Integrable hierarchies, dispersionless limit and string equations

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Abstract

The notion of string equations was discovered in the end of the eighties, and has been studied in the language of integrable hierarchies. String equations in the KP hierarchy are nowadays relatively well understood. Meanwhile, systematic studies of string equations in the Toda hierarchy started rather recently. This article presents the state of art of these issues from the author's point of view.

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1 Introduction

The end of the eighties was a turning point of studies on integrable hierarchies. Before that time, the most popular special solutions of integrable hierarchies had been rational, trigonometric (soliton) and quasi-periodic solutions. Various algebro-geometric methods for these solutions were developed in the seventies [1]. A characteristic of these solutions is the existence of a "spectral curve" for each solution. Spectral curves of quasi-periodic solutions are nonsingular Riemann surfaces, and the solutions can be written explicitly in terms of a Riemann theta function and abelian differentials. Spectral curves of trigonometric and rational solutions are singular, but algebro-geometric methods still turned out to be useful in that case, too. In the early eighties, the theory of τ-functions [2] elucidated the structure of "general solutions" in the language of infinite dimensional Grassmannian manifolds. The algebro-geometric methods were transplanted into this new framework, and more profound and universal structures behind these algebro-geometric solutions have been revealed [3].

In the end of eighties, a new family of special solutions emerged out of studies on low dimensional quantum gravity and string theories [4]. Remarkably, those solutions are related to a moduli space of curves, rather than a single curve. The new type of solutions are characterized by the so called "string equations", which arise as constraints to an integrable hierarchy of KdV or KP type [5].

Since the early nineties, the author has been attempting to elucidate mathematical meanings of string equations and related issues. This article is a review of the author's recent researches as well as basic notions concerning this issue [6].

This article is organized as follows. In Section 2, we review the most fundamental string equation of the KP hierarchy, i.e., the Douglas equation of two-dimensional quantum gravity coupled to conformal matters, in comparison with algebro-geometric solutions of the KP hierarchy. In Section 3, the notion of "dispersionless limit" is illustrated for the KP hierarchy and its string equations. Section 4 is an introduction to the Toda lattice hierarchy and its "dispersionless" analogue. In Section 5, the state of art of studies on string equations in the Toda hierarchies is presented. Section 5 is the concluding section.

2 String equations in KP hierarchy

String equations in the KP hierarchy have several different (but equivalent) expressions. The most convenient expression for our present purpose is the Douglas equation [5]

$$[Q, P] = 1, \tag{2.1}$$

where P and Q are ordinary differential operators of order p and q,

$$\begin{aligned} P &= \partial_x^p + a_2 \partial_x^{p-2} + \cdots + a_p, \\ Q &= \partial_x^q + b_2 \partial_x^{q-2} + \cdots + b_q. \end{aligned} \tag{2.2}$$

This equation describes the so called "(p, q) model" of two-dimensional quantum gravity coupled to conformal matters. One will immediately notice an analogy with canonical commutation relations in quantum mechanics. In fact, this equation is also frequently treated in the $\hbar$-dependent form

$$[Q, P] = \hbar, \tag{2.3}$$

where P and Q are given by

$$\begin{aligned} P &= (\hbar\partial_x)^p + a_2(\hbar\partial_x)^{p-2} + \cdots + a_p, \\ Q &= (\hbar\partial_x)^q + b_2(\hbar\partial_x)^{q-2} + \cdots + b_q. \end{aligned} \tag{2.4}$$

Remarkably, quasi-periodic solutions and their degeneration (trigonometric and rational solutions) of the KP hierarchy, too, are characterized by a similar commutator equation of the form

$$[Q, P] = 0. \tag{2.5}$$

This equation has a rather long history of researches [7]. Let us call it the "Burchnall-Chaundy equation". A fundamental fact is that such a commuting pair of ordinary differential operators satisfy a polynomial relation with constant coefficients:

$$R(P, Q) = 0. \tag{2.6}$$

This relation defines a complex algebraic curve, $R(X, Y) = 0$, which can be compactified by adding a point at infinity. This is exactly the "spectral curve". Conversely, given a complex algebraic curve (Riemann surface) with a set of additional data, one can construct a commuting pair of ordinary differential operators [1].

The Burchnall-Chaundy equation is related to the Lax formalism of the KP hierarchy as follows. The Lax operator of the KP hierarchy is a pseudo-differential operator of the form

$$L = \partial_x + \sum_{n=1}^{\infty} u_{n+1}\partial_x^{-n}, \tag{2.7}$$

and obeys the Lax equations

$$\frac{\partial L}{\partial t_n} = [B_n, L]. \tag{2.8}$$

The Zakharov-Shabat operators B_n are given by

$$B_n = \left(L^n\right)_{\geq 0}, \tag{2.9}$$

where "$(\quad)_{\geq 0}$" denotes the projection onto the space spanned by nonnegative powers of ∂_x. Now suppose that P and Q are written

$$P = f(L), \quad Q = g(L), \tag{2.10}$$

where f and g are Laurent series with constant coefficients of the form

$$\begin{aligned} f(L) &= L^p + p_2 L^{p-2} + p_3 L^{p-3} + \dots, \\ g(L) &= L^q + q_2 L^{q-2} + q_3 L^{q-3} + \dots. \end{aligned} \tag{2.11}$$

They obviously commute. To give a solution of the Burchnall-Chaundy equation, P and Q are required to be differential operators. This requirement can be rewritten

$$(P)_{\leq -1} = 0, \quad (Q)_{\leq -1} = 0. \tag{2.12}$$

where "$(\quad)_{\leq -1}$" denotes the projection onto the space spanned by negative powers of ∂_x. These conditions are conserved under all t_n flows of the KP hierarchy, hence may be interpreted as a constraint to the KP hierarchy. Algebro-geometric methods for the problem of a commuting pair of differential operators [1] are thus reformulated in the language of the KP hierarchy [3].

Although looking very similar, the Douglas equation has quite different properties. First of all, P and Q no longer satisfy an algebraic relation of the form $R(P,Q) = 0$ (hence there is no analogue of spectral curves). One might have an impression that the relation between the Douglas and Burchnall-Chaundy equations is reminiscent of quantum and classical mechanics, because classical mechanics deals with commuting quantities (c-numbers) whereas quantum mechanical quantities are non-commutative (q-numbers). This analogy, however, is somewhat misleading. According to the ordinary prescription of classical limit in quantum mechanics, the quantum mechanical commutator will be replaced by a Poisson bracket as

$$\hbar^{-1}[Q,P] \to \{Q,P\} \tag{2.13}$$

in the limit of $\hbar \to 0$, and the Douglas equation will turn into a Poisson bracket relation of the form

$$\{Q,P\} = 1 \tag{2.14}$$

rather than the Burchnall-Chaundy equation.

To translate the Douglas equation into the language of the KP hierarchy, we need an extended Lax formalism developed by Orlov and Shulman [9]. This contains, besides the standard Lax operator L, another operator M (Orlov-Shulman operator) of the form

$$M = \sum_{n=2}^{\infty} n t_n L^{n-1} + x + \sum_{n=1}^{\infty} v_n L^{-n-1} \tag{2.15}$$

that obey the Lax equations

$$\frac{\partial M}{\partial t_n} = [B_n, M] \tag{2.16}$$

and the canonical commutation relation

$$[L, M] = 1. \tag{2.17}$$

Let us now consider the operators P and Q of the form

$$\begin{aligned} P &= L^p, \\ Q &= -\frac{1}{p} M L^{1-p} + \frac{p-1}{2p} L^{-p} + L^q, \end{aligned} \tag{2.18}$$

which automatically obey the canonical commutation relation

$$[Q,P]=1. \tag{2.19}$$

We further put constraints

$$(P)_{\le -1}=0, \quad (Q)_{\le -1}=0, \tag{2.20}$$

and restrict the range of time variables to

$$t_{p+q}=t_{p+q+1}=\cdots=0. \tag{2.21}$$

P and Q then become differential operators of the required form, and give a solution of the Douglas equation. From the standpoint of the KP hierarchy, therefore, it is the above constraints rather than the Douglas equation itself that plays a more fundamental role. In our terminology, "string equations" mean such constraints.

This correspondence with the KP hierarchy allows one to use many powerful tools developed for the study of the KP hierarchy, such as the Sato Grassmannian, the Hirota equations, $W_{1+\infty}$ algebras, etc. [8, 10, 11, 12].

The case of $(p,1)$ or $(1,q)$ models has been studied with particular interest. A main reason is that this is case where the models are expected to describe "topological gravity" [13]. Furthermore, unlike the other (p,q) models, the string equation of the $(p,1)$ models can be solved explicitly in terms of a "matrix Airy function" (or "Kontsevich integral") [14, 15, 16]. This is a matrix integral of the form

$$Z(\Lambda)=C_N(\Lambda)\int dM \exp \mathrm{Tr}(\Lambda^p M-\frac{1}{p+1}M^{p+1}), \tag{2.22}$$

where the integral is over the space of $N\times N$ Hermitian matrices, Λ is an $N\times N$ Hermitian matrix variable, and $C_N(\Lambda)$ is a normalization factor that also plays an important role. Kontsevich [14] pointed out two distinct interpretations of this integral. According to the first interpretation, this function can be viewed as a τ function of the KP hierarchy,

$$Z(\Lambda)=\tau(t), \tag{2.23}$$

by the so called "Miwa transformation":

$$t_n=\frac{1}{n}\mathrm{Tr}\,\Lambda^{-n}=\frac{1}{n}\sum_{i=1}^{N}\lambda_i^{-n}. \tag{2.24}$$

Here $\lambda_1, \ldots, \lambda_N$ are eigenvalues of Λ. According to the second interpretation (which is proven only for $p = 2$), this integral is a generating function of cohomological intersection numbers on moduli spaces of Riemann surfaces (in other words, the partition function of topological gravity). Kontsevich thus proved physicists' conjecture [13] at least in the case of $p = 2$.

The above p-th generalized Kontsevich model are further generalized by Kharchev et al. [17].

3 Dispersionless KP hierarchy

Let us illustrate the concept of "dispersionless limit" in the case of the KdV equation. To this end, it is convenient to write the KdV equation as

$$u_t + uu_x + \epsilon^2 u_{xxx} = 0. \tag{3.1}$$

Dispersionless limit means the limit as $\epsilon \to 0$. Formally, the equation then becomes

$$u_t + uu_x = 0. \tag{3.2}$$

This equation is well known to develop a shock wave in finite time. In the presence of the nonzero dispersion term $\epsilon^2 u_{xxx}$, this purely nonlinear effect is balanced with dispersion effect, so that stable waves like solitons can exist. We shall not go into such analytical issues, and only consider formal aspects of dispersionless limit. We are, for instance, interested in the fact that the dispersionless equation can be solved by the so called "hodograph transformation"

$$x - tu = f(u). \tag{3.3}$$

Here $f(u)$ is an arbitrary function, and $u = u(t, x)$ is understood to be defined implicitly by this equation.

The small parameter ϵ actually plays the role of the Planck constant $\hbar$ that we have observed in the classical limit of the Douglas equation. The same idea can be applied to the KP hierarchy itself. Let us replace $\partial_x \to \hbar\partial_x$ in the formulation of the KP hierarchy. The Lax and Orlov-Shulman operators are then given by

$$L = \hbar\partial_x + \sum_{n=1}^{\infty} u_{n+1}(\hbar\partial_x)^{-n},$$

$$M = \sum_{n=2}^{\infty} n t_n L^{n-1} + x + \sum_{n=1}^{\infty} v_n L^{-n-1}, \tag{3.4}$$

where the coefficients are now allowed to depend on $\hbar$ as well as (t, x). The Lax equations and the canonical commutation relations, too, are modified as:

$$\hbar \frac{\partial L}{\partial t_n} = [B_n, L], \quad \hbar \frac{\partial M}{\partial t_n} = [B_n, M] \tag{3.5}$$

and

$$[L, M] = \hbar. \tag{3.6}$$

Let us further assume that the coefficients u_n and v_n have a smooth limit $u_n^{(0)}$ and $v_n^{(0)}$ as $\hbar \to 0$. (In fact, this is a rather strong condition. One can obtain a solution of the above $\hbar$-dependent KP hierarchy from the $\hbar$-independent hierarchy just by rescaling $t_n \to \hbar^{-1} t_n$ and $x \to \hbar^{-1} x$, however it does not satisfy the last condition in general. Meanwhile, the solutions given by the generalized Kontsevich models do satisfy this condition, and have a dispersionless limit. We shall give a direct construction of this solution below.) As $\hbar \to 0$, operators and commutators are replaced by their "classical" counterparts,

$$\hbar \partial_x \to p, \quad \hbar^{-1}[A, B] \to \{A, B\}. \tag{3.7}$$

The Poisson bracket is given by

$$\{A, B\} = \partial_p A \cdot \partial_x B - \partial_x A \cdot \partial_p B. \tag{3.8}$$

L and M are accordingly replaced by the Laurent series

$$\begin{aligned} \mathcal{L} &= p + \sum_{n=1}^{\infty} u_{n+1}^{(0)} p^{-n}, \\ \mathcal{M} &= \sum_{n=2}^{\infty} n t_n \mathcal{L}^{n-1} + x + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n-1}, \end{aligned} \tag{3.9}$$

which satisfy the Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\} \tag{3.10}$$

and the canonical relation

$$\{\mathcal{L}, \mathcal{M}\} = 1 \tag{3.11}$$

with respect to the above Poisson bracket. Furthermore, $\mathcal{B}_n$ are given by

$$\mathcal{B}_n = \left(\mathcal{L}^n\right)_{\ge 0}, \tag{3.12}$$

where "$(\quad)_{\ge 0}$" now stands for the projection of Laurent series of p into the polynomial part. This hierarchy is called the "dispersionless KP hierarchy" [18]. As we have shown, its essence is rather a kind of classical limit of a quantum mechanical system (i.e., the full KP hierarchy).

In fact, this analogy with classical limit of quantum mechanics stems from a more profound structure. The Lax equations and the canonical commutation relations are derived as Frobenius integrability conditions of the linear system

$$\hbar \frac{\partial \Psi}{\partial t_n} = B_n \Psi, \quad \lambda \Psi = L \Psi, \quad \hbar \frac{\partial \Psi}{\partial \lambda} = M \Psi. \tag{3.13}$$

It is a "WKB analysis" of this quantum mechanical linear system that underlies the classical limit of the Lax formalism presented above [18]. Furthermore, the τ function, too, turns out to exhibit certain asymptotic behavior as $\hbar \to 0$, which is also crucial for understanding the meaning of the "free energy" function in various matrix models and topological string theories [6].

String equations of the (p, q) model, too, have a natural counterpart in the dispersionless limit. In the present $\hbar$-dependent formulation, P and Q are given by

$$\begin{aligned} P &= L^p, \\ Q &= -\frac{1}{p} M L^{1-p} + \hbar \frac{p-1}{2p} L^{-p} + L^q. \end{aligned} \tag{3.14}$$

In particular, the second term on the right hand side turns out to be a "quantum correction", and disappears in the dispersionless limit. In the limit to the dispersionless KP hierarchy, these operators turn into the Laurent series

$$\begin{aligned} \mathcal{P} &= \mathcal{L}^p, \\ \mathcal{Q} &= -\frac{1}{p} \mathcal{M} \mathcal{L}^{1-p} + \mathcal{L}^q, \end{aligned} \tag{3.15}$$

and the constraints (string equations) become

$$(\mathcal{P})_{\leq -1} = 0, \quad (\mathcal{Q})_{\leq -1} = 0. \tag{3.16}$$

If $q = 1$, these constraints can be solved explicitly by a hodographic method [19]. This solution coincides with the A_{p-1} "topological strings" [20, 21]. (Actually, a similar hodographic method can be used for topological strings of D-type [22].) One can thus identify the A_{p-1} model of topological strings as a dispersionless limit (or "spherical limit" in the terminology of string theory) of the $(p, 1)$ model, i.e., the p-th generalized Kontsevich model [23, 24].

4 Toda lattice hierarchy and its dispersionless limit

We now turn to the Toda lattice hierarchy [25]. The Lax and Orlov-Shulman operators of the Toda lattice hierarchy [26, 27] are difference operators of the form

$$\begin{aligned} L &= e^{\partial_s} + \sum_{n=0}^{\infty} u_{n+1} e^{-n\partial_s}, \\ M &= \sum_{n=1}^{\infty} n t_n L^n + s + \sum_{n=1}^{\infty} v_n L^{-n}, \\ \bar{L} &= \tilde{u}_0 e^{\partial_s} + \sum_{n=0}^{\infty} \tilde{u}_{n+1} e^{(n+2)\partial_s}, \\ \bar{M} &= -\sum_{n=1}^{\infty} n \bar{t}_n \bar{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n, \end{aligned} \tag{4.1}$$

where $e^{n\partial_s}$ are the shift operators that act on a function of s as $e^{n\partial_s} f(s) = f(s+n)$. The coefficients u_n, v_n, $\tilde{u}_n$ and $\bar{v}_n$ are functions of $(t, \bar{t}, q)$, $u_n = u_n(t, \bar{t}, s)$, etc. (The bar "¯" does not mean complex conjugate.) These operators obey the twisted canonical commutation relations

$$[L, M] = L, \quad [\bar{L}, \bar{M}] = \bar{L} \tag{4.2}$$

and the Lax equations

$$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad \frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L],$$

$$\frac{\partial M}{\partial t_n} = [B_n, M], \qquad \frac{\partial M}{\partial \bar{t}_n} = [\bar{B}_n, M],$$
$$\frac{\partial \bar{L}}{\partial t_n} = [B_n, \bar{L}], \qquad \frac{\partial \bar{L}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{L}],$$
$$\frac{\partial \bar{M}}{\partial t_n} = [B_n, \bar{M}], \qquad \frac{\partial \bar{M}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{M}], \tag{4.3}$$

where the Zakharov-Shabat operators B_n and $\bar{B}_n$ are given by

$$B_n = (L^n)_{\geq 0}, \quad \bar{B}_n = (\bar{L}^{-n})_{<0}, \tag{4.4}$$

and $(\quad)_{\geq 0, <0}$ denotes the projection

$$(\sum_n a_n e^{n\partial_s})_{\geq 0} = \sum_{n\geq 0} a_n e^{n\partial_s}, \quad (\sum_n a_n e^{n\partial_s})_{<0} = \sum_{n<0} a_n e^{n\partial_s}. \tag{4.5}$$

Note that ML^{-1} and $\bar{M}\bar{L}^{-1}$ give canonical conjugate "momenta" of L and $\bar{L}$:

$$[L, ML^{-1}] = 1, \quad [\bar{L}, \bar{M}\bar{L}^{-1}] = 1. \tag{4.6}$$

The Toda lattice hierarchy, like the KP hierarchy has a "dispersionless" analogue [28, 29]. (As we shall show below, this is in fact a "long-wave limit", but the hierarchy is now widely called the "dispersionless Toda hierarchy".)

Let us show what this limit looks like in the lowest two-dimensional sector of the hierarchy, i.e., the two-dimensional Toda field theory. In terms of two-dimensional fields $u(s) = u(t_1, \bar{t}_1, s)$, $s \in \mathbf{Z}$, the equations of motion can be written

$$\partial_{t_1}\partial_{\bar{t}_1} u(s) = \exp\Big(u(s+1) - u(s)\Big) - \exp\Big(u(s) - u(s-1)\Big). \tag{4.7}$$

We now rescale the field and space-time variables as

$$\partial_{t_1} \to \hbar\partial_{t_1}, \quad \partial_{\bar{t}_1} \to \hbar\partial_{\bar{t}_1},$$
$$s \pm 1 \to s \pm \hbar, \quad u(s) \to \hbar^{-1}u(s). \tag{4.8}$$

The equations of motion then become

$$\partial_{t_1}\partial_{\bar{t}_1} u(s) = \frac{1}{\hbar}\left(\exp\Big(\frac{u(s+1) - u(s)}{\hbar}\Big) - \exp\Big(\frac{u(s) - u(s-1)}{\hbar}\Big)\right), \tag{4.9}$$

and reduce, as $\hbar \to 0$, to

$$\partial_{t_1}\partial_{\bar{t}_1} u = \partial_s \exp(\partial_s u). \tag{4.10}$$

This is the long-wave limit of the two-dimensional Toda equations. Remarkably, this equation coincides with a dimensional reduction of the self-dual Einstein equation [30].

To extend this limit to the hierarchy itself, we need an $\hbar$-dependent formulation of the Toda lattice hierarchy. This can be achieved by inserting $\hbar$ in front of all derivatives in the previous equations as:

$$\frac{\partial}{\partial t_n} \to \hbar\frac{\partial}{\partial t_n}, \quad \frac{\partial}{\partial \bar{t}_n} \to \hbar\frac{\partial}{\partial \bar{t}_n}, \quad e^{\partial_s} \to e^{\hbar\partial_s}. \tag{4.11}$$

The discrete variable s now takes values in $\hbar\mathbf{Z}$. Accordingly, Lax and Orlov-Shulman operators are difference operators of the form

$$\begin{aligned}
L &= e^{\hbar\partial_s} + \sum_{n=0}^{\infty} u_{n+1} e^{-n\hbar\partial_s}, \\
M &= \sum_{n=1}^{\infty} n t_n L^n + s + \sum_{n=1}^{\infty} v_n L^{-n}, \\
\bar{L} &= \tilde{u}_0 e^{\hbar\partial_s} + \sum_{n=0}^{\infty} \tilde{u}_{n+1} e^{(n+2)\hbar\partial_s}, \\
\bar{M} &= -\sum_{n=1}^{\infty} n\bar{t}_n \bar{L}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n,
\end{aligned} \tag{4.12}$$

and obey the twisted canonical commutation relations

$$[L, M] = \hbar L, \quad [\bar{L}, \bar{M}] = \hbar\bar{L}, \tag{4.13}$$

and Lax equations

$$\hbar\frac{\partial L}{\partial t_n} = [B_n, L], \quad \hbar\frac{\partial L}{\partial \bar{t}_n} = [\bar{B}_n, L], \quad \text{etc..} \tag{4.14}$$

We can now proceed as in the KP hierarchy. In this $\hbar$-dependent formulation, the long-wave limit is nothing but the "classical limit":

$$e^{\hbar\partial_s} \to P = e^p, \quad \hbar^{-1}[A, B] \to \{A, B\}. \tag{4.15}$$

The Poisson bracket takes a somewhat unusual form,

$$\{A, B\} = P\partial_P A \cdot \partial_s B - \partial_s A \cdot P\partial_P B. \tag{4.16}$$

Thus, in particular,

$$\{P, s\} = P, \tag{4.17}$$

or, equivalently,

$$\{P, sP^{-1}\} = 1. \tag{4.18}$$

Counterparts of the Lax and Orlov-Shuman operators are given by the Laurent series

$$\begin{aligned}
\mathcal{L} &= P + \sum_{n=0}^{\infty} u_{n+1}^{(0)} P^{-n}, \\
\mathcal{M} &= \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + s + \sum_{n=1}^{\infty} v_n^{(0)} \mathcal{L}^{-n}, \\
\bar{\mathcal{L}} &= \tilde{u}_0^{(0)} P + \sum_{n=0}^{\infty} \tilde{u}_{n+1}^{(0)} P^{n+2}, \\
\bar{\mathcal{M}} &= -\sum_{n=1}^{\infty} n \bar{t}_n \bar{\mathcal{L}}^{-n} + s + \sum_{n=1}^{\infty} \bar{v}_n^{(0)} \bar{\mathcal{L}}^n,
\end{aligned} \tag{4.19}$$

where $u_n^{(0)} = u_n|_{\hbar\to 0}$, etc. They obey the twisted canonical relations

$$\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}, \quad \{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}} \tag{4.20}$$

and the Lax equations

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{L}\}, & \frac{\partial \mathcal{L}}{\partial \bar{t}_n} &= \{\bar{\mathcal{B}}_n, \mathcal{L}\}, \\
\frac{\partial \mathcal{M}}{\partial t_n} &= \{\mathcal{B}_n, \mathcal{M}\}, & \frac{\partial \mathcal{M}}{\partial \bar{t}_n} &= \{\bar{\mathcal{B}}_n, \mathcal{M}\}, \\
\frac{\partial \bar{\mathcal{L}}}{\partial t_n} &= \{\mathcal{B}_n, \bar{\mathcal{L}}\}, & \frac{\partial \bar{\mathcal{L}}}{\partial \bar{t}_n} &= \{\bar{\mathcal{B}}_n, \bar{\mathcal{L}}\}, \\
\frac{\partial \bar{\mathcal{M}}}{\partial t_n} &= \{\mathcal{B}_n, \bar{\mathcal{M}}\}, & \frac{\partial \bar{\mathcal{M}}}{\partial \bar{t}_n} &= \{\bar{\mathcal{B}}_n, \bar{\mathcal{M}}\}
\end{aligned} \tag{4.21}$$

with respect to the above Poisson bracket. This hierarchy is called "dispersionless Toda hierarchy".

Various results on the dispersionless KP hierarchy are extended to this hierarchy [6].

5 String equations in Toda lattice hierarchy

As for the Toda lattice hierarchy, our knowledge on string equations is more fragmental, but simultaneously suggests richer possibilities.

An important family of string equations is provided by Hermitian (one- and multi-)matrix models. These matrix models give a special solution of the Toda lattice hierarchy satisfying extra "Virasoro constraints" [31]. In fact, Virasoro constraints are string equations in disguise. This fact lies in the heart of the matrix model approach to two-dimensional quantum gravity. To arrive at the continuum limit (two-dimensional gravity), however, one has to take the so called "double scaling limit" [4, 5]. This scaling limit destroys the structure of the Toda lattice hierarchy, and the final outcome is rather the KP hierarchy and the string equations of (p, q) models.

In recent years, several new, and more natural examples of string equations in the Toda lattice hierarchy have come to be studied. These examples are mostly related to string theories with a one-dimensional target space: $c = 1$ strings [32, 33, 34, 35], two-dimensional topological strings [36, 37, 38, 39, 40], the topological CP^1 sigma model and its variations related to affine Coxeter groups [41, 42, 43].

String equations in those models are formulated in the form of a canonical transformation between the two canonical pairs (L, M) and $(\bar{L}, \bar{M})$. For instance, string equations of the one-matrix model can be written

$$L = \bar{L}^{-1}, \quad ML^{-1} = -\bar{M}\bar{L}, \tag{5.1}$$

and those of the two-matrix model are given by

$$L = -\bar{M}\bar{L}, \quad ML^{-1} = -\bar{L}^{-1}. \tag{5.2}$$

The string equations of the one-matrix model are coordinate-to-coordinate and momentum-to-momentum relations, and those of the two-matrix model mix coordinates and momenta.

A remark will be now in order: String equations in the literature may have extra terms on the right hand side such as $V'(L)$ where V is a matrix model potential. These extra terms can be absorbed into redefinition of M and $\bar{M}$ as

$$M \to M + f(L), \quad \bar{M} \to \bar{M} + g(\bar{L}), \tag{5.3}$$

where $f(L)$ and $g(\bar{L})$ are Laurent series of L and $\bar{L}$ with constant coefficients. The above equations should be understood in this manner.

Actually, most examples of string equations in the Toda lattice hierarchy are variants of the above two. For instance, a straightforward generalization of the string equations of the one-matrix model will be given by

$$L^p = \bar{L}^{-\bar{p}}, \quad \frac{1}{p} M L^{-p} = \frac{-1}{\bar{p}} \bar{M} \bar{L}^{\bar{p}}. \tag{5.4}$$

This is indeed the model of Aoyama and Kodama [44], which contains several interesting models of topological strings as special solutions. (More precisely, they consider these string equations in the dispersionless limit.)

Similarly, a generalization of the string equations of the two-matrix model will be of the form

$$L^p = \frac{1}{\bar{p}} \bar{M} \bar{L}^{\bar{p}}, \quad -\frac{1}{p} M L^{-p} = \bar{L}^{-\bar{p}}. \tag{5.5}$$

String equations of compactified $c = 1$ strings are indeed of this type [34]:

$$\begin{aligned} L^{\beta} &= \Big(-\frac{1}{\beta}\bar{M} - \hbar\frac{\beta+1}{2\beta} + 1\Big)\bar{L}^{\beta}, \\ \bar{L}^{-\beta} &= \Big(-\frac{1}{\beta}M + \hbar\frac{\beta-1}{2\beta} + 1)L^{-\beta}. \end{aligned} \tag{5.6}$$

Here β is a positive integer parameter related to the compactification redius of strings. In this case, p and $\bar{p}$ are given by $p = -\bar{p} = \beta$. The extra terms other than $\bar{M}\bar{L}^{\beta}$ and $ML^{-\beta}$ on the right hand side, as noted above, can be absorbed into (or reproduced from)redefinition of M and $\bar{M}$.

The last example can be generalized as follows [45]: Given a pair of integers $(p, \bar{p})$ with $p\bar{p} \neq 0$, one can construct a solution of the Toda lattice hierarchy that obeys the string equations

$$\begin{aligned} L^p &= \frac{1}{\bar{p}}\bar{M}\bar{L}^{-\bar{p}} - \hbar\frac{\bar{p}-1}{2\bar{p}}\bar{L}^{-\bar{p}} + \bar{L}^{\bar{p}}, \\ \bar{L}^{\bar{p}} &= -\frac{1}{p}ML^{-p} + \hbar\frac{p-1}{2p}L^{-p} + L^{\bar{p}}. \end{aligned} \tag{5.7}$$

Let us show an outline of the construction of the solution [45]. The solution is given in terms of the τ function $\tau = \tau(t, \bar{t}, s)$. The τ function of any solution of the Toda lattice hierarchy can be written (at least formally) as a semi-infinite determinant of the form

$$\tau(t,\bar{t},s) = \det\Big(u_{ij}(t,\bar{t})\ (-\infty < i,j < s)\Big). \tag{5.8}$$

Here $\mathbf{U}(t,\bar{t}) = \Big(u_{ij}(t,\bar{t})\Big)$ is an infinite ($\mathbf{Z}\times\mathbf{Z}$) matrix connected with its "initial value" $\mathbf{U} = \mathbf{U}(0,0)$ as

$$\mathbf{U}(t,\bar{t}) = \exp(\sum_{n=1}^{\infty} t_n \mathbf{\Lambda}^n)\mathbf{U}(0,0)\exp(-\sum_{n=1}^{\infty}\bar{t}_n\mathbf{\Lambda}^{-n}). \tag{5.9}$$

$\mathbf{\Lambda}^n$ are the shift matrices

$$\mathbf{\Lambda}^n = \Big(\delta_{i,j-n}\Big). \tag{5.10}$$

Thus the matrix $\mathbf{U}$ determines a solution of the Toda lattice hierarchy. In the construction of a solution of the above string equations, we define the matrix elements u_{ij} to be coefficients of asymptotic expansion of a set of generating functions. Let us now consider the case of $p+\bar{p}>0$. (The case of $p+\bar{p}=0$ coincides with the $c=1$ strings mentioned above. The case of $p+\bar{p}<0$ requires a slightly different treatment.) The generating functions $u_j(\lambda)$ ($j\in\mathbf{Z}$) are given by integrals of the form

$$\begin{aligned} u_j(\lambda) &= c(\lambda)\int d\mu\mu^{(\bar{p}-1)/2-j-1}\exp\hbar^{-1}\Big(\lambda^p\mu^{\bar{p}} - \frac{\bar{p}}{p+\bar{p}}\mu^{p+\bar{p}}\Big), \\ c(\lambda) &= \text{const.}\ \lambda^{(p-1)/2}\exp\hbar^{-1}\Big(\frac{-p}{p+\bar{p}}\lambda^{p+\bar{p}}\Big). \end{aligned} \tag{5.11}$$

The integrals are over a complex path passing through $\mu=\lambda$. The detailed form of the path is irrelevant; the essence is that it is only a neighborhood of $\mu=\lambda$ (saddle point) that eventually contributes to asymptotic expansion as $\lambda\to+\infty$. One can show by the standard saddle point calculation that these functions have asymptotic expansion of the following form:

$$u_j(\lambda) \sim \sum_{i=j}^{\infty}\lambda^{-i-1}u_{ij} \quad (\lambda\to+\infty). \tag{5.12}$$

The matrix elements u_{ij} of $\mathbf{U}$ are thus determined. In particular, $\mathbf{U}$ is a lower triangular matrix with nonvanishing diagonal elements. This allows us to express the τ function as a finite determinant under the Miwa transformation of time variables. Furthermore, these matrix elements obey a set of linear relations. Translated into the language of the Lax and Orlov-Shulman operators, these linear relations give rise to the string equations above.

This construction is almost parallel to the case of the (generalized) Kontsevich models in the KP hierarchy In particular, the p-th generalized Kontsevich model itself can be reinterpreted as a solution of the Toda lattice hierarechy of this type with $(p, \bar{p}) = (p, -1)$. Apart from this case and the case of $\bar{p} = 1$, however, the solution does not have a matrix integral representation of Kontsevich type. Such a matrix integral representation is crucial for identifying the model with a topological field theory. Because of the absence of such a matrix integral representation, it is unclear if these generalized string equations have any physical interpretation (hopefully, as a kind of $c = 1$ string theory).

6 Conclusion

Since the late eighties, string theories have provided a variety of new material to studies on integrable hierarchies. The concept of string equations has been in the center of this new trend. Of course, several key ingredients had been prepared in advance in the early eighties:

1. The theory of τ functions [2] forms a theoretical foundation of all the subsequent progress.

2. The theory of Orlov and Shulman [9], which was developed for a slightly different purpose, is now the most natural and useful framework for formulating string equations.

3. String equations are also related to isomonodromy deformations and Painlevé transcendents [46]. The string equation of $(2, 3)$ model, for instance, is nothing but the first Painlevé equation in disguise. Furthermore, although we have omitted in this review, there is another family of matrix models, the "unitary matrix models", and their double scaling limit [31]. In the simplest case, this gives the the second Painlevé equation. All these models have an interpretation as isomonodromy deformations [47].

4. The theory of dispersionless integrable hierarchies and related "hydrodynamic integrable systems" is very useful in both technical and conceptual aspects.

The last point deserves to be mentioned in more detail. As pointed out by Dubrovin [48] in a far broader scope, this type of integrable structures can be found in many models of two-dimensional topological field theories. The dispersionless KP and Toda hierarchies are just the simplest examples, and moreover, of "genus zero" type. "Higher genus" counterparts (i.e., systems related to a moduli space of higher genus Riemann surfaces) are given by the so called "Whitham equations". These equations were first derived as equations for modulation of quasi-periodic solutions in soliton equations [49], and later reformulated in a more abstract way by Dubrovin [50] and Krichever [51]. Very recently, it is pointed out that this type of equations also emerge in four dimensional $N = 2$ supersymmetric gauge theories [52]. This seems to indicate a new link between integrable hierarchies and solvable quantum field theories (and underlying string theories).

References

[1] B.A. Dubrovin, V.B. Matveev and S.P. Novikov, Non-linear equations of Korteweg-de Vries type, finite-zone linear operators and Abelian varieties, Russ. Math. Surveys **31:1** (1976), 59-146.
I.M. Krichever, Methods of algebraic geometry in the theory of nonlinear equations, Russ. Math. Surveys **32:6** (1977), 185-213.
D. Munford, An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg-de Vries equation and related non-linear equations, in: Proc. International Symposium of Algebraic Geometry, Kyoto 1977, pp. 115-153 (Kinokuniya, Tokyo, 1978).

[2] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, in *Nonlinear Partial Differential Equations in Applied Science; Proceedings of the U.S.-Japan Seminar, Tokyo, 1982*, Lect. Notes in Num. Anal. **5** (1982), 259-271.
M. Sato and M. Noumi, Soliton equations and the universal Grassmann manifolds, Sophia Univ. Kokyuroku in Math. **18** (1984), in Japanese.

E. Date, M. Kashiwara, M. Jimbo and T. Miwa, Transformation groups for soliton equations, in *Nonlinear Integrable Systems – Classical Theory and Quantum Theory* (World Scientific, Singapore, 1983), 39-119.
G. Segal and G. Wilson, Loop groups and equations of KdV type, IHES Publ. Math. **63** (1985) 1-64.

[3] M. Mulase, Cohomological structure in soliton equations and Jacobian varieties, J. Diff. Geom. **19** (1984), 403-430; Category of vector bundles on algebraic curves and infinite dimensional Grassmannians, Int. J. Math. **1** (1990), 293-342.
Shiota, T., Characterization of Jacobian varieties in terms of soliton equations, Inv. Math. **83** (1986), 333-382.

[4] E. Brezin and V.A. Kazakov, Exactly solvable field theories of closed strings, Phys. Lett. **B236** (1990), 144-150.
M.R. Douglas and S.H. Shenker, Strings in less than one-dimension, Nucl. Phys. **B335** (1990), 635-654.
D.J. Gross and A.A. Migdal, Nonperturbative two-dimensional quantum gravity, Phys. Rev. Lett. **64** (1990), 127-130.

[5] M. Douglas, Strings in less than one-dimension and the generalized K-dV hierarchies, Phys. Lett. **238B** (1990) 176-180.

[6] Papers dealing with string equations:
K. Takasaki and T. Takebe, SDiff(2) KP hierarchy, in: A. Tsuchiya, T. Eguchi and T. Miwa (eds.), *Infinite Analysis*, RIMS Research Project 1991, Int. J. Mod. Phys. **A7**, Suppl. 1, pp.889-922 (World Scientific, Singapore, 1992).
K. Takasaki, Integrable hierarchy underlying topological Landau-Ginzburg models of D-type, Lett. Math. Phys. **29** (1993), 111-121.
K. Takasaki, Dispersionless Toda hierarchy and two-dimensional string theory, Commun. Math. Phys. **170** (1995), 101-116.
T. Nakatsu, K. Takasaki and S. Tsujimaru, Quantum and classical aspects of deformed $c = 1$ strings, Nucl. Phys. **B443** (1995), 15-197.
K. Takasaki, Toda lattice hierarchy and generalized string equations, KUCP-0079, hep-th/9506089.
Related topics (mostly, dispersionless limit):
K. Takasaki and T. Takebe, Integrable hierarchies and dispersionless

limit, UTMS 94-35, hep-th/9405096, Rev. Math. Phys. (to appear), and references cited therein.

[7] J.L. Burchnall and T.W. Chaundy, Commutative ordinary differential operators, Proc. London Math. Soc. **21** (1922), 420-440; Proc. Royal So. London **A118** (1928), 557-583; ibid. **134** (1931), 471-485.

[8] A. Schwarz, On solutions to the string equations, Mod. Phys. Lett. **A6** (1991), 2713-2726.
V. Kac and A.Schwarz, Geometric interpretation of partition function of 2D gravity, Phys. Lett. **B257** (1991), 329-334.

[9] A.Yu. Orlov and E.I. Schulman, Additional symmetries for integrable equations and conformal algebra representation, Lett. Math. Phys. **12** (1986), 171-179.
A.Yu. Orlov, Vertex operators, $\bar{\partial}$-problems, symmetries, variational indentities and Hamiltonian formalism for $2+1$ integrable systems, in *Plasma Theory and Nonlinear and Turbulent Processes in Physics* (World Scientific, Singapore, 1988).
P.G. Grinevich and A.Yu. Orlov, Virasoro action on Riemann surfaces, Grassmannians, $\det \bar{\partial}_j$ and Segal Wilson τ function, in *Problems of Modern Quantum Field Theory* (Springer-Verlag, Berlin, 1989).

[10] J. Goeree, W constraints in 2d quantum gravity, Nucl. Phys. **B358** (1991), 737-757.

[11] M. Fukuma, H. Kawai and R. Nakayama, Infinite dimensional Grassmannian structure of two dimensional string theory, Commun. Math. Phys. **143** (1991), 371-403.

[12] T. Yoneya, Toward a canonical formalism of non-perturbative two-dimensional gravity, Commun. Math. Phys. **144** (1992), 623-639.

[13] E. Witten, On the structure of the topological structure of the topological phase of two dimensional gravity, Nucl. Phys. **B340** (1990), 281.
R. Dijkgraaf and E. Witten, Mean field theory, topological field theory, and multi-matrix models, Nucl. Phys. **B342** (1990), 486.
E. Verlinde and H. Verlinde, A solution of two-dimensional topological quantum gravity, Nucl. Phys. **B348** (1991), 457.

E. Witten, Two dimensional gravity and intersection theory on moduli spaces, Surveys in Diff. Geom. 1 (1991), 243.
R. Dijkgraaf, Intersection theory, integrable hierarchies, and topological field theory, in *New Symmetry Principles in Quantum Field Theory*, NATO ASI Cargèse 1991 (Plenum, 1991).

[14] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Commun. Math. Phys. **147** (1992), 1-23.

[15] M. Adler and P. van Moerbeke, A matrix integral solution to two-dimensional W_p-gravity, Commun. Math. Phys. **147** (1992), 25-56.

[16] C. Itzykson, and J.B. Zuber, Combinatorics of the modular group II: the Kontsevich integrals, Intern. J. Mod. Phys. **A7** (1992), 5661-5705.

[17] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, Towards unified theory of 2d gravity, Nucl. Phys. **B380** (1992), 181-240.

[18] Y. Kodama, A method for solving the dispersionless KP equation and its exact solutions, Phys. Lett. **129A** (1988), 223-226.
Y. Kodama and J. Gibbons, A method for solving the dispersionless KP hierarchy and its exact solutions, II, Phys. Lett. **135A** (1989), 167-170.
Y. Kodama and J. Gibbons, Integrability of the dispersionless KP hierarchy in: *Proceedings of Workshop "Nonlinear and Turbulent Processes in Physics"*, Kiev 1989 (World Scientific, Singapore, 1990).

[19] K. Takasaki and T. Takebe, SDiff(2) KP hierarchy [6]

[20] R. Dijkgraaf, H. Verlinde E. Verlinde, Topological strings in $d < 1$, Nucl. Phys. **B352** (1991), 59-86; Loop equations and Virasoro constraints in non-perturbative 2d quantum gravity, Nucl. Phys. **B348** (1991), 435-456.

[21] I.M. Krichever, The dispersionless Lax equations and topological minimal models, Commun. Math. Phys. **143** (1991), 415-426.

[22] K. Takasaki, Integrable hierarchy underlying . . . [6].

[23] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, Landau-Ginzburg topological theories in the framework of GKM and equivalent hierarchies, Mod. Phys. Lett. **A8** (1993), 1047-1062.

[24] T. Nakatsu, A. Kato, M. Noumi and T. Takebe, Topological strings, matrix integrals, and singularity theory, Phys. Lett. **B322** (1994), 192-197.

[25] K. Ueno and K. Takasaki, Toda lattice hierarchy, in *Group Representations and Systems of Differential Equations*, K. Okamoto ed., Advanced Studies in Pure Math. **4** (North-Holland/Kinokuniya, 1984).

[26] K. Takasaki and T. Takebe, Quasi-classical limit of Toda hierarchy and W-infinity symmetries, Lett. Math. Phys. **28** (1993), 165-176.

[27] M. Adler, T. Shiota and P. van Moerbeke, A Lax Representation for the Vertex Operator and the central extension, Commun. Math. Phys. **171** (1995), 547-588.

[28] Y. Kodama, Solutions of the dispersionless Toda equation, Phys. Lett. **147A** (1990), 477-482.

[29] K. Takasaki and T. Takebe, SDiff(2) Toda equation - hierarchy, tau function and symmetries, Lett. Math. Phys. **23** (1991), 205-214.

[30] C. Boyer and J.D. Finley, Killing vectors in self-dual, Euclidean Einstein spaces, J. Math. Phys. **23** (1982), 1126-1128.
J.D. Gegenber and A. Das, Stationary Riemannian space-times with self-dual curvature, Gen. Rel. Grav. **16** (1984), 817-829.

[31] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, Matrix models of 2D gravity and Toda theory, Nucl. Phys. **B357** (1991), 565-618.
E.J. Martinec, On the origin of integrability in matrix models, Commun. Math. Phys. **138** (1991), 437-450.
L. Alvarez-Gaumé, C. Gomez and J. Lacki, Integrability in random matrix models, Phys. Lett. **B253** (1991), 56-62.
L. Bonora and C.S. Xiong, Matrix models without scaling limit, Int. J. Mod. Phys. **A8** (1993), 2973-2992; Multimatrix models without continuum limit, Nucl. Phys. **B405** (1993), 191-227.
S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, Generalized Kontsevich model versus Toda hierarchy and discrete matrix models, Nucl. Phys. **B397** (1993), 339-378.

[32] R. Dijkgraaf, G. Moore and R. Plesser, The partition function of 2D string theory, Nucl. Phys. B394 (1991), 356-382.

[33] A. Marshakov, On the string field theory for $c \leq 1$, in *Pathways to Fundamental Theories*, Geteborg, Sweden, 1992 (World Scientific, Singapore, 1993).

[34] T. Nakatsu, On the string equation at $c = 1$, Mod. Phys. Lett. **A9** (1994), 3313-3324.

[35] T. Nakatsu, K. Takasaki and S. Tsujimaru, Quantum and Classical ... [6].

[36] D. Ghoshal and S. Mukhi, Topological Landau-Ginzburg model of two-dimensional string theory, Nucl. Phy. **B425** (1994), 173-190.

[37] A. Hanany, Y. Oz and R. Plesser, Topological Landau-Ginzburg formulation and integrable structure of 2d string theory, Nucl. Phys. **B425** (1994), 150-172.

[38] T. Eguchi and H. Kanno, Toda lattice hierarchy and the topological description of $c = 1$ string theory, Phys. Lett. **B331** (1994), 330-334.

[39] K. Takasaki, Dispersionless Toda hierarchy ... [6].

[40] L. Bonora and C.S. Xiong, Two-matrix model and $c = 1$ string theory, Phys. Lett. **B347** (1995), 41-48; Extended Toda lattice hierarchy, extended two-matrix model and $c = 1$ string theory, Nucl. Phys. **B434** (1995), 408-444.

[41] T. Eguchi and S.-K. Yang, The topological CP^1 model and the large N matrix integral, Mod. Phys. Lett. **A9** (1994), 2893-2902.

[42] H. Kanno and Y. Ohta, Topological strings with scaling violation and Toda lattice hierarchy, Nucl. Phys. **B442** (1995), 179-204.

[43] T. Eguchi, K. Hori and S.-K. Yang, Topological sigma models and large N matrix integral, UT-700, hep-th/9503017.

[44] S. Aoyama and Y. Kodama, Topological conformal field theory with a rational W potential and the dispersionless KP hierarchy, Mod. Phys. Lett. **A9** (1994), 2481; Topological Landau-Ginzburg theory with a rational potential and the dispersionless KP hierarchy, hep-th/9505122.

[45] K. Takaski, Toda lattice hierarchy . . . [6].

[46] H. Flaschka and A.C. Newell, Monodromy- and spectrum-preserving deformations I, Commun. Math. Phys. **76** (1980), 65-116.
M. Mimbo, T. Miwa and K. Ueno, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I and II, Physica **2D** (1981), 306-352, 407-448.
T. Miwa, Painlevè property of monodromy preserving equations and the analyticity of τ-functions, Publ. RIMS **17** (1981), 703-721.

[47] G. Moore, Geometry of the string equations, Commun. Math. Phys. 133 (1990), 261-304; Matrix models of 2D gravity and isomonodromic deformation, in: *Common Trends in Mathematics and Quantum Field Theories*, Prog. Theor. Phys. Suppl. **102** (1990), 255-285.

[48] B.A. Dubrovin, Integrable systems in topological field theory, Nucl. Phys. **B379** (1992), 627-689; Geometry of 2D topological field theories, SISSA-89/94/FM, hep-th/9407018.

[49] See the following reviews and references cited therein:
B.A. Dubrovin and S.P. Novikov, Hydrodynamics of weakly deformed soliton lattices: Differential geometry and Hamiltonian theory, Russ. Math. Surveys **44:6** (1989), 35-124.
P.D. Lax, C.D. Levermore and S. Venakides, The generation and propagation of oscillations in dispersive initial value problems and their limiting behavior, in: *Important Developments in Soliton Theory*, A.S. Fokas and V.E. Zakharov (eds.), Series in Nonlinear Dynamics (Springer-Verlag, Berlin, 1994).

[50] B.A. Dubrovin, Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models, Commun. Math. Phys. **145** (1992), 195-207; Integrable systems in topological field theory [48].

[51] I.M. Krichever, Topological minimal models and soliton equations, Talk at the 1st Sakharov Conference, Moscow 1991; The τ-Function of the universal Whitham hierarchy, matrix models and topological field theories, Commun. Pure and Appl. Math. **XLVII** (1994), 437-475.

[52] A.Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Integrability and Seiberg-Witten exact solution, Phys. Lett. **B355** (1995), 466-474.