powers of λ one obtains the infinite system of equations

$$(3.7)_2 \quad -\partial_3 u_1 = 0, \quad \partial_2 u_1 = 0,$$

$$(3.7)_n \quad -\partial_3 u_{n-1} + \partial_1 u_n = 0, \quad \partial_2 u_{n-1} + \partial_4 u_n = 0 \quad (n = 1, 0, -1, \dots),$$

which gives an equivalent representation of linear problem (3.1). We now solve these equations successively applying Frobenius' theorem. It should be noted that this argument is almost parallel to the construction of wave functions for self-dual Yang-Mills fields due to Chau, Prasad and Sinha [9].

To start with, let us note that Eq. (3.2) splits into the three equations

$$(3.8a) \quad [\partial_2, \partial_3] = -C_1 \partial_3 + D_1 \partial_2,$$

$$(3.8b) \quad [\partial_1, \partial_2] - [\partial_3, \partial_4] = -C_0 \partial_3 + C_1 \partial_1 + D_0 \partial_2 + D_1 \partial_4,$$

$$(3.8c) \quad [\partial_1, \partial_4] = C_0 \partial_1 + D_0 \partial_4.$$

Since Eq. (3.8a) is satisfied, Frobenius' theorem ensures the existence of two

solutions, u_1^1 and u_1^2 , of linear system $(3.7)_2$ with $du_1^1 \wedge du_1^2 \neq 0$.

The construction of u_n^1 and u_n^2 ($n=0, -1, -2, \dots$) can be achieved by induction. Namely, we suppose that u_n^1, u_{n+1}^1, \dots and u_n^2, u_{n+1}^2, \dots are already given as solutions of Eqs. $(3.7)_{n+1}, (3.7)_{n+2}, \dots$, and then show that Eqs. $(3.7)_n$ certainly have a solution for $u_n = u_n^1$ and u_n^2 , respectively. Eqs. $(3.7)_n$ may be considered an inhomogeneous linear system with an unknown function u_{n-1} , and in view of Eq. (3.8a) Frobenius' theorem again ensures the existence of a solution provided that the integrability condition

$$(3.9) \quad (C_1 \partial_1 + D_1 \partial_4 + \partial_2 \partial_1 + \partial_3 \partial_4)u_n = 0$$

is satisfied. One can actually check, using Eqs. $(3.7)_{n+1}, (3.8b)$ and $(3.8c)$, that u_n^1 and u_n^2 certainly satisfy the above integrability condition. Thus the induction process turns out to surely work, and yields a pair of solutions u^1 and u^2 of linear problem (3.1) with $du_1^1 \wedge du_1^2 \neq 0$.

What remains is the check of the condition $du_1^1 \wedge du_1^2 \wedge du_0^1 \wedge du_0^2 \neq 0$. This is actually an immediate consequence of the condition $du_1^1 \wedge du_1^2 \neq 0$ already fulfilled and of the following equation to be derived from $(3.7)_1$ and $(3.7)_2$:

$$(3.10) \quad \det(\partial_a u_1^1, \partial_a u_1^2, \partial_a u_0^1, \partial_a u_0^2)_{1 \leq a \leq 4} = [\det(\partial_a u_1^i)_{a=1,4}]^2.$$

Indeed, from the assumption ∂_1 and ∂_4 are in particular linearly independent, so that the right side of the above equation does not vanish because of the condition that $du_1^1 \wedge du_1^2 \neq 0$; therefore the generalized Jacobian determinant on the left side also does not vanish, and this implies $du_1^1 \wedge du_1^2 \wedge du_0^1 \wedge du_0^2 \neq 0$. This completes the proof of the "if" part of the proposition.

We next consider the "only if" part. For this purpose one may as well take $(u_1^1, u_1^2, u_0^1, u_0^2)$ as new space-time coordinates; note that this is certainly permitted because the condition $du_1^1 \wedge du_1^2 \wedge du_0^1 \wedge du_0^2 \neq 0$ is assumed. The "only if" part then readily follows from the following fact.

Proposition (3.11). *Suppose that linear problem (3.1) has a pair of solutions u^1 and u^2 as shown in the statement of Proposition (3.6). Take $(p, q, x, y) \equiv (u_1^1, u_1^2, u_0^1, u_0^2)$ as space-time coordinates and regard other u_n^{α} 's as their functions. Then there is a $GL(2, C)$ -valued function $l = (l_{\alpha}^{\beta}(p, q, x, y))$ ($\alpha, \beta = 1, 2$) satisfying the equations*

$$(3.11a) \quad \begin{pmatrix} \partial_3 & \partial_1 \\ -\partial_2 & \partial_4 \end{pmatrix} = l \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial p} + \frac{\partial u_1^1}{\partial x} \frac{\partial}{\partial x} + \frac{\partial u_1^2}{\partial x} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial q} + \frac{\partial u_1^1}{\partial y} \frac{\partial}{\partial x} + \frac{\partial u_1^2}{\partial y} \frac{\partial}{\partial y} \end{pmatrix},$$

$$(3.11b) \quad \left[-\lambda \frac{\partial}{\partial x} + \frac{\partial}{\partial p} + \frac{\partial u^1_1}{\partial x} \frac{\partial}{\partial x} + \frac{\partial u^2_1}{\partial x} \frac{\partial}{\partial y}, -\lambda \frac{\partial}{\partial y} + \frac{\partial}{\partial q} + \frac{\partial u^1_1}{\partial y} \frac{\partial}{\partial x} + \frac{\partial u^2_1}{\partial y} \frac{\partial}{\partial y} \right] = 0.$$

Proof. Eqs. (3.7)₂ imply that the vector fields ∂_2 and ∂_3 take the form of linear combination of $\partial/\partial x$ and $\partial/\partial y$; therefore for some functions l^β_α ($\alpha, \beta=1, 2$) $\partial_3 = l^1_1 \partial/\partial x + l^2_1 \partial/\partial y$ and $-\partial_2 = l^1_2 \partial/\partial x + l^2_2 \partial/\partial y$. Then inserting these formulas into Eqs. (3.7)₁ one obtains the equation

$$(3.12) \quad \det(l^\beta_\alpha) = \det(\partial_a u^b_0, \partial_a u^b_0)_{a=2,3} = \det(\partial_a u^1_1, \partial_a u^2_1)_{a=1,4},$$

whose right side does not vanish as we have remarked above. Now let us check that the matrix $l = (l^\beta_\alpha)$ indeed fulfills all the requirements in the proposition. To this end, we define:

$$(3.13) \quad \begin{pmatrix} \tilde{\partial}_3 & \tilde{\partial}_1 \\ -\tilde{\partial}_2 & \tilde{\partial}_4 \end{pmatrix} \equiv l^{-1} \begin{pmatrix} \partial_3 & \partial_1 \\ -\partial_2 & \partial_4 \end{pmatrix},$$

from the construction, $\tilde{\partial}_3 = \partial/\partial x$ and $-\tilde{\partial}_2 = \partial/\partial y$. Furthermore, since Eqs. (3.7) are also valid for $\tilde{\partial}_1, \dots, \tilde{\partial}_4$,

$$(3.14)_n \quad -\tilde{\partial}_3 u_{n-1} + \tilde{\partial}_1 u_n = 0, \quad \tilde{\partial}_2 u_{n-1} + \tilde{\partial}_4 u_n = 0 \quad (u_n = u^n_1, u^n_2).$$

In particular, from Eqs. (3.14)₁ it turns out that $\tilde{\partial}_1 p = 1, \tilde{\partial}_1 q = 0, \tilde{\partial}_4 p = 0$ and $\tilde{\partial}_4 q = 1$. This implies that $\tilde{\partial}_1 = \partial/\partial p + A\partial/\partial x + B\partial/\partial y$ and $\tilde{\partial}_4 = \partial/\partial q + C\partial/\partial x + D\partial/\partial y$ for some functions A, B, C and D . Inserting these formulas into Eqs. (3.14)₀ one finds:

$$A = \frac{\partial u^1_1}{\partial x}, \quad B = \frac{\partial u^2_1}{\partial x}, \quad C = \frac{\partial u^1_1}{\partial y}, \quad D = \frac{\partial u^2_1}{\partial y},$$

thus one obtains Eq. (3.11a). Eq. (3.11b) follows from Eqs. (3.14)₋₁ after simple calculations. This completes the proof of the proposition.

3.3. Further Generalizations

We now turn to the general aspects of the present issue. As we have illustrated for the most simplest case, a number of nonlinear equations that generalize the self-dual Einstein equations from the point of view of integrability can be obtained as the integrability conditions of linear problems of the following type,

$$(3.15) \quad D_\alpha(\lambda)u = 0, \quad \alpha = 1, \dots, r,$$

with some constraints if necessary, where $D_\alpha(\lambda)$'s are vector fields with prescribed dependence on the parameter λ and play the role of unknown functions. The integrability of such a linear problem means, roughly speaking, that it has a maximal number (=space dimensions $- r$) of functionally independent solutions. The integrability conditions in this sense takes the form of commutator relations such as:

$$(3.16) \quad [D_\alpha(\lambda), D_\beta(\lambda)] = \sum_{\gamma=1}^r C_{\alpha\beta}^\gamma(\lambda) D_\gamma(\lambda),$$

with appropriate coefficients $C_{\alpha\beta}^\gamma(\lambda)$ to be determined by $D_\alpha(\lambda)$'s.

For instance, a series of higher dimensional generalizations including the previous example (3.2) can be derived by setting

$$(3.17) \quad D_\alpha(\lambda) = -\lambda \partial_\alpha + \partial_{\alpha+r}, \quad \alpha = 1, \dots, r,$$

where $\partial_1, \dots, \partial_{2r}$ are linearly independent vector fields in a $2r$ dimensional space. The case of $r=2$ agrees with the previous example, and the results obtained therein can be extended to this series of equations just the same way. The results for the present case can be summarized as follows:

i) The linear problem has a set of Laurent series solutions $u^\alpha = \sum_{n=-\infty}^1 u_n^\alpha \lambda^n$ ($\alpha = 1, \dots, r$) satisfying the condition

$$(3.18) \quad du_1^1 \wedge du_1^2 \wedge \dots \wedge du_1^r \wedge du_0^1 \wedge du_0^2 \wedge \dots \wedge du_0^r \neq 0,$$

if and only if the following equations are satisfied:

$$(3.19) \quad [-\lambda \partial_\alpha + \partial_{\alpha+r}, -\lambda \partial_\beta + \partial_{\beta+r}] = \sum_{\gamma=1}^r (C_{\alpha\beta,0}^\gamma + C_{\alpha\beta,1}^\gamma)(-\lambda \partial_\gamma + \partial_{\gamma+r}),$$

where $C_{\alpha\beta,0}^\gamma$ and $C_{\alpha\beta,1}^\gamma$ are some functions independent of λ to be determined by the vector fields $\partial_1, \dots, \partial_{2r}$.

ii) In terms of the 1-forms e^1, \dots, e^{2r} defined by the duality relation $\langle e^a, \partial_b \rangle = \delta_b^a$, Eqs. (3.19) are equivalent to the exterior differential equations

$$(3.20) \quad d(e^\alpha + \lambda e^{\alpha+r}) \wedge (e^1 + \lambda e^{r+1}) \wedge \dots \wedge (e^r + \lambda e^{2r}) = 0 \quad (\lambda \in \mathbb{C})$$

iii) The above two representations of the integrability conditions are form-invariant under transformations of the vector fields and 1-forms such as:

$$(3.21) \quad \begin{pmatrix} \partial_1 & \partial_{r+1} \\ \vdots & \vdots \\ \partial_r & \partial_{2r} \end{pmatrix} \longrightarrow l \begin{pmatrix} \partial_1 & \partial_{r+1} \\ \vdots & \vdots \\ \partial_r & \partial_{2r} \end{pmatrix}, \quad \begin{pmatrix} e^1 & e^{r+1} \\ \vdots & \vdots \\ e^r & e^{2r} \end{pmatrix} \longrightarrow {}^t l^{-1} \begin{pmatrix} e^1 & e^{r+1} \\ \vdots & \vdots \\ e^r & e^{2r} \end{pmatrix},$$

with $l = (l_\alpha^\beta)$ ($\alpha, \beta = 1, \dots, r$) a $GL(r, \mathbb{C})$ -valued function. With an appropriate

choice of such transformations the vector fields and the 1-forms in question can be set into a canonical form similar to that one shown in Proposition (3.11).

Furthermore, there is a systematic way to generalize the nonlinear constraint employed for the case of the self-dual Einstein equations. For example, for the case of the above series of equations, we take a complex Lie subgroup G of $GL(r, \mathbf{C})$ and consider

Constraint (3.22): The $r \times r$ matrix $(\partial_\beta u^\alpha)$ takes values in G .

This certainly gives a nonlinear constraint as mentioned above; if in particular $r=2$ and $G=SL(2, \mathbf{C})$, this reproduces the self-dual Einstein equations. A variety of reductions of the “master equations” (3.19) and (3.20) can be derived as such. It should be also remarked that (3.22) is, in fact, equivalent to

Constraint (3.23): The $r \times r$ matrix $(\partial u^\alpha / \partial x^\beta)$ takes values in G .

Here we have taken $(p^1, \dots, p^r, x^1, \dots, x^r) \equiv (u_1^1, \dots, u_1^r, u_0^1, \dots, u_0^r)$ as new space coordinates and regarded u^α as functions of these coordinates and λ . The group-theoretical meaning of the above constraint will be clarified in §4.

We now conclude this section with a few remarks:

i) It would be worth mentioning that Zakharov and Shabat [7] already argued equations like (3.16). They pointed out that such equations may be interpreted as the condition of the vanishing of “obstruction” for extending the notion of “zero-curvature representations” from ordinary flat spaces to curved spaces. Eq. (3.2), for example, is connected in this sense with the zero-curvature representation (2.14) of the self-dual Yang-Mills equations, and if Eq. (3.2) is satisfied (namely “obstruction” vanishes), the curved space version of (2.14) as shown by Zakharov and Shabat enables one to describe self-dual Yang-Mills fields on such a curved space-time after a manner almost parallel to the case of flat space-time. This leads to essentially the same description of self-dual Yang-Mills fields on conformally self-dual space-times as presented by Atiyah, Hitchin and Singer [24], though the latter takes a more geometric formulation and, in addition, conformally self-dual space-times to be characterized by Eq. (3.2) are somewhat special ones. See also the work of Torres del Castillo [25] for a related topic.

ii) Thus it would be quite natural to expect that other interesting examples of Eqs. (3.16) can be likewise obtained in connection with various analogues of the self-dual Yang-Mills equations. In fact, Eqs. (3.19) can be derived in the

above-mentioned sense from the curved space version of the "A-series" in higher dimensional integrable gauge-field equations introduced and classified by Ward [11]. Furthermore Ward's classification includes, besides the "A-series", three other classes too; the latter also yield examples of Eqs. (3.16) in their curved space version. (The formulation of the examples derived from the "C-series" and "D-type" in Ward's classification requires multidimensional parameters $(\lambda_1, \dots, \lambda_m)$ rather than a single λ .)

iii) An important question would be whether there are interesting examples of superspace-analogues of Eqs. (3.16). Perhaps some examples will be found in connection with the curved superspace version of the supersymmetric Yang-Mills (constraint) equations; this issue will be discussed elsewhere.

§ 4. Group-Theoretical Structures in Curved Twistor Construction

4.1. Curved Twistor Construction

The Riemann-Hilbert problem has become a powerful solution technique of nonlinear integrable systems [26]. As Boyer and Plebanski stressed [17, 18], the curved twistor construction has a structure very similar to the Riemann-Hilbert problem. The analogy, however, seems to be still incomplete; indeed, the Riemann-Hilbert problem can be formulated as a sort of decomposition problem in a group whereas the curved twistor construction, in its original form, does not take such a form. The purpose of this section is to present a reformulation of the curved twistor construction as a decomposition problem in a "group-like" structure, which consequently enables one to compare these two solution techniques on an equal footing. We start with a brief review of the curved twistor construction here.

This construction, due to Penrose [16], produces self-dual Einstein metrics from a class of three dimensional complex manifolds called "curved twistor spaces". Each curved twistor space \mathcal{T} here is endowed with a fibration $\pi: \mathcal{T} \rightarrow \mathbf{P}^1$ and a "twisted" 3-form that defines on each fiber of π a symplectic structure. The role to space-time is played by the space of holomorphic sections of π (i.e. holomorphic maps $s: \mathbf{P}^1 \rightarrow \mathcal{T}$ with $\pi \circ s = \text{identity map}$) satisfying an additional condition, which ensures that this space has complex dimensions four. Finding the explicit form of this space is the most difficult part of this construction. Because of this difficulty, carrying out the construction to the