Direct linearization and Sato Grassmannian

Kanehisa Takasaki
OCAMI, Osaka Metropolitan University

Contents

Reference

K.T., Direct linearization, Cauchy matrix and Sato Grassmannian, arXiv:2608.24538 (62pp).

Direct linearization (DL)

— Direct approach to integrable systems, in particular, discrete systems, without using Lax equations, inverse scattering, Riemann-Hilbert problem, etc.

- Using singular integral equations

  Fokas & Ablowitz 1981, 1983, Nijhoff, van der Linden, Quispel, Capel & Velthuizen 1982, 1984, Nijhoff, Quispel & Capel 1983

- Using infinite matrices

  Fu & Nijhoff 2017, 2018, Fu 2018

— Closely related to Cauchy matrix approach

Nijhoff, Atkinson & Hietarinta 2009, D.J. Zhang & S.L. Zhao 2013, W. Feng and S.L. Zhao 2013, D.D. Xu, D.J. Zhang & S.L. Zhao 2014, W. Feng & S. Zhao 2022.

1. Fu and Nijhoff's DL scheme

Infinite matrix algebra

- Special $\mathbb{Z} \times \mathbb{Z}$ matrices

$$\Lambda^k = \sum_{i \in \mathbb{Z}} E_{i,i+k}, \quad \mathcal{O}_k = \sum_{i=0}^{k-1} E_{i,k-i-1},$$
$$\Omega = \sum_{i=0}^{\infty} E_{i,-i-1}, \quad E_{kl} = (\delta_{ik}\delta_{jl})_{i,j \in \mathbb{Z}}.$$

and algebraic relation $\mathcal{O}_k = \Omega\Lambda^k - \Lambda^{-k}\Omega$.

- Relation to Cauchy kernel

$$\sum_{i,j \in \mathbb{Z}} \Omega_{ij} w^i z^j = \sum_{i=0}^{\infty} \frac{w^i}{z^{i+1}} = \frac{1}{z-w} \quad \text{for } |z| > |w|.$$

1. Fu and Nijhoff's DL scheme

U-system (Fu & Nijhoff 2017, 2018)

Fu and Nijhoff employ the evolution equations

$$\frac{\partial U}{\partial t_k} = \Lambda^k U - U\Lambda^{-k} - U\mathcal{O}_k U, \quad k = 1, 2, \ldots,$$

of a $\mathbb{Z} \times \mathbb{Z}$ matrix $U = (u_{ij})_{i,j\in\mathbb{Z}}$ for direct linearization of the KP hierarchy and its reductions.

• This system has quadratic nonlinearity in the last term $U\mathcal{O}_k U$.

• This system can be transformed, at least formally, to the linear system

$$\frac{\partial C}{\partial t_k} = \Lambda^k C - C\Lambda^{-k}, \quad k = 1, 2, \cdots$$

by the transformation $U = C(E + \Omega C)^{-1}$ (E is the unit matrix).

• Special solutions of the Gramian type can be written with the Cauchy matrix $\mathcal{M}$ as

$$u_{ij} = {}^{\mathrm{t}}\mathbf{r}K^i(E_N + A\mathcal{M})^{-1}AL^j\mathbf{s} = {}^{\mathrm{t}}\mathbf{r}K^iA(E_M + \mathcal{M}A)^{-1}L^j\mathbf{s}$$

where

$$A = (a_{ij})_{1\le i\le N, 1\le j\le M},\ \ \mathcal{M} = \left(\frac{\sigma(\lambda_i)\rho(\kappa_j)}{\kappa_j - \lambda_i}\right)_{1\le i\le M, 1\le j\le N},$$

$$K = \mathrm{diag}(\kappa_1, \ldots, \kappa_N),\ \ L = \mathrm{diag}(\lambda_1, \ldots, \lambda_M),$$

$$\mathbf{r} = (\rho(\kappa_i))_{i=1}^{N},\ \ \mathbf{s} = (\sigma(\lambda_i))_{i=1}^{M},$$

$$\rho(z) = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right),\ \ \sigma(z) = \exp\left(-\sum_{k=1}^{\infty} t_k z^{-k}\right).$$

$u_{ij} = S^{(ij)}$ in the Cauchy matrix approach.

2. KP hierarchy

Infinite matrices and Sato Grassmannian (Sato & Sato 1982)

The Sato Grassmannian can be described as a coset space:

$$\begin{gathered}
\mathrm{Gr} \simeq \mathrm{Fr}(\mathbb{Z}, \mathbb{Z}_{<0})/\mathrm{GL}(\mathbb{Z}_{<0}) \simeq \mathrm{GL}(\mathbb{Z}_{\geq 0})\backslash \mathrm{Fr}(\mathbb{Z}_{\geq 0}, \mathbb{Z}), \\
\mathrm{Fr}(\mathbb{Z}, \mathbb{Z}_{<0}) = \{\xi \mid \mathbb{Z} \times \mathbb{Z}_{<0} \text{ matrix of full rank}\}, \\
\mathrm{Fr}(\mathbb{Z}_{\geq 0}, \mathbb{Z}) = \{\eta \mid \mathbb{Z}_{\geq 0} \times \mathbb{Z} \text{ matrix of full rank}\}, \\
\mathbb{Z}_{<0} = \{-1, -2, \ldots\}, \quad \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}.
\end{gathered}$$

Points of the top cell of Gr can be represented by the following matrices:

$$\begin{gathered}
\xi = \left(\frac{\delta_{ij}}{w_{i,-j-1}} \right) \in \mathrm{Fr}(\mathbb{Z}, \mathbb{Z}_{<0}), \\
\eta = \left(\; -w_{i,-j-1} \; \middle| \; \delta_{ij} \; \right) \in \mathrm{Fr}(\mathbb{Z}_{\geq 0}, \mathbb{Z}).
\end{gathered}$$

$\{w_{ij}\}_{i,j=0}^{\infty}$ are affine coordinates of the top cell.

Evolution equations for affine coordinates w_{ij}'s (T. 1985, 1989)

The KP hierarchy can be viewed as a dynamical system on Gr (Sato & Sato 82). The motion of a point of the top cell of Gr can be described by the evolution equations

$$\frac{\partial w_{ij}}{\partial t_k} = w_{i+k,j} - w_{i,j+k} - \sum_{l=0}^{k-1} w_{il} w_{k-l-1,j}, \quad i,j \geq 0.$$

for the affine coordinates w_{ij}. Note that these equations have quadratic nonlinear terms.

Remark: This system can be cast into the matrix forms

$$\frac{\partial \xi}{\partial t_k} = \Lambda^k \xi - \xi \mathcal{B}_k, \quad \frac{\partial \eta}{\partial t_k} = \mathcal{C}_k \eta - \eta \Lambda^k.$$

Key observation

The evolution equations of u_{ij}'s for $i, j \geq 0$ coincide with the evolution equations of w_{ij}'s.

$$\frac{\partial u_{ij}}{\partial t_k} = u_{i+k,j} - u_{i,j+k} - \sum_{l=0}^{k-1} u_{il} u_{k-l-1,j} \quad \text{for } i, j \in \mathbb{Z}$$

$$\frac{\partial w_{ij}}{\partial t_k} = w_{i+k,j} - w_{i,j+k} - \sum_{l=0}^{k-1} w_{il} w_{k-l-1,j} \quad \text{for } i, j \geq 0.$$

Interpretation

The KP hierarchy may be thought of as a subsystem of the U-system by identifying $w_{ij} = u_{ij}$ for $i, j \geq 0$.

Another description of KP hierarchy in U-system

Gr allows for another coset space representation

$$\mathrm{Gr} \simeq \mathcal{P} \backslash \mathrm{GL}(\mathbb{Z}),$$

where $\mathcal{P}$ is the subgroup of $\mathrm{GL}(\mathbb{Z})$ consisting of invertible matrices of the form

$$p = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$$

The infinite matrix (playing a role in Fu and Nijhoff's work)

$$g = E - U\Omega = \left(\begin{array}{c|c} \delta_{ij} - u_{i,-j-1} & 0 \\ \hline -u_{i,-j-1} & \delta_{ij} \end{array}\right)$$

represents the same point as η in the previous coset space representation of $\mathrm{Gr} \simeq \mathrm{GL}(\mathbb{Z}_{\geq 0}) \backslash \mathrm{Fr}(\mathbb{Z}_{\geq 0}, \mathbb{Z})$.

2. KP hierarchy

The motion of this point can be captured by the following differential equations:

- g satisfies the differential equations

$$\frac{\partial g}{\partial t_k} = (\Lambda^k - U\mathcal{O}_k)g - g\Lambda^k, \quad k = 1, 2, \cdots .$$

- The matrix $\Lambda^k - U\mathcal{O}_k$ has the block-triangular structure

$$\Lambda^k - U\mathcal{O}_k = \left(\begin{array}{c|c} * & * \\ \hline 0 & \mathcal{C}_k \end{array}\right).$$

The block-triangular structure of $\Lambda^k - U\mathcal{O}_k$ implies that these differential equations are consistent with the differential equations $\partial\eta/\partial t_k = \mathcal{C}_k\eta - \eta\Lambda^k$ of η.

Extension of U-system by negative flows

The U-system can be extended by the negative flows

$$\frac{\partial U}{\partial t_{-k}} = \Lambda^{-k} U - U\Lambda^{k} - U\mathcal{O}_{-k} U, \quad k = 1, 2, \ldots,$$

where $\mathcal{O}_{-k} = -\sum_{l=1}^{k} E_{-l,l-k-1}$. A new Grassmannian structure is hidden behind these flows. To describe it, we introduce the new infinite matrices

$$\bar{\Omega} = -\sum_{i=0}^{\infty} E_{-i-1,i},$$

$$\bar{g} = E - U\bar{\Omega} = \left(\begin{array}{c|c} \delta_{ij} & u_{i,-j-1} \\ \hline 0 & \delta_{ij} + u_{i,-j-1} \end{array} \right).$$

2. KP hierarchy

$\bar{g}$ represents a point of the complementary Grassmannian

$$\overline{\mathrm{Gr}} \simeq \overline{\mathcal{P}} \backslash \mathrm{GL}(\mathbb{Z}).$$

The stabilizer $\overline{\mathcal{P}} \subset \mathrm{GL}(\mathbb{Z})$ consists of invertible matrices of the form

$$p = \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array} \right).$$

The negative flows of U defines a dynamical system of $\bar{g}$ on $\overline{\mathrm{Gr}}$. This is another copy of the KP hierarchy.

The extended U-system with two sets of time variables $\{t_k\}_{k=1}^{\infty}$ and $\{t_{-k}\}_{k=1}^{\infty}$ can be reformulated by the extended η matrix

$$\eta^{(2)} = \left(\begin{array}{c|c} g & \bar{g} \end{array} \right)$$

and turns out to be equivalent to the two-component KP hierarchy.

3. AKNS hierarchy

Constrained multi-component U-system

• The AKNS hierarchy and its higher rank generalizations can be captured by the U-matrix $U = (u_{ij})_{i,j\in\mathbb{Z}}$ whose components are $r \times r$ matrices

$$u_{ij} = (u_{ij,\alpha\beta})^r_{\alpha,\beta=1}$$

($r = 2$ for the AKNS hierarchy).

• This matrix is required to satisfy the constraint

$$(E_r \otimes \Lambda)U - U(E_r \otimes \Lambda^{-1}) - U(E_r \otimes \mathcal{O}_1)U = 0.$$

Remark: Action of the tensor product $A \otimes B$ of $r \times r$ matrix A and $\mathbb{Z} \times \mathbb{Z}$ matrix $B = (b_{ij})$ is defined as

$$(A \otimes B)U = \left(\sum_{k\in\mathbb{Z}} b_{ik} A u_{kj}\right), \quad U(A \otimes B) = \left(\sum_{k\in\mathbb{Z}} u_{ik} A b_{kj}\right).$$

3. AKNS hierarchy

• Evolution equations for U (u_{ij} are 2×2 matrices) are

$$\frac{\partial U}{\partial t_k} = (a \otimes \Lambda^k)U - U(a \otimes \Lambda^{-k}) - U(a \otimes \mathcal{O}_k)U \quad k = 1, 2, \dots,$$

where

$$a = \sigma_3 = \mathrm{diag}(1, -1).$$

• These equations can be, at least formally, transformed to the linear equations

$$\frac{\partial C}{\partial t_k} = (a \otimes \Lambda^k)C - C(a \otimes \Lambda^{-k})$$

by the transformation $U = C(E + \Omega C)^{-1}$. Moreover, C satisfies the constraint

$$(E_2 \otimes \Lambda)C - C(E_2 \otimes \Lambda) = 0.$$

Component representation

- Evolution equations

$$\frac{\partial u_{ij}}{\partial t_k} = a u_{i+k,j} - u_{i,j+k} a - \sum_{l=0}^{k-1} u_{il} a u_{k-l-1,j} \quad \text{for } i, j \in \mathbb{Z}.$$

- Constraints

$$u_{i+1,j} - u_{i,j+1} - u_{i,0} u_{0,j} = 0, \quad \text{for } i, j \in \mathbb{Z}.$$

Equations for $i, j \geq 0$ form a close subsystem for $\{u_{ij}\}_{i,j=0}^{\infty}$. This subset of u_{ij}'s can be identified with 2×2 matrix-valued affine coordinates $\{w_{ij}\}_{i,j=0}^{\infty}$ of the two-component Sato Grassmannian $\mathrm{Gr}^{(2)}$.

ξ and η matrices

A point of $\mathrm{Gr}^{(2)}$ can be represented by the ξ and η matrices

$$\xi = \left(\frac{\delta_{ij} E_r}{w_{i,-j-1}} \right), \quad \eta = \left(\begin{array}{c|c} -w_{i,-j-1} & \delta_{ij} E_r \end{array} \right)$$

with 2×2 matrix-valued affine coordinates w_{ij}. The constraints

$$w_{i+1,j} - w_{i,j+1} - w_{i,0} w_{0,j} = 0 \quad \text{for } i, j \geq 0,$$

imply the constraints

$$(E_r \otimes \Lambda)\xi = \xi \mathcal{B}, \quad \eta(E_r \otimes \Lambda) = \mathcal{C}\eta$$

for ξ, η.

Solution of constraints (T. 1984 for ASDYM equation)

Under the constraints for w_{ij}'s, the two-variate generating function

$$G(z,y) = \sum_{i,j\geq 0} w_{ij} y^{-i-1} z^{-j-1}$$

can be expressed as

$$G(z,y) = \frac{W(y)^{-1}W(z) - E_r}{z-y}$$

where

$$W(z) = E_2 - \sum_{j=0}^{\infty} w_{0j} z^{-j-1}$$

and w_{0j}'s are matrix-valued free parameters.

Evolution equations of $W(z)$

Under the identification $w_{ij} = u_{ij}$ for $i, j \geq 0$, the evolution equations for $\{u_{ij}\}_{i,j=0}^{\infty}$'s can be eventually converted to differential equations of the form

$$\frac{\partial W(z)}{\partial t_k} = A_k(z)W(z) - W(z)az^k, \quad k = 1, 2, \ldots,$$

where $A_k(z)$ are polynomials in z of the form

$$A_k(z) = az^k + a_{k1}z^{k-1} + \cdots + a_{kk}$$

with 2×2 matrix coefficients. These are the AKNS version of the Sato-Wilson equations in the KP hierarchy. Thus the constrained two-component U-system turns out to contain the AKNS hierarchy as a subsystem.

U-system for ASDYM hierarchy

The ASDYM hierarchy has two sets of space-time coordinates $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$. The U-system for this case consists of the evolution equations

$$\frac{\partial U}{\partial x_k} = (E_r \otimes \Lambda - U(E_r \otimes \mathcal{O}_1))\, \frac{\partial U}{\partial x_{k-1}},$$
$$\frac{\partial U}{\partial y_k} = (E_r \otimes \Lambda - U(E_r \otimes \mathcal{O}_1))\, \frac{\partial U}{\partial y_{k-1}}$$

and the constraint

$$(E_r \otimes \Lambda)U - U(E_r \otimes \Lambda^{-1}) - U(E_r \otimes \mathcal{O}_1)U = 0.$$

4. ASDYM hierarchy

Component representation

- Evolution equations

$$\frac{\partial u_{ij}}{\partial x_k} = \frac{\partial u_{i+1,j}}{\partial x_{k-1}} - u_{i0}\frac{\partial u_{0j}}{\partial x_{k-1}},$$
$$\frac{\partial u_{ij}}{\partial y_k} = \frac{\partial u_{i+1,j}}{\partial y_{k-1}} - u_{i0}\frac{\partial u_{0j}}{\partial y_{k-1}} \quad \text{for } i,j \in \mathbb{Z}.$$

- Constraints

$$u_{i+1,j} - u_{i,j+1} - u_{i,0}u_{0,j} = 0 \quad \text{for } i,j \in \mathbb{Z}.$$

Just as in the case of the AKNS hierarchy, $\{u_{ij}\}_{i,j=0}^{\infty}$ can be identified with $r \times r$ matrix-valued affine coordinates $\{w_{ij}\}_{i,j=0}^{\infty}$ of the r-component Sato Grassmannian $\mathrm{Gr}^{(r)}$.

4. ASDYM hierarchy

Evolution equations for ξ, η and $W(z)$ (generalization of T. 1984)

The differential equations for $u_{ij} = w_{ij}$, $i, j \geq 0$, can be converted to the differential equations

$$\frac{\partial \xi}{\partial x_k} = (E_r \otimes \Lambda)\frac{\partial \xi}{\partial x_{k-1}} - \xi \mathcal{P}_k, \quad \frac{\partial \xi}{\partial y_k} = (E_r \otimes \Lambda)\frac{\partial \xi}{\partial y_{k-1}} - \xi \mathcal{Q}_k$$

for ξ, η (defined in the same form as the case of the AKNS hierarchy), and eventually turn into the equations

$$\begin{aligned} \frac{\partial W(z)}{\partial x_k} - z\frac{\partial W(z)}{\partial x_{k-1}} + \frac{\partial w_1}{\partial x_{k-1}} W(z) = 0, \\ \frac{\partial W(z)}{\partial y_k} - z\frac{\partial W(z)}{\partial y_{k-1}} + \frac{\partial w_1}{\partial y_{k-1}} W(z) = 0, \end{aligned}$$

of the Sato-Wison type for the Laurent series

$$W(z) = E_r - \sum_{j=0}^{\infty} w_{0j} z^{-j-1}, \quad w_1 = -w_{00}.$$

Conclusion

• Fu and Nijhoff's U-system literally contains the KP hierarchy as a subsystem. To explain this fact, part of the matrix elements of U are identified with affine coordinates of the top cell of the Sato Grassmannian.

• The U-system can be extended by negative flows. The negative flows are related to another Grassmannian. The extended U-system is substantially equivalent to the two-component KP hierarchy.

• The multi-component U-matrix yields the AKNS and ASDYM hierarchies by imposing a constraint. The constraint can be solved with the aid of a two-variate generating function. The solution can be described by a Laurent series $W(z)$ with matrix coefficients. The evolution equations of U can be converted to evolution equations of $W(z)$, which are analogues of the Sato-Wilson equations in the KP hierarchy.