3D Young diagrams and Gromov-Witten theory of $\mathbb{CP}^1$

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1. Motivation and goal

Gromov-Witten theory of all genera

— integrable Hamiltonian systems (Dubrovin & Zhang)

— quantization of symplectic geometry (Givental)

— computable cases (point, $\mathbb{CP}^1$, etc.)

Gromov-Witten theory of $\mathbb{CP}^1$ (Okounkov & Pandharipande)

— relation to Hurwitz numbers

— fermionic (semi-infinite wedge product) formalism

— relation to Toda lattice

— several versions (absolute; relative; equivariant)

My interest

— search for combinatorial and integrable structures in computable cases

This talk presents an approach to Gromov-Witten theory of $\mathbb{CP}^1$ from a statistical model of 3D Young diagrams. This melting crystal model is a kind of q-deformation of the Gromov-Witten invariants of $\mathbb{CP}^1$, and also the simplest case of instanton partition functions of 5D supersymmetric gauge theories on $\mathbb{R}^4 \times S^1$.

I will show

— prescrption of $q \to 1$ (5D $\to$ 4D) limit

— quantum spectral curve in this limit

— integrable structure in this limit

Reference: K.T., Quantum curve and 4D limit of melting crystal model, arXiv:1704.02750 [math-ph] (to be revised)

2. $\mathbb{CP}^1$ Gromov-Witten theory in a nutshell

GW invariants

The (connected) correlators of the descendants $\tau_k(\omega)$, $k = 0, 1, \ldots$, of the Kähler class ω are defined by intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d)$ of stable maps to $\mathbb{CP}^1$:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_{g,d}^{\circ} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,d)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\omega)$$

where

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, d) \to \mathbb{CP}^1, \ (f, C, p_1, \ldots, p_n) \mapsto f(p_i),$$

$$\psi_i = c_1(L_i), \ (L_i)_{(f,C,p_1,\cdots,p_n)} = T^*_{p_i} C$$

GW/Hurwitz correspondence (Okounkov-Pandharipande)

The (disconnected) correlators of $\tau_k(\omega)$'s can be expressed in terms of the Hurwitz numbers of $\mathbb{CP}^1$:

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega) \right\rangle_{g,d}^{\bullet} = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \prod_{i=1}^{n} \frac{p_{k_i+1}(\lambda)}{(k_i+1)!}$$

where

$$\lambda = (\lambda_1, \lambda_2, \ldots) \quad (\text{partition}),$$

$$\lambda \vdash d \iff |\lambda| = \sum_{i \geq 1} \lambda_i = d,$$

$$p_k(\lambda) = \sum_{i\geq 1} \Big((\lambda_i - i + 1/2)^k - (-i + 1/2)^k \Big) + (1 - 2^{-k})\zeta(-k),$$

$$\frac{\dim \lambda}{|\lambda|!} = \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)},$$

$$h(i,j) = \lambda_i + {}^{t}\lambda_j - i - j + 1 \quad (\text{hook length})$$

Remark:

$$\dim \lambda = \dim(\text{irred. rep. of } \mathcal{S}_d \text{ of type } \lambda)$$

$$= \#\{\text{standard Young tableaux of shape } \lambda\}$$

Generating function of GW invariants

$$\left\langle \exp\left(\sum_{k=0}^{\infty} \tau_k(\omega) t_k\right)\right\rangle^{\bullet} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1,\dots,k_n=0}^{\infty} \left\langle \prod_{i=1}^{n} \tau_{k_i}(\omega)\right\rangle^{\bullet} \prod_{i=1}^{n} t_{k_i}$$

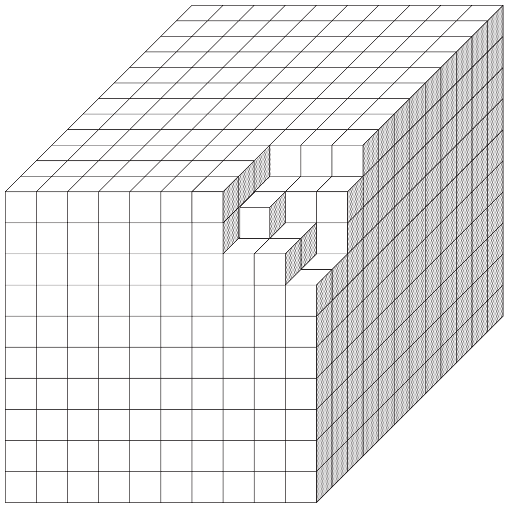
$$= \sum_{\lambda\in\mathcal{P}} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 \exp\left(\sum_{k=0}^{\infty} \frac{p_{k+1}(\lambda)}{(k+1)!} t_k\right)$$

where $\mathcal{P}$ is the set of all partitions (of arbitrary length).

Remark: This sum resembles the instanton partition functions of 4D $\mathcal{N} = 2$ supersymmetric gauge theories (Losev, Marshakov & Nekrasov).

3. Melting crystal model

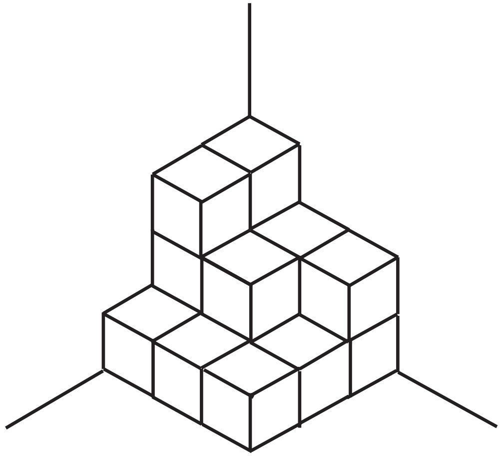
Statistical model (Okounkov, Reshetikhin & Vafa)



crystal corner

complement

$\longleftrightarrow$



3D Young diagram

The melting crystal model is a statistical model of random **3D Young diagrams**. The 3D Young diagrams are represented by the **plane partitions** $\pi = (\pi_{ij})_{i,j=1}^{\infty}$ (π_{ij} = height of (i, j)-th column).

Undeformed partition function

$$Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|} \qquad \text{(sum over plane partitions)}$$

where q is a parameter in the range $0 < q < 1$, $\mathcal{PP}$ denotes the set of all plane paritions, and

$$|\pi| = \sum_{i,j=1}^{\infty} \pi_{ij} \quad \text{(volume of 3DYD).}$$

Undeformed partition function (cont'd)

By the method of diagonal slicing (Okounkov, Reshetikin, Vafa), this sum over plane partitions can be converted to a sum over ordinary partitions:

$$Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|} = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2,$$

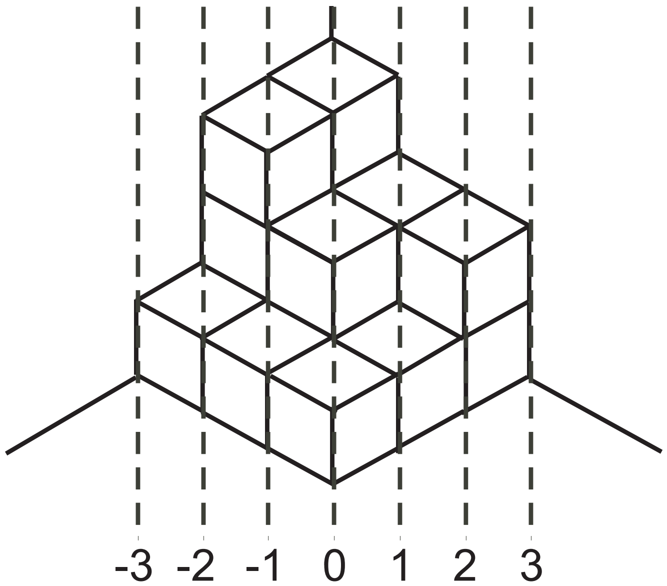
where $s_\lambda(q^{-\rho})$ is the special value of the infinite-variate Schur function $s_\lambda(x)$, $x = (x_1, x_2, \ldots)$, at

$$x = q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{i-1/2}, \ldots).$$

Remark: Z coincides with the instanton partition functions for 5D supersymmetric $U(1)$ gauge theories.

Diagonal slicing

$$m\text{-th diagonal slice } \pi(m) = \begin{cases} (\pi_{i,i+m})_{i=1}^{\infty} & \text{if } m \geq 0, \\ (\pi_{j-m,j})_{j=1}^{\infty} & \text{if } m < 0 \end{cases}$$



Hook-length formulae

$$s_\lambda(q^{-\rho}) = \frac{\dim_q \lambda}{|\lambda|!} = \frac{q^{-\kappa(\lambda)/4}}{\prod_{(i,j)\in\lambda}(q^{-h(i,j)/2} - q^{h(i,j)/2})},$$

$$\kappa(\lambda) = \sum_{i=1}^{\infty} \lambda_i(\lambda_i - 2i + 1).$$

This is a q-deformation of the quantity

$$\frac{\dim \lambda}{|\lambda|!} = \frac{1}{\prod_{(i,j)\in\lambda} h(i,j)}$$

that emerges in the GW/Hurwitz correspondence.

Deformed partition function

$$Z(t) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(t,\lambda)},$$

$$\phi(t,\lambda) = \sum_{k=1}^{\infty} t_k \phi_k(\lambda),$$

$$\phi_k(\lambda) = \sum_{i=1}^{\infty} \left(q^{k(\lambda_i - i + 1)} - q^{k(-i+1)} \right).$$

Q is a new parameter, and $t = (t_1, t_2, \ldots)$ are coupling constants of the external potentials $\phi_k(\lambda)$.

Deformed partition function

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$Z(t)$ is a tau function of the KP hierarchy (Nakatsu & T.).

Proof: Use fermions (infite wedge products) and a quantum torus algebra.

Tau functions of KP hierarchy

$$\tau(t) = \sum_{\lambda \in \mathcal{P}} S_\lambda(t) W_\lambda,$$

where $S_\lambda(t)$'s, $t = (t_1, t_2, \ldots)$, are the Schur functions of the power sum variables ($t_k = p_k/k$),

$$S_\lambda(t) = \det(S_{\lambda_i - i + j}(t))_{i=1}^{n}, \quad \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = \sum_{m=0}^{\infty} S_m(t) z^m,$$

and W_λ's are the Plücker coordinates of a point W of the Sato Grassmannian. The tau functions satisfy an infinite number of bilinear (Hirota) differential equations (a formulation of the KP hierarchy) that amount to the Plücker relations.

4D partition function

$$Z_{4\mathrm{D}}(t) = \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} e^{\phi_{4\mathrm{D}}(t,\lambda)},$$

$$\phi_{4\mathrm{D}}(t,\lambda) = \sum_{k=1}^{\infty} t_k \phi_k^{4\mathrm{D}}(\lambda),$$

$$\phi_k^{4\mathrm{D}}(\lambda) = \sum_{i=1}^{\infty} \left((\lambda_i - i + 1)^k - (-i + 1)^k \right).$$

This is essentially the same as the generating function of the GW invariants of $\mathbb{CP}^1$ (though slightly modified for comparison with $Z(t)$).

4. Formulation of 4D limit

What is 4D limit?

Nekrasov's instanton partition functions of 5D gauge theories are derived for theories on $\mathbb{R}^4 \times S^1$. The partition functions are expected to turn into those of 4D gauge theories on $\mathbb{R}^4$ as the radius R of S^1 tends to 0.

Key relation: $q = e^{-R\hbar}$

R-dependent parametrization of q, Q

The 4D limit $Z(0) \to Z_{\mathrm{4D}}(0)$ of the undeformed partition function can be achieved in the following well-known manner:

— q and Q are parametrized as

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2.$$

— As $R \to 0$, the Boltzmann weights behave nicely:

$$\lim_{R \to 0} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} = \left(\frac{\dim \lambda}{|\lambda|}\right)^2 \left(\frac{\Lambda}{\hbar}\right)^{2|\lambda|}.$$

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We want to extend this prescription to the deformed partition function $Z(t)$.

Prescription for external potentials

5D external potentials

$$\phi_k(\lambda) = \sum_{i=1}^{\infty} \left(q^{k(\lambda_i - i + 1)} - q^{k(-i+1)} \right)$$

4D external potentials

$$\phi_k^{\text{4D}}(\lambda) = \sum_{i=1}^{\infty} \left((\lambda_i - i + 1)^k - (-i + 1)^k \right)$$

Question: How $\phi_k^{\text{4D}}(\lambda)$ can be derived from $\phi_k(\lambda)$ in the limit as $R \to 0$? (Recall that $q = e^{-R\hbar}$)

Hint: Take linear combinations of $\phi_k(\lambda)$'s.

Key relation:

$$\sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \phi_j(\lambda) = \phi_k^{\mathrm{4D}}(\lambda)(-R\hbar)^k + O(R^{k+1})$$

This relation implies the identity

$$\lim_{R\to 0} \sum_{k=1}^{\infty} \frac{T_k}{(-R\hbar)^k} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \phi_j(\lambda) = \sum_{k=1}^{\infty} T_k \phi_k^{\mathrm{4D}}(\lambda).$$

Since

$$\sum_{k=1}^{\infty} \frac{T_k}{(-R\hbar)^k} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \phi_j(\lambda) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k} \phi_j(\lambda),$$

the 4D limit $\phi(t, \lambda) \to \phi^{4\mathrm{D}}(T, \lambda)$ is achieved if t_k's are parametrized by T_k's as $t_j = \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k}$.

Letting $R \to 0$ under the R-dependent parametrization

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2, \quad t_j = \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^{k-j} T_k}{(-R\hbar)^k},$$

we have the correct 4D limit $Z(t) \to Z_{4\mathrm{D}}(T)$.

5. Quantum spectral curve

Single-variate specialization $Z(x)$ of $Z(t)$

Substituting

$$t_k = -\frac{q^{-k/2}x^k}{k}, \quad k = 1, 2, \ldots,$$

in $e^{\phi(t,\lambda)}$ gives

$$e^{\phi(t,\lambda)} = \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2}x}{1 - q^{-i+1/2}x}.$$

$Z(t)$ thereby reduces to

$$Z(x) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2}x}{1 - q^{-i+1/2}x}.$$

Generating operator G of admissible basis $\{\Phi_i(x)\}_{i=0}^{\infty}$

The point W of the Sato Grassmannian relevant to $Z(t)$ can be realized in a space of formal Laurent series:

$$W = \mathrm{Span}(\Phi_i(x), i \geq 0) \subset V = \mathbb{C}((x)),$$

$$\Phi_i(x) = G x^{-i}.$$

The generating operator G can be expressed as

$$G = \prod_{i=1}^{\infty}(1-q^{i-1/2}x)\cdot q^{-(D-1/2)^2/2}\cdot \prod_{i=1}^{\infty}(1+q^{i-1/2}x)(1+Qq^{i-1/2}x)$$

where $D = x\frac{d}{dx}$.

q-difference equations for $\Phi_i(x)$'s

$$A\Phi_i(x) = q^{-i}\Phi_i(x)$$

where

$$A = G \cdot q^D \cdot G^{-1}, \quad D = x\frac{d}{dx}.$$

A is a q-difference operator of the form

$$A = (1 - q^{1/2}x)q^D + (1 + Q)q^{1/2}x + Qx^2(1 - q^{-1/2}x)^{-1}q^{-D}.$$

Remark: $q^D = e^{D \log q}$ is a q-shift operator: $q^D f(x) = f(qx)$.

Quantum spectral curve of melting crystal model

Since $Z(x) = \text{const.}\Phi_0(x)$, $Z(x)$ satisfies the following equation of the quantum spectral curve:

$$A(x, q^D)Z(x) = Z(x).$$

Remarks:

1. In the classical limit as $q \to 1$, $q^D \to y$, this equation turns into the equation $A_{\text{cl}}(x, y) = 1$ of an ordinary curve.

2. Quantum curves of this type are also known for topological string theory of the so called strip geometry.

4D limit of $Z(x)$

As $R \to 0$ under the R-dependent parametrization

$$q = e^{-R\hbar}, \quad Q = (R\Lambda)^2, \quad x = e^{R(X-\hbar/2)},$$

$Z(x)$ converges to

$$Z_{4\mathrm{D}}(X) = \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} \prod_{i=1}^{\infty} \frac{X - (\lambda_i - i + 1)\hbar}{X - (-i + 1)\hbar}.$$

Proof:

$$\frac{1 - q^{\lambda_i - i + 1/2} x}{1 - q^{-i+1/2} x} = \frac{X - (\lambda_i - i + 1)\hbar}{X - (-i + 1)\hbar}(1 + O(R)).$$

4D limit of quantum spectral curve

$Z_{\rm 4D}(X)$ satisfies the difference equation

$$\left((X-\hbar)(e^{-\hbar d/dX}-1)+\frac{\Lambda^2}{X}e^{\hbar d/dX}\right)Z_{\rm 4D}(X)=0.$$

Proof:

$$A-1=\left(-(X-\hbar)(e^{-\hbar d/dX}-1)-\frac{\Lambda^2}{X}e^{\hbar d/dX}\right)R+O(R^2).$$

4D limit of quantum spectral curve (cont'd)

Dunin-Barkowski et al. derived the foregoing difference equation by a different method, and pointed out that

$$\Psi(X) = \exp\left(B\left(-\hbar\frac{d}{dX}\right)\frac{X - X\log X}{\hbar}\right)\cdot Z_{\mathrm{4D}}(X+\hbar)$$

$(B(z) = z/(e^z - 1))$ satisfies the equation

$$\left(e^{-\hbar d/dX} + \Lambda^2 e^{\hbar d/dX} - X\right)\Psi(X) = 0$$

of the quantum spectral curve of $\mathbb{CP}^1$ GW theory.

6. Bilinear equations of Fay type

Bilinear equations for tau functions of KP hierarchy

The shifted tau functions

$$\tau(t, x_1, \ldots, x_N) = \tau\left(t_1 + \sum_{j=1}^{N} x_j, \ldots, t_k + \sum_{j=1}^{N} \frac{x_j^k}{k}, \ldots\right)$$

satisfy the bilinear equation

$$\begin{aligned}
&(x_1 - x_2)(x_3 - x_4)\tau(t, x_1, x_2)\tau(t, x_3, x_4) \\
&- (x_1 - x_3)(x_2 - x_4)\tau(t, x_1, x_3)\tau(t, x_2, x_4) \\
&+ (x_1 - x_4)(x_2 - x_3)\tau(t, x_1, x_4)\tau(t, x_2, x_3) = 0
\end{aligned}$$

of the Fay type.

Bilinear equations for tau functions of KP hierarchy

The shifted tau functions

$$\tau(t, x_1, \ldots, x_N) = \tau\left(t_1 + \sum_{j=1}^{N} x_j, \ \ldots, \ t_k + \sum_{j=1}^{N} \frac{x_j^k}{k}, \ \ldots\right)$$

satisfy the bilinear equation

$$\begin{aligned}
&(x_1 - x_2)(x_3 - x_4)\tau(t, x_1, x_2)\tau(t, x_3, x_4) \\
&- (x_1 - x_3)(x_2 - x_4)\tau(t, x_1, x_3)\tau(t, x_2, x_4) \\
&+ (x_1 - x_4)(x_2 - x_3)\tau(t, x_1, x_4)\tau(t, x_2, x_3) = 0
\end{aligned}$$

of the Fay type. Conversely, this equation implies all equations of the KP hierarchy.

Bilinear equations for $Z(t, x_1, \ldots, x_N)$

The shifted partition functions

$$Z(t, x_1, \ldots, x_N) = Z\Big(\ldots, t_k - \sum_{j=1}^{N} \frac{q^{-k/2} x_j^k}{k}, \ldots\Big)$$

may be thought of as a shifted KP tau function.

$Z(t)$ satisfies the Fay-type bilinear equation

$$\begin{aligned} &(x_1 - x_2)(x_3 - x_4) Z(t, x_1, x_2) Z(t, x_3, x_4) \\ &- (x_1 - x_3)(x_2 - x_4) Z(t, x_1, x_3) Z(t, x_2, x_4) \\ &+ (x_1 - x_4)(x_2 - x_3) Z(t, x_1, x_4) Z(t, x_2, x_3) = 0. \end{aligned}$$

4D limit of $Z(t, x_1, \ldots, x_N)$

The shifted partition function can be expressed as

$$Z(t, x_1, \ldots, x_N) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|} e^{\phi(t,\lambda)} \prod_{j=1}^{N} \prod_{i=1}^{\infty} \frac{1 - q^{\lambda_i - i + 1/2} x_j}{1 - q^{-i+1/2} x_j}.$$

The 4D limit $Z(t, x_1, \ldots, x_N) \to Z_{4\mathrm{D}}(T, X_1, \ldots, X_N)$ can be achieved by letting $R \to 0$ under the same R-dependent parametrization of q, Q, t as for $Z(t)$ and the substitution

$$x_j = e^{R(X_j - \hbar/2)}.$$

of the x_j's.

4D limit of $Z(t, x_1, \ldots, x_N)$ (cont'd)

The 4D limit $Z_{4\mathrm{D}}(T, X_1, \ldots, X_N)$ can be expressed as

$$Z_{4\mathrm{D}}(T, X_1, \ldots, X_N)$$

$$= \sum_{\lambda \in \mathcal{P}} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \left(\frac{\Lambda}{\hbar} \right)^{2|\lambda|} e^{\phi_{4\mathrm{D}}(T,\lambda)} \prod_{j=1}^{N} \prod_{i=1}^{\infty} \frac{X_j - (\lambda_i - i + 1)\hbar}{X_j - (-i + 1)\hbar}$$

$$= Z_{4\mathrm{D}}\Big(\ldots,\ T_k - \sum_{j=1}^{N} \frac{h^k}{k X_j^k},\ \ldots\Big).$$

Remark: X_j's show up in the denominator of the shift terms, in contrast to the case of $Z(t, x_1, \ldots, x_N)$:

$$Z(t, x_1, \ldots, x_N) = Z\Big(\ldots,\ t_k - \sum_{j=1}^{N} \frac{q^{-k/2} {x_j}^k}{k},\ \ldots\Big).$$

4D limit of three-term bilinear equation

$$
\begin{aligned}
&(X_1 - X_2)(X_3 - X_4) Z_{4D}(T, X_1, X_2) Z_{4D}(T, X_3, X_4) \\
&- (X_1 - X_3)(X_2 - X_4) Z_{4D}(T, X_1, X_3) Z_{4D}(T, X_2, X_4) \\
&+ (X_1 - X_4)(X_2 - X_3) Z_{4D}(T, X_1, X_4) Z_{4D}(T, X_2, X_3) = 0
\end{aligned}
$$

Proof: As $R \to 0$, $Z(t, x_i, x_j)$ converges to $Z_{4D}(T, X_i, X_j)$ and

$$
x_i - x_j = R(X_i - X_j) + O(R).
$$

Corollary:

$Z_{4D}(T)$ is a tau function of the KP hierarchy.

Concluding remarks

Both $Z(t)$ and $Z_{4\mathrm{D}}(t)$ can be extended to $Z(s,t)$ and $Z_{4\mathrm{D}}(s,t)$, $s \in \mathbb{Z}$. The following facts are known:

- **— $Z(s,t)$ is, up to simple factors, a tau function of the Toda hierarchy (Nakatsu and T., 2009). This is proven with the aid of fermions and the quantum torus algebra.**

- **— A similar statement for $Z_{4\mathrm{D}}(s,t)$ was conjectured around 2000 (the Toda conjecture) and proved later on by several different methods (Getzler; Dubrovin & Zhang; Milanov).**

It will be possible to reprove the Toda conjecture by the 4D limit.