Issues of Multi-Dimensional Integrable Systems

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0. Introduction

At present it is a very hard task to give an exact definition to the notion of integrability or integrable system. What we have in hand now is rather an enormous list of examples accumulated for many years of intensive studies from both the physical and mathematical sides (cf. Jimbo, Miwa [1] and references cited therein). Most of this list is occupied by the so-called soliton equations, which are so named because of their origin in soliton phenomena. Mathematically, soliton equations describe nonlinear waves propagating in one-dimensional space like a canal. Even the KP (Kadomtsev–Petviashvili) equation should be considered as such (Sato and Sato [2]) though physically it was introduced as a two-space-dimensional generalisation of the KdV (Korteweg–de Vries) equation. A natural question coming then, would be whether there are multi-dimensional analogues of soliton equations. Frankly speaking, naive attempts at such a generalisation have almost all failed up to now; they could at best just (re)produce a new type of soliton equations.

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A few examples of what can be really called multi-dimensional integrable systems have been discovered from a somewhat distinct point of view, that is, twistor theory. The equations of motion of self-dual connections (Yang–Mills fields) [3], self-dual metrics (gravitational fields) [4] and their extensions to dimensions greater than four (e.g., hyper-Kähler versions) [5, 6] provide such examples. Of course this assertion would be meaningless unless the notion of integrability is made more definite, though regrettably an ultimate definition of integrability with mathematical rigor still lies far beyond our scope. A series of my work since 1983 [7] has aimed at the analysis of in what sense the nonlinear systems mentioned above are integrable, and how that can be understood from a more general principle. This article presents an intermediate summary of this research, including current interest and some outlook towards the future.

1. Self-Dual Connections, Linear System, and Twistors

From here through Section 5 we examine self-dual connections from various aspects. The integrability of the equations of motion of self-dual connections becomes most manifest in a complexified setting. Let $\mathbf{x} = (y, z, \bar{y}, \bar{z})$ be a set of complex coordinates in $\mathbf{C}^4$ with complex metric $ds^2 = dy\, d\bar{y} + dz\, d\bar{z}$ and $\nabla = d + A$, $A = A_y\, dy + A_z\, dz + A_{\bar{y}}\, d\bar{y} + A_{\bar{z}}\, d\bar{z}$, a connection with a complex structure group, say, $GL(r, \mathbf{C})$. Its coordinate components are $\nabla_u = \partial_u + A_u$, $u \in \mathbf{x}$, the A_u's being the so-called gauge potentials that play the role of unknown functions. The curvature form $F = dA + A \wedge A$ has six components $F_{uv} = [\partial_u + A_u, \partial_v + A_v]$, $u, v \in \mathbf{x}$. The equations of motion of self-dual connections are

$$F_{yz} = 0, \qquad F_{\bar{y}\bar{z}} = 0, \qquad F_{y\bar{y}} + F_{z\bar{z}} = 0.$$

With the aid of an auxiliary parameter λ (called a *spectral parameter* after the terminology of inverse scattering theory) these equations are written more compactly as

$$[-\lambda \nabla_y + \nabla_{\bar{z}}, \lambda \nabla_z + \nabla_{\bar{y}}] = 0.$$

This nonlinear system, the so-called zero-curvature representation, is the integrability condition in the sense of Frobenius of the following linear system (Belavin, Zakharov [8], Pohlmeyer [9], Chau, Prasad, Sinha [10]):

$$(-\lambda \nabla_y + \nabla_{\bar{z}})\, W = 0, \qquad (\lambda \nabla_z + \nabla_{\bar{y}})\, W = 0,$$

where $W = W(\mathbf{x}, \lambda)$ is an unknown function taking values in $GL(r, \mathbf{C})$. Another way of stating this fact is by using the Pfaffian system of (cf. Gindikin [11])

$$dW + AW = 0,$$

$$dy + \lambda\, d\bar{z} = 0, \qquad dy + \lambda\, d\bar{z} = 0, \qquad d\lambda = 0,$$

in $GL(r, \mathbf{C}) \times \mathbf{C}^4 \times \mathbf{P}^1$, $(W, \mathbf{x}, \lambda)$ being coordinates in this manifold. Its integrability condition can be written

$$(dA + A \wedge A) \wedge (dy + \lambda\, d\bar{z}) \wedge (dz - \lambda\, d\bar{y}) \wedge d\lambda = 0,$$

giving another equivalent expression of the previous zero-curvature representation. The last three equations of the above Pfaffian system form a Pfaffian system integrable in itself, whose first integrals $y + \lambda\bar{z}$, $z - \lambda\bar{y}$, λ may be thought of as taking values in the twistor space $\mathbf{P}^3$. (As we shall see later, twistor theory of self-dual metrics is nothing other than an extension of this picture into curved manifolds.) With this correspondence, self-dual connections can be described in terms of vector bundles on the twistor space. This story has now become very familiar to us (cf. Atiyah [12] and references therein).

2. Riemann–Hilbert Transformations

Riemann-Hilbert transformations (RHT) had been studied for the case of soliton equations, but it was Ueno and Nakamura [13] who first introduced them to the case of self-dual connections. We shall not repeat the construction here, and refer details to their original paper, but the reason why such transformations exist may be understood without difficulty from the following heuristic argument.

Given a self-dual connection ∇, suppose there are two solutions $W = W(\mathbf{x}, \lambda)$, $\hat{W} = \hat{W}(\mathbf{x}, \lambda)$ of the linear system which are holomorphic functions of $(\mathbf{x}, \lambda)$ but with different domains of definition with respect to λ, D and $\hat{D}$, which are discs covering $\mathbf{P}^1$ and centered at respectively $\lambda = \infty(D)$ and $\lambda = 0(\hat{D})$. Then evidently the product $g = W^{-1}\hat{W}$ satisfies

$$(-\lambda\partial_y + \partial_{\bar{z}})g = 0, \qquad (\lambda\partial_z + \partial_{\bar{y}})g = 0,$$

which means that g is a function (with values in $GL(r, \mathbf{C})$) of $(y + \lambda\bar{z}, z - \lambda\bar{y}, \lambda)$, λ running over the anulus $D \cap \hat{D}$. Thus naturally arise the three variables of the twistor picture. In fact this g can be used as a transition function to construct a vector bundle on the twistor space, and this is exactly the correspondence of self-dual connections and vector bundles mentioned above. The inverse correspondence is achieved by factorising a given $GL(r, \mathbf{C})$-valued holomorphic function g of $(y+\lambda\bar{z}, z-\lambda\bar{y}, \lambda)$, $\lambda \in D \cap \hat{G}$, into the product of two functions as

$$g = g(y+\lambda\bar{z}, z-\lambda\bar{y}, \lambda) = W(\mathbf{x}, \lambda)^{-1}\hat{W}(\mathbf{x}, \lambda),$$

where W and $\hat{W}$ are required to have the same analyticity properties as above. It is indeed not difficult to see that the W and $\hat{W}$ thus obtained from g do satisfy the linear system of self-dual connections for an appropriate choice of A. Strictly speaking, there are some cases where the above argument breaks down because of some topological reasons, but in a generic case this gives a complete description of self-dual connections.

In the above setting, RHT's are nothing other than the left and right multiplication of the above g with $GL(r, \mathbf{C})$-valued functions of the same analytical properties as

$$g \to g_{\mathrm{L}} g g_{\mathrm{R}}^{-1}, \qquad g_{\mathrm{L(R)}} = g_{\mathrm{L(R)}}(y+\lambda\bar{z}, z-\lambda\bar{y}, \lambda).$$

This action on g by $(g_{\mathrm{L}}, g_{\mathrm{R}})$ causes a transformation of the triple $(A, W, \hat{W})$ (to be more precise, its gauge equivalence class), thus there are in fact *two* types of RHT's, *left* and *right* (cf. Wu [14]). Note that they mutually commute, that is, the result of the successive action of g_{L} and g_{R} does not depend on the order. What Ueno and Nakamura considered were only one of them, say left RHT's. A simplification gained by considering one-sided RHT's is that they can be described in terms of just two components, (A, W) (left case) or $(A, \hat{W})$ (right case), of the triple $(A, W, \hat{W})$. To be more precise, under the action of, say, a left RHT g_{L} the result of transformation of A and W is independent of $\hat{W}$, so that one can recognise g_{L} simply as acting on the pair (A, W); just the same is true for g_{R} and $(A, \hat{W})$. For details and related topics, cf. Takasaki [7, 1985].

3. Cauchy Problem

The following observation is also a variation of the factorisation problem mentioned above, but before my work [7] no one has noticed such a point of view.

We first note that the $GL(r, \mathbf{C})$-valued function g is uniquely determined by the "Cauchy data" $g^{(0)}(y, z, \lambda) = g(\bar{y} = \bar{z} = 0)$. The linear system of g

with the initial condition $g(\bar{y}=\bar{z}=0)=g^{(0)}$ can be solved as

$$g=\exp(\bar{z}\lambda\,\partial_y-\bar{y}\lambda\,\partial_z)g^{(0)}.$$

Here evidently $(\bar{y},\bar{z})$ play the role of "time" variables, and (y,z) that of "space" variables, though this has nothing to do with the original physical interpretation of these variables.

What about the W? To make clear this point, one has to "fix the gauge" as $W(\lambda=\infty)=1$. Then from the linear system of W the gauge potentials are fixed as

$$A_y=0,\qquad A_z=0,\qquad A_{\bar{y}}=-\partial_z W_1,\qquad A_{\bar{z}}=\partial_y W_1,$$

where W_1 denotes the next-to-leading coefficient of the Laurent expansion of W around $\lambda=\infty$, $W=1+\sum_{n\geq 1}W_n\lambda^{-n}$, $W_n=W_n(\mathbf{x})$. Taking back these expressions of the gauge potentials, one finds that the linear system for W becomes now a nonlinear system for the W_n's, which read

$$-\partial_y W_n+\partial_{\bar{z}}W_{n+1}+(\partial_y W_1)\,W_n=0,$$

$$\partial_z W_n+\partial_{\bar{y}}W_{n+1}-(\partial_z W_1)\,W_n=0.$$

With a simple argument one can check that each solution (for example, holomorphic) of this system is uniquely determined by the "Cauchy data" $W_n^{(0)}=W_n(\bar{y}=\bar{z}=0)$ and, besides, the Cauchy data can be given arbitrarily, thus providing a set of good "functional coordinates" in the solution space of the above nonlinear system.

Recovering W from $W^{(0)}$ is not as simple as in the case of g, but can be reduced to a factorisation problem of the same nature as before. Eliminating g from the previous relations, one can indeed deduce the relation

$$\exp(\bar{z}\lambda\,\partial_y-\bar{y}\lambda\,\partial_z)\,W^{(0)-1}=W^{-1}V,$$

where $V=\hat{W}\cdot\exp(\bar{z}\lambda\,\partial_y-\bar{y}\lambda\,\partial_z)\,\hat{W}(\bar{y}=\bar{z}=0)^{-1}$. The explicit form of V is irrelevant; the point is that $V=V(\mathbf{x},\lambda)$ is holomorphic with respect to λ in $\hat{D}$ (at least for sufficiently small values of $\bar{y}$ and $\bar{z}$). Thus we encounter another factorisation problem, which describes the "time evolution" from $W^{(0)}$ to W.

The point of view of the Cauchy problem, though it appears fairly artificial at first glance, is one of the indispensable elements of the method developed in [7]. Without this, any comparison with soliton equations will remain at a superficial level. We shall see some other aspects of this in the following.

4. Grassmann Manifold

The goal set up in [7, 1983–85] was to give a framework for reformulating self-dual connections à la Sato [2]. An infinite-dimensional Grassmann manifold plays a basic role in Sato's approach, as a parameter (or moduli) space of solutions to soliton equations. As a result, both an infinite hierarchy of time evolutions and a variety of transformations of solutions (Bäcklund transformations etc.) can be understood on an equal footing, as part of the large general linear group $GL(\infty)$ acting on the Grassmann manifold. This, of course, clearly explains why soliton equations are related to infinite-dimensional Lie algebras (cf. Jimbo and Miwa [1]).

A conclusion presented in [7, 1983–85] is that almost the same structure exists in the case of self-dual connections, so that this case, too, may be judged as "integrable." The essence is as follows.

We construct an infinite matrix $\xi = (\xi_{ij})_{i \in \mathbf{Z}, j \in \mathbf{N}^c}$ of size $\mathbf{Z} \times \mathbf{N}^c$, $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbf{N}^c = \{-1, -2, \ldots\}$, from the Laurent coefficients of W as

$$\xi_{ij} = \sum_{k<0} W^*_{i-k} W_{k-j},$$

where the W^*_n's stand for the Laurent coefficients of W^{-1} in the sense of formal series, $W^{-1} = 1 + \sum_{n \geq 1} W^*_n \lambda^{-n}$. The entries of ξ then satisfy the algebraic relations

$$\xi_{ij} = \delta_{ij} \text{ (Kronecker's delta)} \qquad \text{for } i<0, j<0,$$

$$\xi_{i+1,j} = \xi_{i,j-1} + \xi_{i,-1}\xi_{0j} \qquad \text{for } i \in \mathbf{Z}, j<0,$$

which conversely characterise a matrix ξ that can be obtained from some W as above.

Such a matrix ξ of maximal rank, in general, can be thought of as representing a vector subspace V of a fixed larger vector space $\mathbf{V}$ formed by column vectors of size $\mathbf{Z}$; the V being spanned by the columns of ξ. Actually the correspondence between ξ and V is by no means one-to-one; ξ and ξh, where h is an invertible (in an appropriate sense) $\mathbf{N}^c \times \mathbf{N}^c$ matrix, should be considered equivalent to the effect that they correspond to the same V. One can thus freely replace a ξ-matrix by an equivalent one. An extreme choice, allowed as far as the $\mathbf{N}^c \times \mathbf{N}^c$ part $\xi_{(-)} = (\xi_{ij})_{i,j<0}$ of ξ is invertible, is to retake ξ to be such that $\xi_{ij} = \delta_{ij}$ for $i, j < 0$, by multiplying $\xi_{(-)}^{-1}$ from the right. The ξ-matrix constructed above from W is exactly of that nature.

In terms of the above ξ, the action of RHT's and the solution of the Cauchy problem discussed in the preceding sections can be represented as follows:

$$g_{\mathrm{L}} \circ \xi = g_L(y + \Lambda \bar{z}, z - \Lambda \bar{y}, \Lambda)\xi \cdot \text{(invertible matrix)},$$

$$\xi = \exp(\bar{z}\Lambda\partial_y - \bar{y}\Lambda\partial_z)\xi^{(0)} \cdot \text{(invertible matrix)},$$

where $g_{\mathrm{L}} \circ \xi$ and $\xi^{(0)}$ denote the ξ-matrices corresponding to, respectively, $g_{\mathrm{L}} \circ W$ (= the action of a left RHT g_{L} on W) and $W^{(0)}$ (= the Cauchy data of W on $\bar{y} = \bar{z} = 0$); Λ stands for the shift matrix $(\delta_{i+1,j})_{i,j \in \mathbf{Z}}$ that acts on the entries of, say ξ, as $\Lambda\xi = (\xi_{i+1,j})_{i \in \mathbf{Z}, j<0}$. The above relations are not of symbolic nature, but provide a practical solution process [7, 1983-84].

A description of bi-sided RHT's along a similar line is argued in [7, 1985], in which case one has to enlarge the ξ-matrix so as to incorporate both W and $\hat{W}$ in its structure.

The method to Grassmann manifold presented here has also been applied to some other equations by Suzuki [15], Harnad, Jacques [16], Nagatomo [17], and Nakamura [18].

5. Enlarged Groups and Lie Algebras

The content of this section belongs to a research just beginning [19].

As we have seen above, the method of Grassmann manifold provides a framework to unify two seemingly distinct objects, that is, RHT's (= the action of matrix functions $g_{\mathrm{L}}(y + \Lambda\bar{z}, z - \Lambda\bar{y}, \Lambda)$ on ξ, or more preferably that of $g_{\mathrm{L}}(y, z, \Lambda)$ on $\xi^{(0)}$) and time evolution (= the action of operators such as $\exp(\bar{z}\Lambda\partial_y - \bar{y}\Lambda\partial_z)$ on $\xi^{(0)}$). In view of the work of Sato and Sato [2], a natural question would be what kind of groups or Lie algebras lie behind this picture.

To make the situation simpler, we here focus on the infinitesimal version of the above objects and formulate them in an abstract setting.

Let R be a differential ring on which ∂_y and ∂_z act as derivations; for example, $R = \mathbf{C}[[y, z]]$ if one wishes to deal with formal power series solutions. Then the Lie algebra of infinitesimal one-sided (say, left) RHT's can be identified with the formal loop algebra $\mathrm{gl}(r, R((\lambda^{-1}))) = \mathrm{gl}(r, \mathbf{C}) \otimes R((\lambda^{-1}))$, where $R((\lambda^{-1})) = \{\sum a_n\lambda^n;\ a_n \in R (n \in \mathbf{Z}),\ a_n = 0\ \ (n \gg 0)\}$. This should be supplemented by another Lie algebra including the infinitesimal generators $\lambda\partial_y$ and $\lambda\partial_z$ of time evolutions in $(\bar{y}, \bar{z})$; a candidate would be $R[\lambda]\partial_y + R[\lambda]\partial_z$. Thus as a Lie algebra relevant to $GL(r, \mathbf{C})$-self-dual connections, one may take the following:

$$\mathfrak{g}_{\mathrm{SDYM}} = \mathfrak{gl}(r, R((\lambda^{-1}))) + R[\lambda]\partial_y + R[\lambda]\partial_z.$$

This is not a simple Lie algebra in the sense that the first component on the right side forms a nontrivial ideal.

The second and third components include a mutually commutative set of derivations $\lambda^n\partial_y, \lambda^n\partial_z$, $n = 1, 2, \ldots$, which generate an infinite hierarchy of simultaneous time evolutions parallel to the case of soliton equations. A similar idea was also presented by Nakamura [18].

Since the Lie algebra $\mathfrak{g}_{\mathrm{SDYM}}$ incorporates these time evolutions in a natural manner, one may expect to develop by use of this an orbit method of the Kostant–Kirillov type. In fact, it appears hard to give a variational formulation, but the argument of Flaschka *et al.* [20] on an abstract Poisson bracket structure described for a class of soliton equations can be extended to the present setting.

Another, more challenging issue would be whether there is a natural extension $\hat{\mathfrak{g}}_{\mathrm{SDYM}}$ of $\mathfrak{g}_{\mathrm{SDYM}}$ with the exact sequence $0 \to \mathfrak{o} \to \hat{\mathfrak{g}}_{\mathrm{SDYM}} \to \mathfrak{g}_{\mathrm{SDYM}} \to 0$, $\mathfrak{o}$ being an abelian ideal (but possibly not central). Such a Lie algebra, if it exists, provides an interesting multi-dimensional analogue of Kac-Moody Lie algebras with central charge.

6. Issue of Multi-Dimensional Spectral Parameters

We now discuss a somewhat delicate subject related to the problem mentioned at the beginning of this article.

The nonlinear system of self-dual connections is by no means an isolated example. What we have seen up to here can be readily extended to a general setting that also covers, for example, the hyper-Kähler version of self-dual connections in $4k$, $k = 1, 2, \ldots$, dimensions included in the table of Ward [5]. A common feature of these nonlinear systems is the presence of an associated linear system with a spectral parameter, λ, which reproduces the relevant nonlinear system as its integrability conditions in the sense of Frobenius. In fact, as we shall see later, this is still insufficient to ensure "integrability" in the same sense as self-dual connections are integrable, but our primary subject in the following is the case where a nonlinear system has an associated linear system with *more than one* spectral parameter.

Several examples of such nonlinear systems with multi-dimensional spectral parameters are already known, all related to gauge theory, which are: (*i*) eight-dimensional gauge fields introduced by Witten [21] and Isenberg *et al.* [22], and discussed from the point of view of Grassmann manifold by Suzuki [15]; (*ii*) constraint equations of super-Yang–Mills fields in four dimensions studied in the context of integrability by Volovich [23], Devchand [24], Chau, Ge, Popowicz [25], Harnad and Jacques [16] on the basis of Witten's twistorial interpretation [21]; (*iii*) super-Yang–Mills fields in ten dimensions by Harnad and Schnider [26]; (*iv*) examples presented by Ward [5], etc. In fact, all the above papers on case (*ii*) are based on a linear system with just one spectral parameter, which is derived through a certain process from another linear system with two spectral parameters that Witten's argument originally suggests (cf. Devchand [27]); this however still inherits a serious problem as we shall see later.

The problem is, *whether the equations above are really integrable.* The answer to this question may depend on what one expects as the contents of integrability. If one requires as a basic ingredient of integrability the presence of a large (e.g., transitive) transformation group or Lie algebra acting on the solution space, there are several evidences suggesting that the above examples are *non-integrable*; at present there is no hope to construct such a group or Lie algebra.

A heuristic argument to support this observation is as follows. According to Ward [5] and Isenberg *et al.* [22], solving these equations is equivalent to finding certain (family of) trivial vector bundles on a complex manifold Σ; spectral parameters are coordinates of Σ. For example, Σ is $\mathbf{P}^1 \times \mathbf{P}^1$ in cases (*i*) and (*ii*), a nonsingular quadratic in $\mathbf{P}^9$ in case (*iii*), and case (*iv*) includes examples with $\Sigma = \mathbf{P}^m$. Such a vector bundle is described by a finite covering $\{U_i\}$ of Σ and a set of patching functions $\{g_{ij};\ U_i \cap U_j \neq \varnothing\}$ with the cocyclic conditions $g_{ij}g_{jk} = g_{ik}$ on nonempty intersections $U_i \cap U_j \cap U_k$. The case of $\dim \Sigma = 1$ (e.g., $\Sigma = \mathbf{P}^1$) is exceptional; one can take the covering in such a way that three distinct patches never intersect, so that there are in fact no cocycle conditions to be imposed on the g_{ij}'s. The set of such g_{ij}'s acquires a structure of a group of componentwise matrix multiplication as $\{g_{ij}\}, \{h_{ij}\} \to \{g_{ij}h_{ij}\}$. This is exactly the origin of RHT's (cf. Section 2). If $\dim \Sigma > 1$, there is no such natural group structure of patching functions retaining cocycle conditions, so probably no analogue of RHT's.

One might, however, still expect a loophole in the direction indicated in [23–25], that is, by means of a linear system with just one spectral parameter derived by putting some relations among multi-dimensional spectral parameters. Regrettably, this leads to no substantial improvement of the situation. The reduced linear system in, for example, case (*ii*) becomes

$$(\nabla^i_1+\lambda\nabla^i_2)W=0, \qquad (\bar{\nabla}_{1i}+\lambda^2\bar{\nabla}_{2i})W=0,$$

$$(\nabla_1+\lambda\nabla_2+\lambda^2\nabla_3+\lambda^3\nabla_4)W=0,$$

where ∇^i_1, etc., are components of a super-connection; their precise form is irrelevant here. What is crucial is the absence of the λ-term in the second set of equations. Because of this, it turns out that RHT's do not act on this linear system, since the action of a RHT in general produces a new λ-term therein. This difficulty has been also noticed by several people [28].

Of course these observations never reduce the importance of the nonlinear systems mentioned above, their "integrability" however thus being considerably problematical. They should be understood, rather, as defining a subset of the solution space of some other nonlinear systems which *are* integrable.

7. Self-Dual Metrics, or Deformation of Integrable *G*-Structures

We now turn to another, less familiar class of nonlinear systems, which are related to metrics or G-structures. A typical one, in a position to be compared with self-dual connections, is provided by self-dual metrics in four dimensions. Also in this case, a complexified setting is much more convenient than the ordinary Riemannian geometry, in order to see aspects of integrability.

We start from a complex metric ds^2 (= nondegenerate holomorphic bilinear form on the holomorphic tangent bundle) of a four-dimensional complex manifold X; actually we focus on local geometry. In a local coordinate patch, one can take a set of linearly independent 1-forms e_1, e_2, $\bar{e}_1$, $\bar{e}_2$ (the bar does not mean complex conjugation) with which ds^2 can be written

$$ds^2=e_1\bar{e}_1+e_2\bar{e}_2.$$

One may recognise this as generalising the representation $ds^2 = dy\,d\bar{y} + dz\,d\bar{z}$ encountered in Section 1. Self-duality is defined, originally, by a set of linear equations among the components of the Riemann curvature R (cf. Eguchi *et al.* [29]) (compare this with the case of self-duality of Yang–Mills connections), a rather cumbersome object. Fortunately there is another way to handle this notion just in terms of the 1-forms $e_1, \ldots, \bar{e}_2$ without referring to R; it is due to the fact (cf. Boyer [30], Gindikin [31]) that ds^2 is self-dual iff after an appropriate linear transformation of $e_1, \ldots, \bar{e}_2$, the following exterior differential equations are satisfied:

$$d(e_1 \wedge e_2) = 0, \qquad d(\bar{e}_1 \wedge \bar{e}_2) = 0, \qquad d(e_1 \wedge \bar{e}_1 + e_2 \wedge \bar{e}_2) = 0.$$

The use of a spectral parameter λ, again, brings a remarkable simplification. The above equations can be gathered up to become the following:

$$\hat{d}((e_1 + \lambda\bar{e}_2) \wedge (e_2 - \lambda\bar{e}_1)) \wedge \hat{d}\lambda = 0,$$

where $\hat{d}$ denotes the total differentiation on $X \times \mathbf{P}^1$, $\hat{d} = d + d_{\mathbf{P}^1}$, introduced so as to distinguish it from $d = d_X$ on X. One may therefore write the above equation as

$$d((e_1 + \lambda\bar{e}_2) \wedge (e_2 - \lambda\bar{e}_1)) = 0,$$

λ then being considered a parameter. Anyway, as Darboux's theorem ensures, this is nothing other than the integrability condition for the existence, at each point $(\mathbf{x}_0, \lambda_0) \in X \times \mathbf{P}^1$, of a pair of functions ("canonical variables") u_1 and u_2 defined in a neighborhood of $(\mathbf{x}_0, \lambda_0)$ for which

$$(e_1 + \lambda\bar{e}_2) \wedge (e_2 - \lambda\bar{e}_1) = du_1 \wedge du_2.$$

These functions (u_1, u_2) play the same role as a solution W of the linear system of self-dual connections (cf. [7, 1986]).

An analogue of the notion of RHT's was already pointed out by Boyer and Plebanski [32]; the main part of [7, 1986] is devoted to an analysis based on their results, leading to a close analogy with the results on self-dual connections in [7, 1984–85]. In the above setting, the existence of such a transformation group can be seen as follows. Suppose there are two pairs (u_1, u_2) and $(\hat{u}_1, \hat{u}_2)$ of "canonical variables" with different analytical properties almost parallel to W and $\hat{W}$ in Section 2, that is, $\lambda^{-1}u_A$ and $\hat{u}_A$ are holomorphic in, respectively, D and $\hat{D}$ with regard to λ. Since $du_1 \wedge du_2 = d\hat{u}_1 \wedge d\hat{u}_2$, there should be a two-dimensional parametric canonical transformation $f=(f_1(y,z,\lambda), f_2(y,z,\lambda))$, $df_1\wedge df_2 = dy\wedge dz$, $\lambda\in D\cap\hat{D}$, such that

$$u_A = f_A(\hat{u}_1, \hat{u}_2, \lambda), \qquad A=1,2.$$

This canonical transformation f provides a counterpart of the transition function g in Section 2; in an appropriate setting, indeed, a complete description of (local) self-dual metrics can be given along this line. Evidently such f's form a (pseudo)group by composition of maps, and give rise to an analogue of the notion of RHT's. The role of matrix groups such as $GL(r, \mathbf{C})$ in the case of conventional integrable systems, is thus played by a (pseudo)group of (local) diffeomorphisms here.

One can recognise the present situation from a geometrical point of view (cf. Goldschmidt [33]), as a sort of deformation of integrable G-structures; this leads to various possibilities of generalisation. Details and further progress in this direction will be presented in a forthcoming series of papers [34]. In the context of Riemannian or Kählerian geometry, basically the same structure is usually understood as deformation of complex structures, whose parameter space is exactly the corresponding twistor space (cf. Hitchin *et al.* [6]). In our setting the twistor space can be derived as follows (cf. Gindikin [11, 31]). Let us consider the Pfaffian system on $X\times\mathbf{P}^1$:

$$e_1 + \lambda\bar{e}_2 = 0, \qquad e_2 - \lambda\bar{e}_1 = 0, \qquad \hat{d}\lambda = 0.$$

This Pfaffian system is integrable in the sense of Frobenius if the previous equations to $e_1, \ldots, \bar{e}_2$ are satisfied, the triple (u_1, u_2, λ) then giving a set of first integrals. The set of maximal integral manifolds (leaves of the corresponding complex foliation) is nothing other than the twistor space. The canonical transformation f above (in fact, one has to "twist it" [7, 1986]) plays the role of transition function gluing together coordinate patches to form the twistor space.

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