Cauchy matrix approach and Sato Grassmannian

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Contents

1. Cauchy matrix approach

Sylvester equation

Given the matrices A, B, C, find a matrix X such that

$$AX - XB = C. \quad \text{(Sylvester equation)}$$

Existence and uniqueness of a solution is ensured under some conditions on A and B.

Example If

$$A = \mathrm{diag}(\alpha_1, \ldots, \alpha_N), \quad B = \mathrm{diag}(\beta_1, \ldots, \beta_N), \quad \alpha_i \neq \beta_j$$

and $C = (C_{ij})$, then

$$X = \left(\frac{C_{ij}}{\alpha_i - \beta_j} \right). \quad \text{(Cauchy matrix)}$$

Cauchy matrix approach

— proposed as a direct method for constructing special solutions of discrete integrable systems (Nijhoff-Atkinson-Hietarinta, J. Phys. A: Math. Theor. 42 (2009), 404005).

— may be thought of as an alternative to the degenerate case of the Riemann-Hilbert problem that was developed by Zakharov and Shabat in 1970's. Matrices of the form $\left(\frac{C_{ij}}{\alpha_i-\beta_j}\right)$ play a central role therein.

— mostly used for discrete and continuous integrable systems in $1+1$D and $2+1$D.

— recently applied to the $2+2$D ASDYM (anti-self-dual Yang-Mills) equations (Li-Qu-Yi-Zhang arXiv:2112.06408, Li-Yi-Zhang arXiv:2211.08574).

Goal of this talk

The output of the Cauchy matrix approach is a set of auxiliary (scalar or matrix-valued) functions $S^{(i,j)}$, $i,j=0,1,2,\ldots$ obtained from the Sylvester equation of a particular form. $S^{(i,j)}$'s satisfy algebraic and differential equations with quadratic nonlinearity. I will show that $S^{(i,j)}$ can be identified with the affine coordinates of an infinite dimensional Grassmann manifold in Sato's approach to soliton equations (Sato 1981, Sato-Sato 1982). For illustration, I will focus on the case of the ASDYM equations and its higher flows.

ASDYM equations

Let $x = (y, z, \tilde{y}, \tilde{z})$ be (complexified) space-time coordinates, $A_y, A_z, A_{\tilde{y}}, A_{\tilde{z}}$ the $(n \times n)$-matrix-valued gauge fields, and $\nabla_y = \partial_y + A_y$, etc. the covariant derivatives. The ASDYM equations read

$$[\nabla_y, \nabla_z] = 0, \quad [\nabla_{\tilde{y}}, \nabla_{\tilde{z}}] = 0,$$
$$[\nabla_y, \nabla_{\tilde{y}}] - [\nabla_z, \nabla_{\tilde{z}}] = 0$$

or, equivalently,

$$[\nabla_{\tilde{y}} - \zeta \nabla_z,\ \nabla_{\tilde{z}} - \zeta \nabla_y] = 0$$

with the spectral parameter ζ. This zero-curvature equation gives a Lax form of the ASDYM equation.

2. ASDYM equations and higher flows

Yang and Chalmers-Siegel potentials

Viewing $[\nabla_y, \nabla_z] = 0$ as a partial flatness condition, we can eliminate A_y and A_z by gauge transformations. The zero-curvature equation reduces to

$$[\partial_{\tilde{y}} - \zeta\partial_z + A_{\tilde{y}},\ \partial_{\tilde{z}} - \zeta\partial_y + A_{\tilde{z}}] = 0.$$

$A_{\tilde{y}}$ and $A_{\tilde{z}}$ can be expressed as

$$A_{\tilde{y}} = -\partial_{\tilde{y}} J \cdot J^{-1} = \partial_z K, \quad A_{\tilde{z}} = -\partial_{\tilde{z}} J \cdot J^{-1} = \partial_y K$$

with the matrix-valued potentials J, K (not unique!), which in turn satisfy the Yang and Chalmers-Siegel equations

$$\partial_y(\partial_{\tilde{y}} J \cdot J^{-1}) + \partial_z(\partial_{\tilde{z}} J \cdot J^{-1}) = 0, \quad \text{(Yang eqn)}$$
$$(\partial_y\partial_{\tilde{y}} - \partial_z\partial_{\tilde{z}})K = [\partial_y K, \partial_z K]. \quad \text{(Chalmers-Siegel eqn)}$$

2. ASDYM equations and higher flows

ASDYM hierarchy

The hierarchy comprises the zero-curvature equations

$$
\begin{aligned}
[\partial_{y_j} - \zeta\partial_{y_{j-1}} + A_j,\ \partial_{y_k} - \zeta\partial_{y_{k-1}} + A_k] = 0,\\
[\partial_{z_j} - \zeta\partial_{z_{j-1}} + B_j,\ \partial_{z_k} - \zeta\partial_{z_{k-1}} + B_k] = 0,\\
[\partial_{y_j} - \zeta\partial_{y_{j-1}} + A_j,\ \partial_{z_k} - \zeta\partial_{z_{k-1}} + B_k] = 0
\end{aligned}
$$

with the independent variables $y_k, z_k, \ldots,\ k = 0, 1, 2, \ldots$ (Nakamura 1988; KT 1990; Ablowitz-Chakravarty-Takhtajan 1993). This is an extension of the $2+2$D equations with $y_0 = y$, $z_0 = z$, $y_1 = \tilde{z}$, $z_1 = \tilde{y}$. The potentials J, K are also extended as

$$
A_k = -\partial_{y_k} J \cdot J^{-1} = \partial_{y_{k-1}} K, \quad B_k = -\partial_{z_k} J \cdot J^{-1} = \partial_{z_{k-1}} K
$$

and satisfy a system of the Yang and Chalmers-Siegel types.

2. ASDYM equations and higher flows

Auxiliary linear equations

$$(\partial_{y_k} - \zeta\partial_{y_{k-1}} + A_k)\Psi = 0, \quad (\partial_{z_k} - \zeta\partial_{z_{k-1}} + B_k)\Psi = 0.$$

- If Ψ is defined in a neighborhood of $\zeta = 0$, its value $\Psi(\zeta = 0)$ at $\zeta = 0$ gives J.

- If Ψ can be expanded into a Laurent series of the form

$$W = \mathrm{I} + \sum_{j=1}^{\infty} w_j \zeta^{-j}$$

in a neighborhood of $\zeta = \infty$, then w_1 gives K, hence $A_k = \partial_{y_{k-1}} w_1$, $B_k = \partial_{z_{k-1}} w_1$. The auxiliary linear equations thereby turn into an infinite system of differential equations for w_j's.

3. ASDYM hierarchy in Sato Grassmannian

- KT, Comm. Math. Phys. **94** (1984), 35–59.
- KT, Saitama Math. J. **3** (1985), 11–40.

Dependent variables with two indices

Introduce the dependent variables w_{ij}, $i, j = 0, 1, \ldots$, from $W(\zeta) = \mathrm{I} + \sum_{j=1}^{\infty} w_j \zeta^{-j}$ by the generating function

$$\frac{W(\eta)^{-1} W(\zeta) - \mathrm{I}}{\zeta - \eta} = \sum_{i,j=0}^{\infty} w_{ij} \zeta^{-i-1} \eta^{-j-1}.$$

Remark This generating function shows up in many places, e.g., the Schlesinger transformations of isomonodromic systems (M. Jimbo and T. Miwa, Physica 4D (1981), 26–46). It is also related to a 2D free fermion system.

Properties of w_{ij}'s

- $W(\zeta)$ and $W(\zeta)^{-1}$ can be recovered from w_{ij}'s as

$$W(\zeta) = \mathrm{I} - \sum_{j=0}^{\infty} w_{0j}\zeta^{-j-1}, \quad W(\zeta)^{-1} = \mathrm{I} + \sum_{i=0}^{\infty} w_{i0}\zeta^{-i-1}.$$

- w_{ij}'s satisfy (and are characterized by) the equations

$$\begin{aligned} w_{i+1,j} &= w_{i,j+1} + w_{i0}w_{0j}, \\ \frac{\partial w_{ij}}{\partial y_k} &= \frac{\partial w_{i+1,j}}{\partial y_{k-1}} - w_{i0}\frac{\partial w_{0j}}{\partial y_{k-1}}, \\ \frac{\partial w_{ij}}{\partial z_k} &= \frac{\partial w_{i+1,j}}{\partial z_{k-1}} - w_{i0}\frac{\partial w_{0j}}{\partial z_{k-1}} \end{aligned}$$

with quadratic nonlinearity.

Frame matrix

The $\mathbb{Z} \times \mathbb{Z}_{<0}$ matrix

$$\xi = \begin{pmatrix} \delta_{ij}\mathrm{I} \\ w_{i,-j-1} \end{pmatrix}_{i\in\mathbb{Z}, j<0} = \begin{pmatrix} \ddots & \vdots & \vdots \\ \cdots & \mathrm{I} & 0 \\ \cdots & 0 & \mathrm{I} \\ \cdots & w_{01} & w_{00} \\ \cdots & w_{11} & w_{10} \\ \iddots & \vdots & \vdots \end{pmatrix}$$

represents a point $[\xi]$ of the top cell of the infinite-dimensional Sato Grassmannian $\mathrm{Gr}(\infty, \infty)$. ξ and ξh, $h \in \mathrm{GL}(\infty)$, represent the same point $[\xi] = [\xi h]$. w_{ij}'s are affine coordinates of the top cell.

Equations for frame matrix

The equations for w_{ij}'s can be translated to equations for ξ:

$$\Lambda\xi = \xi\mathcal{C}, \quad \frac{\partial\xi}{\partial y_k} = \Lambda\frac{\partial\xi}{\partial y_{k-1}} - \xi\mathcal{A}_k, \quad \frac{\partial\xi}{\partial z_k} = \Lambda\frac{\partial\xi}{\partial z_{k-1}} - \xi\mathcal{B}_k,$$

where Λ denotes the block-wise shift matrix $(\delta_{i,j-1}\mathrm{I})$ and

$$\mathcal{C} = \begin{pmatrix} \delta_{i,j-1}\mathrm{I} \\ w_{0,-j-1} \end{pmatrix}, \quad \mathcal{A}_k = \begin{pmatrix} 0 \\ \frac{\partial w_{0,-j-1}}{\partial y_{k-1}} \end{pmatrix}, \quad \mathcal{B}_k = \begin{pmatrix} 0 \\ \frac{\partial w_{0,-j-1}}{\partial z_{k-1}} \end{pmatrix}.$$

Remarks 1) The equations for the AKNS hierarchy ($n = 2$) read

$$\Lambda\xi = \xi\mathcal{C}, \quad \frac{\partial\xi}{\partial t_k} = a\Lambda^k\xi - \xi\mathcal{C}_k, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2) The algebraic equation $\Lambda\xi = \xi\mathcal{C}$ ensures the correspondence with a Laurent series of the form $W(\zeta) = \mathrm{I} + \sum_{j=1}^{\infty} w_j \zeta^{-j}$.

3) The differential equations for the ASDYM hierarchy can be rewritten as

$$\frac{\partial\xi}{\partial y_k} = \Lambda^k \frac{\partial\xi}{\partial y_0} - \xi(\mathcal{C}^{k-1}\mathcal{A}_1 + \mathcal{C}^{k-2}\mathcal{A}_2 + \cdots + \mathcal{A}_k),$$
$$\frac{\partial\xi}{\partial z_k} = \Lambda^k \frac{\partial\xi}{\partial z_0} - \xi(\mathcal{C}^{k-1}\mathcal{B}_1 + \mathcal{C}^{k-2}\mathcal{B}_2 + \cdots + \mathcal{B}_k).$$

They can be compared with the differential equations

$$\frac{\partial\xi}{\partial t_k} = a\Lambda^k\xi - \xi\mathcal{C}_k$$

for the AKNS hierarchy. Thus the role of $a\Lambda^k$'s are played by $\Lambda^k\partial_{y_0}$ and $\Lambda^k\partial_{z_0}$.

3. ASDYM hierarchy in Sato Grassmannian

Toroidal extension of AKNS hierarchy

- 2-Torodial (Bogomolny) extension (Ikeda-Kakei-KT 2002)

$$\Lambda\xi = \xi\mathcal{C}, \quad \frac{\partial\xi}{\partial t_k} = a\Lambda^k\xi - \xi\mathcal{C}_k,$$
$$\frac{\partial\xi}{\partial y_k} = \Lambda\frac{\partial\xi}{\partial y_{k-1}} - \xi\mathcal{A}_k.$$

- 3-Torodial (AKNS+ASDYM) extension (Kakei unpublished)

$$\Lambda\xi = \xi\mathcal{C}, \quad \frac{\partial\xi}{\partial t_k} = a\Lambda^k\xi - \xi\mathcal{C}_k,$$
$$\frac{\partial\xi}{\partial y_k} = \Lambda\frac{\partial\xi}{\partial y_{k-1}} - \xi\mathcal{A}_k, \quad \frac{\partial\xi}{\partial z_k} = \Lambda\frac{\partial\xi}{\partial z_{k-1}} - \xi\mathcal{B}_k.$$

Cauchy matrix for ASDYM hierarchy

Let us consider the Sylvester equation

$$MK - LM = s\,{}^{\mathrm{t}}r$$

for the $N \times N$ matrix M. K and L are $N \times N$ constant matrices, and $r = r(x)$ and $s = s(x)$ are $N \times n$ matrices that depend on $x = (y_k, z_k, k \geq 0)$ and satisfy the following differential equations (Li-Hamanaka-Huang-Zhang arXiv:2501.08250):

$$\frac{\partial\,{}^{\mathrm{t}}r}{\partial y_k} = \frac{\partial\,{}^{\mathrm{t}}r}{\partial y_{k-1}}K, \quad \frac{\partial s}{\partial y_k} = L\frac{\partial s}{\partial y_{k-1}},$$
$$\frac{\partial\,{}^{\mathrm{t}}r}{\partial z_k} = \frac{\partial\,{}^{\mathrm{t}}r}{\partial z_{k-1}}K, \quad \frac{\partial s}{\partial z_k} = L\frac{\partial s}{\partial z_{k-1}}.$$

Auxiliary functions $S^{(i,j)}$

For simplicity, we assume that K and L are invertible, diagonalizable and have no common eigenvalue. Define the $(n \times n)$-matrix-valued functions $S^{(i,j)}$, $i, j \in \mathbb{Z}$, as

$$S^{(i,j)} = {}^{\mathrm{t}}r\, K^i M^{-1} L^j s.$$

Remarks

1) We have slightly modified the construction in the literature, exchanging $i \leftrightarrow j$, $K \leftrightarrow L$, $r \leftrightarrow$s (and flipping a sign somewhere).

2) If there are common eigenvalues of K and L, we can choose a non-zero constant matrix C with $CK = LC$ and consider

$$S^{(i,j)} = {}^{\mathrm{t}}r\, K^i (C + M)^{-1} L^j s$$

(cf. Li-Qu-Zhang arXiv:2211.08574).

Properties of $S^{(i,j)}$

$S^{(i,j)}$'s satisfy the algebraic and differential equations

$$S^{(i+1,j)} = S^{(i,j+1)} + S^{(i,0)}S^{(0,j)},$$
$$\frac{\partial S^{(i,j)}}{\partial y_k} = \frac{\partial S^{(i+1,j)}}{\partial y_{k-1}} - S^{(i,0)}\frac{\partial S^{(0,j)}}{\partial y_{k-1}},$$
$$\frac{\partial S^{(i,j)}}{\partial z_k} = \frac{\partial S^{(i+1,j)}}{\partial z_{k-1}} - S^{(i,0)}\frac{\partial S^{(0,j)}}{\partial z_{k-1}},$$

which are identical to the equations for w_{ij}'s. Thus $S^{(i,j)}$ for $i, j = 0, 1, \ldots$, can be identified with the affine coordinates in the Grassmannian description of the ASDYM hierarchy.

Ψ and J

Identifying $S^{(i,j)} = w_{ij}$, we find the solution $\Psi = W(\zeta)$ of the auxiliary linear equation with the help of the geometric series

$$\sum_{j=0}^{\infty} L^j \zeta^{-j-1} = (\zeta \mathrm{I} - L)^{-1}$$

as

$$W(\zeta) = \mathrm{I} - {}^{\mathrm{t}}r(C+M)^{-1}(\zeta \mathrm{I} - L)^{-1}s.$$

Hence we can reach the point $\zeta = 0$ by analytic continuation and obtain the Yang potential

$$J = \Psi(\zeta = 0) = \mathrm{I} + {}^{\mathrm{t}}r(C+M)^{-1}L^{-1}s = \mathrm{I} + S^{(0,-1)}.$$

This coincides with the known formula (up to an opposite sign).

Conclusion

- The Cauchy matrix approach gives rise to an infinite number of auxiliary functions $S^{(i,j)}$, $i, j = 0, 1, \ldots$. They satisfy algebraic and differential equations with quadratic nonlinearity. We have seen, in the case of the ASDYM hierarchy, that they can be identified with the affine coordinates w_{ij} of the top cell of the Sato Grassmannian.
- Apart from $S^{(0,-1)}$ (and $S^{(-1,0)}$), it is not clear what geometric meaning $S^{(i,j)}$'s outside the range $i, j \geq 0$ have.
- Part of the U-matrix in the direct linearization approach (see, e.g., S. Li and D.-j. Zhang, arXiv:2403.06055) can be identified with w_{ij}'s, because they satisfy equations of the same form. It is not clear what geometric meaning the other part of the U-matrix has.