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SINGULAR CAUCHY PROBLEMS FOR A CLASS OF WEAKLY HYPERBOLIC DIFFERENTIAL OPERATORS

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In this paper singular Cauchy problems of Hamada's type are studied in the category of holomorphic functions and hyperfunctions for a class of hyperbolic differential operators with non-involutive multiple characteristics. Integral representations of their solutions are obtained.

§1. Introduction.

Let $P(t,x,D_t,D_x)$ be a differential operator of order m of the form

$$P(t,x,D_t,D_x) = D_t^{\,m} + \Sigma_{i=1}^{m} A_i(t,x,D_x)D_t^{\,m-i},$$

where $D_t = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial t}$, $D_x = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x}$ and $A_i(t,x,D_x)$ is a differential operator at most of order i, not containing D_t, whose coefficients are holomorphic functions defined in a neighborhood of $(t,x) = (0,0)$ in $\mathbb{C} \times \mathbb{C}^n$.

We assume the following conditions:

(A-1) (Degeneracy of characteristic roots) The principal symbol $P_m(t,x,\tau,\xi)$ of $P(t,x,D_t,D_x)$ is given by

$$P_m(t,x,\tau,\xi) = \Pi_{j=1}^{m}(\tau - t^q\lambda_j(\xi)),$$

where q is a non-negative integer and $\lambda_j(\xi)$ $(1 \leq j \leq m)$ are homogeneous holomrphic functions of degree 1 defined in a conic neighborhood Ω_0 of $\xi_0 = (1,0,\ldots,0)$ in $\mathbb{C}^n-0$ such that $\lambda_j(\xi) \neq \lambda_k(\xi)$ if $j \neq k$ and $\xi \in \Omega_0$.

(A-2) (Hyperbolicity) $\lambda_j(\xi)$ $(1 \leq j \leq m)$ are real if ξ is real.

(A-3) (Levi condition) Set

$$A_{i,j,k}(x,\xi) = \frac{1}{k!}\left(\frac{\partial}{\partial t}\right)^k A_{i,j}(t,x,\xi)\Big|_{t=0},$$

where $A_{i,j}(t,x,\xi)$ is the homogeneous part of $A_i(t,x,\xi)$ of degree j with respect to ξ. Then

$$A_{i,j,k} = 0 \qquad \text{for } k < (q+1)j - i. \ \square$$

Using a type of ordinary differential operators with polynomial coefficients, many mathematicians constructed parametrices of the Cauchy problem for weakly hyperbolic operators of the above type (or its generalization)(Alinhac [1], [2], [3], Amano [4], Amano-Nakamura [5], Nakamura-Uryu [11], Nakane [12], Taniguchi-Tozaki [18] and Yoshikawa [20]). Shinkai [16] constructed parametrices by a different (but interesting) method. All of them, except Nakane [12], treated this problem in the category of C^∞ functions.

We shall study, for $0 \leqslant j \leqslant m-1$, the following singular Cauchy problems of Hamada's type

$$(CP)_i^{\mathbb{C}} \begin{cases} P(t,x,D_t,D_x)\, u_i(t,x,y,\xi) = 0, \\ D_t^{\,j}\, u_i|_{t=0} = \delta_{i,j} \langle x-y,\xi\rangle^{-n} \text{ for } 0 \leqslant j \leqslant m-1, \end{cases}$$

$$(CP)_i \begin{cases} P(t,x,D_t,D_x)\, u_i(t,x,y,\xi) = 0, \\ D_t^{\,j}\, u_i|_{t=0} = \delta_{i,j}(\langle x-y,\xi\rangle+\sqrt{-1}0)^{-n} \text{ for } 0 \leqslant j \leqslant m-1, \end{cases}$$

in the category of holomorphic functions and hyperfunctions, respectively. Here $\delta_{i,j}$ is Kronecker's delta. Kashiwara-Kawai [9] studied the Cauchy problem for the general "micro-hyperbolic" operators which include our operators as special ones. But we shall give a more concrete construction suitable for the analysis of the fine structure of solutions.

We shall construct the solutions as infinite series of "Radon integrals" (see (1.2) and Remark 1.1). Our method is similar to those of [11] and [20] which are closely related to the theory of Nishimoto [13] on the turning points of ordinary differential equations with a parameter. But we need more careful estimations.

Notice that fundamental solutions of the Cauchy problem are obtained as the integrals over the unit sphere

$$\int_{|\xi|=1} u_i(t,x,y,\xi)\,\omega(\xi) \qquad (0 \leqslant i \leqslant m-1),$$

where $\omega(\xi) = \Sigma_{j=1}^{n}(-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n$. (See Kashiwara-Kawai [9], §5.)

Main results of the present paper were announced in [17] without a detailed proof.

Before stating main theorems we introduce following notations:

$$\psi_j(t,\xi) = \lambda_j(\xi)t^{q+1}/(q+1),$$
$$\varphi_j(t,x,y,\xi) = \langle x-y,\xi\rangle + \psi_j(t,\xi),$$
$$r_j(t,\xi) = \max_{1\le k\le m} |\psi_j(t,\xi/\xi_1) - \psi_k(t,\xi/\xi_1)|,$$
$$d(t,\xi_1) = (|t^{q+1}| + |\xi_1|^{-1})^{1/(q+1)},$$
$$X = \{ x\in \mathbb{C}^n;\ |x| < a\},$$
$$\Omega = \{ \xi = (\xi_1,\xi')\in \mathbb{C}^n - 0;\ |\xi'| < b|\xi_1|,\ |\arg(\xi_1)| < b\},$$
$$S_\sigma = \{ t\in \mathbb{C}-0;\ |\arg(\sigma t)| < (2q+2)^{-1}\pi - \varepsilon \},$$
$$Z_\sigma = S_\sigma \times X\times \Omega,$$
$$Z = \mathbb{C}\times X\times \Omega,$$
$$D_0(r) = \{ p\in\mathbb{C};\ \mathrm{Im}(p) > r\},$$
$$D_1(d,r,R) = \bigcup_{-b<\theta<b} \{p\in\mathbb{C};\ \mathrm{Im}(pe^{\sqrt{-1}\theta}) > r,\ de^{R(|p|+r)} < 1\},$$

for positive constants a, b, d, r, R and $\sigma = \pm 1$, $1\le j \le m$. These notations are used throughout this paper.

Under assumptions (A-1), (A-2) and (A-3) we have

Theorem 1. For any sufficiently small constant $\varepsilon > 0$ there exist holomorphic functions $u^{(\nu)}_{\sigma,i,j}(t,x,\xi)$ ($\sigma = \pm 1$, $0\le i\le m-1$, $1\le j\le m$, $\nu\ge 0$) defined in the domain Z and positive constants a, b, h such that

(i) A solution of $(CP)^{\mathbb{C}}_i$ is obtained in the form

$$(1.1)\quad u_i(t,x,y,\xi) = \sum_{j=1}^{m} u_{\sigma,i,j,R}(t,x,\xi;\varphi_j(t,x,y,\xi)) + h_{\sigma,i,R}(t,x,y,\xi),$$

where $u_{\sigma,i,j,R}(t,x,\xi;p)$ is defined by

$$(1.2)\quad u_{\sigma,i,j,R}(t,x,\xi;p) = \sum_{\nu=0}^{\infty}\int_{(\nu+1)R}^{\infty} e^{\sqrt{-1}p\rho/\xi_1}\xi_1^{-n} \times u^{(\nu)}_{\sigma,i,j}(t,x,\rho\xi/\xi_1)\rho^{n-1}\,d\rho,$$

and $h_{\sigma,i,R}(t,x,y,\xi)$ is a holomorphic function defined in a neighborhood of $(t,x,y,\xi) = (0,0,0,\xi_0)$ and homogeneous of degree -n with respect to ξ.

(ii) The series (1.2) converges uniformly in every compact subset of the domain

$$\{(t,x,\xi,p)\in Z_\sigma\times\mathbb{C};\ d(t,R)h<1,\ p/\xi_1\in D_0(0)\}$$
$$\bigcup\bigcap_{j=1}^{m}\{(t,x,\xi,p)\in Z\times\mathbb{C};\ d(t,R)h<1,\ p/\xi_1\in D_0(r_j(t,\xi))\},$$

and is homogeneous of degree -n with respect to (ξ,p).

Moreover by deforming the path of integration in the ν-th term of (1.2) into the path

$$C_{\nu,R,\theta} = \{(\nu+1)e^{\sqrt{-1}s\theta}R;\ 0\le s\le 1\}\cup\{(\nu+1)e^{\sqrt{-1}\theta}sR;\ s\ge 1\}$$

$(-b<\theta<b)$, $u_{\sigma,i,j,R}$ is continued to a holomorphic function defined in the domain

$$\{(t,x,\xi,p)\in Z_\sigma\times\mathbb{C};\ p/\xi_1\in D_1(d(t,R)h,0,R)\}$$
$$\bigcup\bigcap_{j=1}^{m}\{(t,x,\xi,p)\in Z\times\mathbb{C};\ p/\xi_1\in D_1(d(t,R)h,r_j(t,\xi),R)\}.\ \square$$

Theorem 2. The solution of $(CP)_i$ is given by the "boundary value" of (1.1). Namely,

$$(1.3)\quad u_i(t,x,y,\xi) = \sum_{j=1}^{m} u_{\sigma,i,j,R}(t,x,\xi;\varphi_j(t,x,y,\xi)+\sqrt{-1}0) + h_{\sigma,i,R}(t,x,y,\xi).$$

The singularity support and the singularity spectrum of u_i, where ξ is regarded as a parameter, are estimated as follows:

$$(1.4)\quad \begin{cases} \text{sing.supp. } u_i \subset \bigcup_{j=1}^{m} \{(t,x,y);\ \varphi_j = 0\}, \\ \text{S.S. } u_i \subset \bigcup_{j=1}^{m} \{(t,x,y;\sqrt{-1}\, d\varphi_j(t,x,y,\xi)\infty);\ \varphi_j = 0\}. \end{cases}$$

(As for the terminologies of hyperfunctions and their singularity spectra, we refer to Sato-Kashiwara-Kawai[15].)

Remark 1.1. We shall construct the functions $u^{(\nu)}_{\sigma,i,j}$ so that they are "semi-homogeneous" of degree $(-i-\nu)/(q+1)$ (see [20]; here we use the complex version of the semi-homogeneity of [20]), i.e., for $c \in \mathbb{C}-0$,

$$(1.5)\quad u^{(\nu)}_{\sigma,i,j}(t/c,x,c^{q+1}\xi) = c^{-i-\nu} u^{(\nu)}_{\sigma,i,j}(t,x,\xi),$$

and that they satisfy "transport equations"

$$(1.6)\quad (D_t^{m} + \sum_{k=1}^{m} A_k^{(0)} D_t^{m-k})(e^{\sqrt{-1}\psi_j} u^{(\nu)}_{\sigma,i,j})$$

$$= - \sum_{k=1}^{m} \Sigma^* \frac{1}{\alpha!}(\partial_\xi^\alpha A_k^{(\kappa)})\cdot D_x^\alpha D_t^{m-k}(e^{\sqrt{-1}\psi_j} u^{(\nu)}_{\sigma,i,j}),$$

and "growth conditions" $(0 \le k \le m-1)$

$$(1.7)\quad |D_t^k u^{(\nu)}_{\sigma,i,j}(t,x,\xi)|$$

$$\le C h^\nu |\xi_1|^{k-i/(q+1)}\, d(t,\xi_1)^{kq}\, d(t,\xi_1/(\nu+1))^{\nu} \times$$

$$\times \begin{cases} d(t\xi_1^{1/(q+1)},1)^{\mu_j(x,\xi)} & \text{if } (t,x,\xi)\in Z_\sigma, \\ d(t\xi_1^{1/(q+1)},1)^{\mu(x,\xi)}\, e^{r_j(t,\xi)|\xi_1|} & \text{if } (t,x,\xi)\in Z, \end{cases}$$

where $\partial_\xi^\alpha = (\frac{\partial}{\partial\xi_1})^{\alpha_1}\cdots(\frac{\partial}{\partial\xi_n})^{\alpha_n}$ and Σ^* denotes the summation

over all (κ,λ,α) such that $\nu > \lambda$ and $\nu = \kappa + \lambda + (q+1)|\alpha|$.

Here we set

$$A_i^{(\nu)}(t,x,\xi) = \sum_{\substack{k\ge 0,\ i \ge j \\ \nu = k-(q+1)j+i}} t^k A_{i,j,k}(x,\xi),$$

$$\pi_j(x,\xi) = -\sum_{i=1}^{m} \{(\tfrac{q}{2})(m-i+1)(m-i)A_{i-1,i-1,qi-q} + \sqrt{-1}A_{i,i-1,qi-q-1}\} \times$$

$$\times\ \lambda_j(\xi)^{m-i} \prod_{k=1(k\ne j)}^{m} (\lambda_j(\xi)-\lambda_k(\xi))^{-1},$$

$$\mu_j(x,\xi) = \mathrm{Re}[\ \pi_j(x,\xi)\ ],$$

$$\mu(x,\xi) = \max_{1\le j\le m} \mu_j(x,\xi).$$

"Transport equations" are the same as those of [11] and [20]. But "growth conditions" are essentially stronger than those of C^∞ theories, which assure the convergence of the series (1.2) in the complex domains. According to Aoki [6] and Kataoka [10] we may well call (1.2) a (series of) "Radon integral(s)" with the "formal symbol" $\Sigma_{\nu=0}^{\infty} u^{(\nu)}_{\sigma,i,j}$. Such a class of formal symbols is regarded as a natural version, in the category of hyperfunctions, of the "Boutet de Monvel class" (see Boutet de Monvel [7] and Yoshikawa [20]). □

Remark 1.2. As pointed out in [4] and [5], our method for the construction of solutions will be effective in the analysis of "the branching of singularities" at multiply characteristic points. The analysis in the case m=2 was carried out in [3], [12] and [18] where special functions of hypergeometric or confluent hypergeometric type were utilized. Hanges [8] and Ôaku [14] studied this problem from different points of view. □

Remark 1.3. (1.4) implies that singularities of the solutions propagate along the union of bicharacteristic strips of $\tau - t^q\lambda_j$ $(1 \leq j \leq m)$ passing through $(t,x,y,\xi) = (0,0,0,\xi\ )$. It seems that (1.4) follows from the results of [9]. But our construction shows us more detailed structures of solutions. □

A brief account of contents of the present paper is as follows:

In §2 and §3 we study some properties of "Radon integrals". Namely, (§2) convergent domains and analytic continuations, (§3) the action formula of differential operators.

In §4 we show how the proof of Theorems 1 and 2 is reduced to the constructions of solutions of transport equations with initial conditions and growth conditions.

In §5 we construct solutions of 0-th transport equations. They are homogeneous ordinary differential equations with respect to t with polynomial coefficients, in which (x,ξ) is contained as a holomorphic parameter. The point $t=\infty$ is its irregular singular point of Poincaré's rank q+1. Hence we can apply general theories of irregular singular points with slightest modifications.

In §6 we construct solutions of ν-th transport equations with growth conditions by the method of

"variations of constants", using the same paths of integrations as those of [13].

§2. Symbolic calculus.

In this and the next sections we study general properties of Radon integrals.

Throughout two sections we fix j $(1 \leq j \leq m)$ and abbreviate φ_j, ψ_j and r_j to φ, ψ and r respectively. We replace μ_j by μ, for it does not affect the following arguments.

Now let us consider generally a sequence of holomorphic functions $u_\nu(t,x,\xi)$ $(\nu \geq 0)$ defined in the domain Z which satisfy, for certain positive constants C, h, δ, ℓ, for $0 \leq k \leq m$ and for $\alpha \geq 0$, the following inequalities

$$(2.1)\quad |D_x^\alpha D_t^k\, u_\nu(t,x,\xi)| \leq C|\alpha|!\, h^\nu \delta^{-|\alpha|} |\xi_1|^{\ell+k}\, d(t,\xi_1)^{kq}\, d(t,\xi_1/(\nu+1))^\nu \times$$

$$\times \begin{cases} d(t\xi_1^{1/(q+1)},1)^{\mu(x,\xi)} & \text{if } (t,x,\xi) \in Z_\sigma, \\ d(t\xi_1^{1/(q+1)},1)^{\mu(x,\xi)}\, e^{r(t,\xi)} & \text{if } (t,x,\xi) \in Z\ . \end{cases}$$

We set, for positive constant R,

$$(2.2)\quad u_R(t,x,\xi;p) = \sum_{\nu=0}^{\infty} \int_{(\nu+1)R}^{\infty} e^{\sqrt{-1}p\rho/\xi_1}\, \xi_1^{-n} \times$$
$$\times\ u_\nu(t,x,\rho\xi/\xi_1)\, \rho^{n-1} d\rho.$$

Remark 2.1. If (2.1) holds for $\alpha = 0$, it follows, by Cauchy's inequality, that (2.1) is valid in $\{(t,x,\xi) \in Z;\ \mathrm{dist}(x,\partial X) > \delta\}$ for every α. □

Now we study convergent domains and analytic continuations of (2.2).

<u>Proposition 2.2.</u> (i) The series (2.2) converges uniformly in every compact subset of the domain

$$(2.3)\quad \{(t,x,\xi,p) \in Z_\sigma \times \mathbb{C};\ d(t,R)h < 1,\ p/\xi_1 \in D_0(0)\}$$
$$\cup \{(t,x,\xi,p) \in Z \times \mathbb{C};\ d(t,R)h < 1,\ p/\xi_1 \in D_0(r(t,\xi))\}.$$

Hence u_R is a holomorphic function defined in (2.3) and homogeneous of degree $(-n)$ with respect to (ξ, p).

(ii) By deforming the path of integration in the ν-th term of (2.2) into $C_{\nu,R,\theta}$, u_R is continued to a holomorphic function defined in the domain

$$(2.4)\quad \{(t,x,\xi,p) \in Z_\sigma \times \mathbb{C};\ p/\xi_1 \in D_1(d(t,R)h,\ 0,R)\}$$
$$\cup \{(t,x,\xi,p) \in Z \times \mathbb{C};\ p/\xi_1 \in D_1(d(t,R)h, r(t,\xi),\ R)\}.$$

<u>Proof.</u> (i) If (t,x,ξ,p) is in the first component of (2.3), the former half of (2.1) implies

$$(2.5)\quad \left| \int_{(\nu+1)R}^{\infty} e^{\sqrt{-1}p\rho/\xi_1} \xi_1^{-n} u_\nu(t,x,\rho\xi/\xi_1)\rho^{n-1} d\rho \right|$$
$$\le Ch^\nu |\xi_1|^{-n} \int_{(\nu+1)R}^{\infty} e^{-\mathrm{Im}(p/\xi_1)\rho} d(t,\tfrac{\rho}{\nu+1})^\nu d(t\rho^{1/(q+1)},1)^{\mu(x,\xi)} \rho^{\ell+n-1} d\rho$$
$$\le Ch^\nu d(t,R)^\nu |\xi_1|^{-n} \int_R^{\infty} e^{-\mathrm{Im}(p/\xi_1)\rho} d(t\rho^{1/(q+1)},1)^{\mu(x,\xi)} \rho^{\ell+n-1} d\rho.$$

In the same way, if (t,x,ξ,p) is in the second component of (2.3),

$$(2.6)\quad \left| \int_{(\nu+1)R}^{\infty} e^{\sqrt{-1}p\rho/\xi_1} \xi_1^{-n} u_\nu(t,x,\rho\xi/\xi_1)\rho^{n-1} d\rho \right|$$
$$\le Ch^\nu d(t,R)^\nu |\xi_1|^{-n} \int_R^{\infty} e^{-\{\mathrm{Im}(p/\xi_1) - r(t,\xi)\}\rho} \times$$
$$\times\ d(t\rho^{1/(q+1)},1)^{\mu(x,\xi)} \rho^{\ell+n-1} d\rho.$$

(Notice that $r(t,\xi)$, $\mu(x,\xi)$ are homogeneous of degree 0.)

Hence (2.2) converges uniformly in every compact subset of (2.3), for $d(t,R)h < 1$ in the domain (2.3).

(ii) It suffices to show that the series

$$(2.7)\quad \sum_{\nu=0}^{\infty} \int_{C_{\nu,R,\theta}} e^{\sqrt{-1}p\rho/\xi_1} \xi_1^{-n} u_\nu(t,x,\rho\xi/\xi_1)\rho^{n-1} d\rho$$

converges, for $-b < \theta < b$, uniformly in every compact subset of the domain

$$(2.8)\quad \{(t,x,\xi,p) \in Z_\sigma \times \mathbb{C};\ p/\xi_1 \in D_{1,\theta}(d(t,R)h,0,R)\}$$
$$\cup \{(t,x,\xi,p) \in Z \times \mathbb{C};\ p/\xi_1 \in D_{1,\theta}(d(t,R)h, r(t,\xi),R)\},$$

where we set

$$D_{1,\theta}(d,r,R) = \{ p \in \mathbb{C};\ e^{(|p|+r)R} d < 1,\ \mathrm{Im}(pe^{\sqrt{-1}\theta}) > r\}.$$

In order to estimate the ν-th term of (2.7), we devide $C_{\nu,R,\theta}$ into two parts i.e., $C'_{\nu,R,\theta} = \{(\nu+1)Re^{\sqrt{-1}s\theta};\ 0 \le s \le 1\}$ and $C''_{\nu,R,\theta} = \{(\nu+1)Rse^{\sqrt{-1}\theta};\ s \ge 1\}$. If (t,x,ξ,p) is in the first component of (2.8),

$$(2.9)\quad \left| \int_{C'_{\nu,R,\theta}} e^{\sqrt{-1}p\rho/\xi_1} \xi_1^{-n} u_\nu(t,x,\rho\xi/\xi_1)\rho^{n-1} d\rho \right|$$
$$\le Cd(t,R)^\nu h^\nu |\xi_1|^{-n} \int_{C'_{\nu,R,\theta}} e^{|p/\xi_1|\rho} d(t\rho^{1/(q+1)},1)^{\mu(x,\xi)} \rho^{\ell+n-1} d\rho$$
$$\le 2\pi C e^{|p/\xi_1|R} |\xi_1|^{-n} d(t\{(\nu+1)R\}^{1/(q+1)},1)^{\mu(x,\xi)} \times$$

$$\times\{(\nu+1)R\}^{\ell+n}\{d(t,R)he^{|p/\xi_1|R}\}^{\nu},$$

and if (t,x,ξ,p) is in the second component of (2.8),

$$(2.10)\quad \left|\int_{C'_{\nu,R,\theta}} e^{\sqrt{-1}p\rho/\xi_1}\xi_1^{-n}u_\nu(t,x,\rho\xi/\xi_1)\rho^{n-1}d\rho\right|$$

$$\leq 2\pi Ce^{(|p/\xi_1|+r(t,\xi))R}|\xi_1|^{-n}d(t\{(\nu+1)R\}^{1/(q+1)},1)^{\mu(x,\xi)}\times$$

$$\times\{(\nu+1)R\}^{\ell+n}\{d(t,R)he^{(|p/\xi_1|+r(t,\xi))R}\}^{\nu}.$$

On the other hand the integral along $C''_{\nu,R,\theta}$ is estimated in the same form as (2.5) and (2.6), except that $\mathrm{Im}(p/\xi_1)$ must be replaced by $\mathrm{Im}(pe^{\sqrt{-1}\theta})$. Hence (2.7) converges uniformly in every compact subset of (2.8). Q.E.D.

Remark 2.3. In the proof of the preceding proposition we encountered integrals of the form

$$\int_R^\infty e^{-\mathrm{Im}(pe^{\sqrt{-1}\theta}/\xi_1)\rho}d(t\rho^{1/(q+1)},1)^{\mu(x,\xi)}\rho^{\ell+n-1}d\rho.$$

One can show easily that this integral blows up at most at the rate of a negative power of p/ξ_1 when p/ξ_1 goes to 0 in the domain $D_{1,\theta}(d,r,R)$. Hence the "boundary value hyperfunction" $u_R(t,x,\xi;\varphi(t,x,y,\xi)+\sqrt{-1}0)$ is actually a distribution.

§3. Symbolic calculus (continued).

In this section we derive an action formula for $P = \Sigma_{i=0}^{m} A_i(t,x,D_x)D_t^{m-i}$ to the Radon integral $u_R(t,x,\xi;\varphi(t,x,y,\xi))$. We assume that each coefficient of P is

defined in an open neighborhood of $\{(t,x)\in\mathbb{C}^{n+1};|t|\leq c, |x|\leq a\}$.

We set, for $R\geq 1$,

$$(3.1)\quad v_\nu = \sum_{k=0}^{m}\Sigma^{**}\frac{1}{\alpha!}(\partial_\xi^\alpha A_k{}^{(\kappa)})e^{-\sqrt{-1}\psi}D_x^\alpha D_t{}^{m-k}(e^{\sqrt{-1}\psi}u_\lambda),$$

$$(3.2)\quad v_R(t,x,\xi;p)=\sum_{\nu=0}^{\infty}\int_{(\lambda+1)R}^{\infty}e^{\sqrt{-1}p\rho/\xi_1}\xi_1^{-n}v_\nu(t,x,\rho\xi/\xi_1)\rho^{n-1}d\rho,$$

where Σ^{**} denotes the summation over all (κ,λ,α) such that $\nu=\kappa+\lambda+(q+1)|\alpha|$.

Proposition 3.1. (i) v_ν is semi-homogeneous of degree $(m-\nu)/(q+1)$ and satisfies the inequalities

$$(3.3)\quad |v_\nu(t,x,\xi)|$$

$$\leq C_1h_1^\nu|\xi_1|^{\ell+m}d(t,\xi_1)^{mq}\,d(t,\xi_1/(\nu+1))^\nu\times$$

$$\times\begin{cases} d(t\xi_1^{1/(q+1)},1)^{\mu(x,\xi)} & \text{if } (t,x,\xi)\in Z_\sigma, \\ d(t\xi_1^{1/(q+1)},1)^{\mu(x,\xi)}\,e^{r(t,\xi)} & \text{if } (t,x,\xi)\in Z, \end{cases}$$

for certain positive constants C_1 and h_1.

(ii) The function $P(t,x,D_t,D_x)u_R(t,x,\xi;\varphi(t,x,y,\xi)) - v_R(t,x,\xi;\varphi(t,x,y,\xi))$ is continued to a holomorphic function defined in the domain

$$(3.4)\quad \{(t,x,y,\xi)\in Z''_\sigma;\ \varphi(t,x,y,\xi/\xi_1)\in D_2(d(t,R)h_1,0,R)\}$$
$$\cup\{(t,x,y,\xi)\in Z'';\ \varphi(t,x,y,\xi/\xi_1)\in D_2(d(t,R)h_1,r(x,\xi),R)\},$$

where we set

$$D_2(d,r,R) = \{\,p\in\mathbb{C};\ e^{(|p|+r)R}\,d<1\}$$

$Z''_\sigma = S_\sigma \times X \times X \times \Omega$, $Z'' = \mathbb{C} \times X \times X \times \Omega$. □

Remark 3.2. The domain (3.4) is a neighborhood of $(t,x,y,\xi) = (0,0,0,\xi_0)$ for sufficiently large R. Hence the above proposition implies that the action of a differential operator to a Radon integral is written in the form of a Radon integral "modulo a holomorphic function defined in a neghborhood of $(0,0,0,\xi_0)$". □

In order to prove the above proposition we prepare

Lemma 3.3. (i) Set

$$(3.5)\quad P^\psi(t,x,D_t,D_x) = \sum_{i=1}^{m} e^{-\sqrt{-1}\psi}\circ P(t,x,D_t,D_x)\circ e^{\sqrt{-1}\psi},$$

where "$\circ$" denotes the composition of operators. Then $P^\psi(t,x,D_t,D_x)$ is a differential operator of order m and satisfies (A-1), (A-2) and (A-3), except that λ_j $(1 \le j \le m)$ are changed.

(ii) Corresponding to P^ψ we define $A^\psi_{i,j,k}(x,\xi)$ and $A_i^{\psi(\nu)}(t,x,\xi)$ as we did $A_{i,j,k}$ and $A_i^{(\nu)}$ using P. Then $A_i^{(\nu)}$ and $A_i^{\psi(\nu)}$ are semi-homogeneous of degree $(i-\nu)/(q+1)$ and satisfy, for certain positive constants C_0 and h_0,

$$(3.6)\quad \left.\begin{array}{l}|\partial_\xi^\alpha A_i^{(0)}| \\ |\partial_\xi^\alpha A_i^{\psi(0)}|\end{array}\right\} \le C_0 |\xi_1|^{i-|\alpha|}\, d(t,\xi)^{qi} \text{ if } (t,x,\xi)\in Z,$$

and

$$(3.7)\quad \left.\begin{array}{l}|\partial_\xi^\alpha A_i^{(\nu)}| \\ |\partial_\xi^\alpha A_i^{\psi(\nu)}|\end{array}\right\} \le C_0 h_0^\nu |\xi_1|^{i-|\alpha|-1} d(t,\xi)^{\nu+qi-q-1} \quad \text{if } (t,x,\xi)\in Z \text{ and } \nu > 0.$$

(iii)

$$(3.8)\quad v_\nu = \sum_{k=0}^{m} \Sigma^{**} \frac{1}{\alpha!} (\partial_\xi^\alpha A_k^{\psi(\kappa)})(D_x^\alpha D_t^{m-k} u_\lambda). \ \square$$

Proof. (i) and (iii) are easily proved. Since arguments are similar, we prove (3.7) only, and omit the proof of (3.6). Notice that we have only to consider the case $|\alpha| \le m$, for $\partial_\xi^\alpha A_i^{(\nu)} = \partial_\xi^\alpha A_i^{\psi(\nu)} = 0$ for $|\alpha| > m$.

By Cauchy's inequality there exist positive constants C_2 and h_2 such that

$$\left.\begin{array}{l}|\partial_\xi^\alpha A_{i,j,k}| \\ |\partial_\xi^\alpha A^\psi_{i,j,k}|\end{array}\right\} \le C_2 h_2^k |\xi_1|^{j-|\alpha|} \text{ if } (t,x,\xi)\in Z \text{ and } |\alpha| \le m.$$

If $\nu > 0$, by virtue of (A-1) and (A-3),

$$A_i^{(\nu)} = \sum_{\substack{i-1\ge j,\ k\ge 0,\\ \nu=k-(q+1)j+i}} t^k A_{i,j,k}.$$

Hence

$$\begin{aligned}|\partial_\xi^\alpha A_i^{(\nu)}| &\le C_2 m^2 |\xi_1|^{(q+1)^{-1}(i-\nu)-|\alpha|} \\ &\quad\times \sum_{k=0}^{\nu+qi-q-1} (h_2|t|\,|\xi_1|^{1/(q+1)})^k \\ &\le C_2 m^2 |\xi_1|^{(q+1)^{-1}(i-\nu)-|\alpha|} \\ &\quad\times [2(h_2^{q+1}|t|^{q+1}|\xi_1|+1)^{1/(q+1)}]^{\nu+qi-q-1}.\end{aligned}$$

The same estimation is valid for $A_i^{\psi(\nu)}$. Thus (3.7) holds for suitable constants C_0 and h_0. Q.E.D.

Proof of Proposition 3.1. (i) If $(t,x,\xi)\in Z_\sigma$, (2.1), (3.6) and (3.7) imply that

$$|v_\nu(t,x,\xi)| \le \sum_{k=0}^{m} \Sigma^{**} \frac{1}{\alpha!} C_0 h_0^{\varkappa} |\xi_1|^{k-|\alpha|} d(t,\xi_1)^{(\varkappa+qk)/(q+1)}$$
$$\times\ Ch^{\lambda}\delta^{-|\alpha|}\alpha!\,|\xi_1|^{\ell+m-k}\, d(t,\xi_1/(\lambda+1))^{\lambda}$$
$$\times\ d(t\xi_1{}^{1/(q+1)},1)^{\mu(x,\xi)} d(t,\xi_1)^{q(m-k)}$$
$$\le C_0 C|\xi_1|^{m+\ell} d(t,\xi_1/(\nu+1))^{\nu}\ d(t\xi_1{}^{1/(q+1)},1)^{\mu(x,\xi)}$$
$$\times\ d(t,\xi_1)^{mq} \sum_{k=0}^{m} \Sigma^{**}\ h_0^{\varkappa} h^{\lambda}\delta^{-|\alpha|}.$$

Here we used the inequality

$$|\xi_1|^{-|\alpha|}\, d(t,\xi_1)^{\varkappa} d(t,\xi_1/(\lambda+1))^{\lambda} \le d(t,\xi_1/(\nu+1))^{\nu},$$

where $\nu = \varkappa+\lambda+(q+1)|\alpha|$ and $\varkappa,\lambda,\alpha \ge 0$.

If we set $h_1 = h_0+h+(n/\delta)^{1/(q+1)}$, the former half of (3.3) follows from the inequality

$$\Sigma^{**}\ h_0^{\varkappa} h^{\lambda}\delta^{-|\alpha|} \le (h_0+h+(n/\delta)^{1/(q+1)})^{\nu}.$$

(ii) We notice that

$$P(t,x,D_t,D_x)\ u_R(t,x,y;\varphi(t,x,y,\xi))$$
$$= \sum_{k=0}^{m}\sum_{\lambda=0}^{\infty}\int_{(\lambda+1)R}^{\infty} e^{\sqrt{-1}\varphi(t,x,y,\xi/\xi_1)\rho}\,\xi_1^{-n} A_k^{\psi}(t,x,D_x+\rho\xi/\xi_1)\times$$
$$\times\ D_t^{m-k}u_\lambda(t,x,\rho\xi/\xi_1)\rho^{n-1}d\rho$$
$$= \sum_{k=0}^{m}\sum_{\nu=0}^{\infty}\Sigma^{**}\int_{(\lambda+1)R}^{\infty} e^{\sqrt{-1}\varphi(t,x,y,\ \xi/\xi_1)\rho}\,\xi_1^{-n}\times$$
$$\times\ \frac{1}{\alpha!}(\partial_\xi^\alpha A_k^{\psi(\varkappa)})(t,x,\rho\xi/\xi_1)(D_x^\alpha D_t^{m-k}u_\lambda)(t,x,\rho\xi/\xi_1)\rho^{n-1}d\rho.$$

Hence it is sufficient to prove that the series

$$(\#)\quad \sum_{\nu=0}^{\infty}\Sigma^{**}\int_{(\lambda+1)R}^{(\nu+1)R} e^{\sqrt{-1}p\rho/\xi_1}\,\xi_1^{-n}\frac{1}{\alpha!}(\partial_\xi^\alpha A_k^{\psi(\varkappa)})(t,x,\rho\xi/\xi_1)\times$$
$$\times\ (D_x^\alpha D_t^{m-k}u_\lambda)(t,x,\rho\xi/\xi_1)\rho^{n-1}d\rho$$

converges uniformly in every compact subset of the domain

$$(\#\#)\ \{(t,x,\xi,p)\in Z_\sigma\times\mathbb{C};\ p/\xi_1\in D_2(d(t,R)h_1,0,R)\}$$
$$\cup\{(t,x,\xi,p)\in Z\times\mathbb{C};\ p/\xi_1\in D_2(d(t,R)h_1,r(x,\xi),R)\}.$$

If (t,x,ξ,p) is in the first component of (##), (2.1), (3.6) and (3.7) imply that

$$|\Sigma^{**}\int_{(\lambda+1)R}^{(\nu+1)R} e^{\sqrt{-1}p\rho/\xi_1}\frac{1}{\alpha!}(\partial_\xi^\alpha A_k^{\psi(\varkappa)})(t,x,\rho\xi/\xi_1)\times$$
$$\times\ (D_x^\alpha D_t^{m-k}u_\lambda)(t,x,\rho\xi/\xi_1)\rho^{n-1}d\rho\ |$$
$$\le C_0 C d(t,R)^{m+\nu}\ \Sigma^{**}\ h_0^{\varkappa} h^{\lambda}\delta^{-|\alpha|}$$
$$\times\int_R^{(\nu+1)R} e^{|p/\xi_1|\rho}\, d(t\rho^{1/(q+1)},1)^{\mu(x,\xi)}\rho^{m+\ell+n-1}d\rho$$
$$\le C_0 C(m+1)e^{|p/\xi_1|R} d(t,R)^m\{d(t,R)h_1 e^{|p/\xi_1|R}\}^{\nu} R\,\nu$$
$$\times\{d(t(R\nu+R)^{\frac{1}{q+1}},1)^{\mu_+}+d(tR^{\frac{1}{q+1}},1)^{\mu_-}\}\{(\nu R+R)^{\ell_+}+R^{\ell_-}\}.$$

Here we set

$$\mu_+ = \max\{\mu(x,\xi),\ 0\},\quad \mu_- = \max\{-\mu(x,\xi),\ 0\},$$
$$\ell_+ = \max\{\ell,0\},\quad \ell_- = \max\{-\ell,0\}.$$

If (t,x,ξ,p) is in the second component of (##), the same estimation is valid, except that $|p/\xi_1|$ is replaced by $|p/\xi_1| + r(t,\xi)$. Thus (ii) follows. Q.E.D.

§ 4. Preliminary remarks on transport equations.

4.a. Results in §2 and §3 imply that the proof of Theorem 1 and Theorem 2 is reduced to the construction of solutions of transport equations with growth conditions (see Remark 1.1.) and initial conditions. Precisely speaking:

Suppose that holomorphic functions $u^{(\nu)}_{\sigma,i,j}$ defined in Z satisfy (1.5), (1.6), (1.7) and the following initial conditions

$$(4.1)\quad \sum_{j=1}^{m} D_t^{\,k}(e^{\sqrt{-1}\psi_j}\, u^{(\nu)}_{\sigma,i,j})|_{t=0} = \delta_{k,i}\delta_{\nu,0}(\sqrt{-1})^{n}/(n-1)!,$$

for $0 \le k \le m-1$. Then from (1.6) one can show by simple calculations that (1.7) holds for $k = m$ for a suitable C. Hence the results in § 2 and § 3 implies that the series $\Sigma_{j=1}^{m} u_{\sigma,i,j,R}(t,x,\xi;\varphi_j(t,x,y,\xi))$ is a solution of the equation $Pu = 0$ "modulo a holomorphic function defined in a neighborhood of $(t,x,y,\xi) = (0,0,0,\xi_0)$". On the other hand we notice that

$$\int_R^\infty e^{\sqrt{-1}p\rho/\xi_1}\xi_1^{-n}\rho^{n-1}d\rho = (n-1)!(\sqrt{-1}p)^{-n} - \int_0^R e^{\sqrt{-1}p\rho/\xi_1}\xi_1^{-n}\rho^{n-1}d\rho,$$

and the second term in the right hand side is holomorphic in $\{\xi_1 \ne 0\}$. Hence (4.1) implies that $\Sigma_{j=1}^{m} u_{\sigma,i,j,R}(t,x,\xi;\varphi_j(t,x,y,\xi))$ satisfies the initial conditions of $(CP)_i^{\mathbb{C}}$ "modulo holomorphic functions defined in a neighborhood of $(x,y,\xi) = (0,0,\xi_0)$".

Now we can find a "compensating" function $h_{\sigma,i,R}$ by the usual Cauchy-Kowalevsky theorem such that the function u_i defined by (1.1) is a solution of $(CP)_i^{\mathbb{C}}$.

Notice that Cauchy data and $\Sigma_{j=1}^{m}u_{\sigma,i,j,R}(t,x,\xi;\varphi_j(t,x,y,\xi))$ are homogeneous of degree $-n$ with respect to ξ. Hence $h_{\sigma,i,R}$ can be chosen to have the same homogeneity. Other parts of Theorems 1 and 2 follow immediately.

4.b. Next we rewrite (1.5), (1.6), (1.7) and (4.1) in the matrix form for the convenience of later uses.

We set

$$(4.2)\quad U^{(\nu)}_{\sigma,k} = \left(D_t^{\,i}(e^{\sqrt{-1}\psi_k}\, u^{(\nu)}_{\sigma,j,k}) \right)_{\substack{0\le i\le m-1,\downarrow \\ 0\le j\le m-1,\rightarrow}},$$

$$(4.3)\quad A^{(\nu)} = \begin{bmatrix} 0, & -\delta_{\nu,0}, & & & \\ & \ddots & \ddots & & \\ & & & 0, & -\delta_{\nu,0} \\ A_m^{(\nu)}, & A_{m-1}^{(\nu)}, & \dots, & A_2^{(\nu)}, & A_1^{(\nu)} \end{bmatrix},$$

$$(4.4)\quad T(t) = \mathrm{Diag}[1,t,t^2,\dots,t^{m-1}] = \begin{bmatrix} 1 & & & \\ & t & & \\ & & t^2 & \\ & & & \ddots \\ & & & & t^{m-1} \end{bmatrix}.$$

Then conditions (1.6), (1.7) and (4.1) are equivalent to the following conditions:

$$(4.5)\quad (D_t + A^{(0)})U^{(\nu)}_{\sigma,j} = -\Sigma^* \frac{1}{\alpha!}(\partial_\xi^\alpha A^{(\kappa)})(D_x^\alpha U^{(\lambda)}_{\sigma,j}),$$

$$(4.6)\quad \sum_{j=1}^{m} U^{(\nu)}_{\sigma,j}\Big|_{t=0} = \delta_{\nu,0}/((n-1)!\,(\sqrt{-1})^n)\cdot I,$$

$$(4.7)\quad |T(|\xi_1|^{-1}d(t,\xi)^{-q})\,e^{-\sqrt{-1}\psi_j}U^{(\nu)}_{\sigma,j}T(|\xi_1|^{1/(q+1)})| \le Ch^\nu d(t,\xi_1/(\nu+1))^\nu$$

$$\times\begin{cases} d(t\xi_1^{1/(q+1)},1)^{\mu_j(x,\xi)} & \text{if } (t,x,\xi)\in Z_\sigma,\\ d(t\xi_1^{1/(q+1)},1)^{\mu(x,\xi)}\, e^{r_j(t,\xi)|\xi_1|} & \text{if } (t,x,\xi)\in Z.\end{cases}$$

Here we used the notation $|X| = \max_{1\le i,j\le m} |x_{i,j}|$ for an $m\times m$ matrix $X = (x_{i,j})_{1\le i,j\le m}$.

(1.5) is equivalent to the condition that $T(\xi_1^{-1/(q+1)})U^{(\nu)}_{\sigma,j}T(\xi_1^{1/(q+1)})$ is componentwise semi-homogeneous of degree $\nu/(q+1)$.

4.c. At last we notice that for the construction of $u^{(\nu)}_{\sigma,i,j}$ or equivalently $U^{(\nu)}_{\sigma,j}$ it is sufficient to consider their restrictions to $\{\xi_1 = 1\}$:

Set

$$\Omega' = \{\xi' = (\xi_2,\dots,\xi_n)\in \mathbb{C}^{n-1};\ |\xi'| < b\},$$
$$S'_\sigma = \{t\in \mathbb{C}-0;\ |\arg(\sigma t)| < (q+1)^{-1}(\tfrac{\pi}{2}+b) - \varepsilon\},$$
$$Z'_\sigma = S'_\sigma\times X\times\Omega',\quad Z' = \mathbb{C}\times X\times\Omega',$$

and denote by $(4.5)_{\xi_1=1}$, $(4.6)_{\xi_1=1}$ and $(4.7)_{\xi_1=1}$ the conditions (4.5), (4.6) and (4.7) in which we set $\xi_1 = 1$. Here, however, domains where inequalities of $(4.7)_{\xi_1=1}$ are assumed are Z'_σ and Z', not $Z_\sigma\cap\{\xi_1 = 1\}$ and $Z\cap\{\xi_1=1\}$. (Hereafter we often use these notaions.)

Then by virtue of the semi-homogeneity of ψ_j and $A_i^{(\nu)}$ the conditions (4.5), (4.6) and(4.7) are compatible with the condition (1.5). Namely, if $U^{(\nu)}_{\sigma,j}(t,x,\xi')$ $(1\le j\le m)$ are holomorphic matrix functions defined in Z' and satisfy $(4.5)_{\xi_1=1}$, $(4.6)_{\xi_1=1}$ and $(4.7)_{\xi_1=1}$, and if we set

$$U^{(\nu)}_{\sigma,j}(t,x,\xi) = \xi_1^{-\nu/(q+1)}T(\xi_1^{1/(q+1)})$$
$$\times\ U^{(\nu)}_{\sigma,j}(t\xi_1^{1/(q+1)},x,\xi'/\xi_1)T(\xi_1^{-1/(q+1)}),$$

then $U^{(\nu)}_{\sigma,j}(t,x,\xi)$ $(1\le j\le m)$ satisfy (4.5), (4,6) and (4.7) , and they are holomorphic in Z. Similar assertions are are also valid for $u^{(\nu)}_{\sigma,i,j}$.

4.d. Thus the proof of Theorems 1 and 2 is reduced to the construction of holomorphic functions $U^{(\nu)}_{\sigma,j}(t,x,\xi')$ defined in Z' (a, b and ε are suitably chosen) which satisfy $(4.5)_{\xi_1=1}$, $(4.6)_{\xi_1=1}$ and $(4.7)_{\xi_1=1}$. In the following sections we shall construct such matrix functions.

§5. Solutions of 0-th transport equations.

In the case $\nu=0$, $(4.5)_{\xi_1=1}$ is a homogeneous ordinary differential equation with respect to t with polynomial coefficients, in which (x,ξ') is contained as a holomorphic parameter. The point $t=\infty$ is an irregular singular point of Poincaré's rank q+1. Hence general theories on irregular singular points (see W. Wasow [19], Chapter IV) are applicable with slightest modifications caused by the existence of the holomorphic parameter (x,ξ').

Let us consider the ordinary differential equation

(5.1) $(D_t + A^{(0)}(t,x,1,\xi'))\,v = 0,\quad v = {}^t(v_0,v_1,\dots,v_{m-1}).$

The transformation

(5.2) $v = T(t^q)w,\ w = {}^t(w_0, w_1, \ldots, w_{m-1})$

changes (5.1) into the equation

(5.3) $(t^{-q}D_t - \hat{A}^{(0)}(t,x,\xi'))\, w = 0,$

where

(5.4) $$\hat{A}^{(0)}(t,x,\xi') = -t^{-q}T(t^{-q})\, A^{(0)}(t,x,1,\xi')\, T(t^q) - qt^{-q-1}\mathrm{Diag}[0,1,2,\ldots,m-1].$$

From (A-1) and (A-3) immediately follows

Proposition 5.1. (i) $\hat{A}^{(0)}$ is a polynimial in t^{-q-1} whose coefficients are holomorphic matrix functions in (x,ξ') defined in a neighborhood of $(x,\xi') = (0,0)$ in $\mathbb{C}^n \times \mathbb{C}^{n-1}$, and are polynomials in ξ' at most of degree m. Moreover

(5.5) $$\hat{A}^{(0)}(t,x,\xi') = \hat{A}^{(0)}(\infty,\xi') + O(t^{-q-1}),$$

where

$$\hat{A}^{(0)}(\infty,\xi') = -\begin{bmatrix} 0, & & -1, & & \\ & \ddots & & \ddots & \\ & & 0, & & -1 \\ A_{m,m,mq}(1,\xi'), & \ldots, & A_{2,2,2q}(1,\xi'), & A_{1,1,q}(1,\xi') \end{bmatrix}$$

(ii) The eigenvalues of $\hat{A}^{(0)}(\infty,\xi')$ coincide with $\lambda_j(1,\xi')$ $(1 \le j \le m)$. In particular they are mutually distinct. □

Since eigenvalues of the matrix $\hat{A}^{(0)}(\infty,\xi')$ are mutually distinct, we can perform the process of "formal diagonalizations" to obtain a formal fundamental matrix solution of the equation (5.3). (See Wasow [19], Section 11, Theorem 12.3.) Detailed arguments are developed in Nakamura-Uryu [11]. The result is

Proposition 5.2. There exists a formal matrix solution of (5.3) of the form

(5.6) $$W = \hat{W}\, t^{\Pi(x,\xi')}\, e^{\sqrt{-1}\,\Psi(t,\xi')},$$

where $\hat{W}$ is a formal power series in t^{-1} of the form

(5.7) $$\hat{W} = \sum_{k=0}^{\infty} W_k(x,\xi')\, t^{-k}, \quad \det W_0(x,\xi') \neq 0,$$

whose coefficients are holomorphic functions in (x,ξ'). $\Pi(x,\xi')$ and $\Psi(t,\xi')$ are $m \times m$ matrices of the form

(5.8) $$\begin{cases} \Pi(x,\xi') = \mathrm{Diag}[\pi_1(x,1,\xi'), \ldots, \pi_m(x,1,\xi')], \\ \Psi(t,\xi') = \mathrm{Diag}[\psi_1(t,1,\xi'), \ldots, \psi_m(t,1,\xi')]. \end{cases}$$ □

Next, we consider the existence of a holomorphic matrix solution $W_\sigma(t,x,\xi')$ of (5.3) such that the asymptotic expansion of $W_\sigma t^{-\Pi} e^{-\sqrt{-1}\Psi}$ in the sector S'_σ coincides with $\hat{W}$, and the asymptotic expansion is uniformly valid with respect to $(\arg(t), x, \xi')$; namely, for any integer $N \ge 0$ there exist positive constants t_N and K_N such that, for $t \in \{t \in S'_\sigma; |t| \ge t_N\}$ and $(x,\xi') \in X \times Z$,

(5.9) $$\left|W_\sigma t^{-\Pi} e^{-\sqrt{-1}\Psi} - \sum_{k=0}^{N} W_k t^{-k}\right| \le K_N |t|^{-N-1}.$$

Hereafter we assume that $b < (q+1)\varepsilon$. Then the positive central angle of S'_σ is strictly smaller than $\pi/(2q+2)$.

If (5.3) contains no parameter, the existence of such a solution follows from well-known results. (See Wasow [19], § 12, Theorem 12.3.) But in our case (5.3) contains a holomorphic parameter (x,ξ'). We must pay attention to the geometric arrangement of the sector S'_σ and directions in which $e^{\sqrt{-1}\psi_j(t,1,\xi')}$ increases at the exponential rate. Now we notice that, from (A-1), (A-2) and the assumption $b<(q+1)\varepsilon$, immediately follows

Proposition 5.3. For sufficiently small positive constants ε and b there exist $\lambda_0\in\mathbb{R}$ and $\alpha_{\sigma,j}$, $\alpha_{\sigma,j,k}\in S'_\sigma$ $(j\neq k,\ \sigma=\pm 1)$ such that $|\alpha_{\sigma,j}| = |\alpha_{\sigma,j,k}| = 1$ and

$$(5.10)\quad \begin{cases} \inf_{\xi'\in\Omega'} \mathrm{Re}\{\sqrt{-1}(\lambda_j(1,\xi')-\lambda_0)\alpha_{\sigma,j}^{\,q+1}\} > 0, \\ \inf_{\xi'\in\Omega'} \mathrm{Re}\{\sqrt{-1}(\lambda_j(1,\xi')-\lambda_k(1,\xi'))\alpha_{\sigma,j,k}^{\,q+1}\} > 0. \ \square \end{cases}$$

In the rest of this paper we develope our arguments under the assumption that statements of Proposition 5.3. are fulfilled and W_k, W_0^{-1}, $\pi_j(x,1,\xi')$ and $\lambda_j(1,\xi')$ are bounded in $X\times\Omega'$ (by shrinking X and Ω', if necessary).

By virtue of this proposition we can show the existence of W_σ as follows:

At first notice that the transformation $w \longmapsto we^{\sqrt{-1}\lambda_0 t^{q+1}/(q+1)}$ changes eigenvalues of $\hat{A}^{(0)}(\infty,\xi')$ into $\lambda_j(1,\xi')-\lambda_0$ $(1\leq j\leq m)$. Hence we have only to consider the case $\lambda_0 = 0$.

We intend to apply the proof of Theorem 12.3 of [19] to our case. Remember that one of the most crucial points of arguments in § 14 of [19] is the choice of paths of integrations in § 14.3. If we can choose suitable paths so that the integral equation (14.16) in § 14.2 make sense, other arguments in § 14 are also entirely valid in our case.

In fact it is possible to choose such paths. Change variables in such a manner that

$$(5.11)\quad \tau = t^{q+1},\quad \xi = x^{q+1},$$

and set

$$(5.12)\quad \delta_j(\xi) = \{\xi+\alpha_{\sigma,j}^{\,q+1}r;\ r\geq 0\} \qquad (1\leq j\leq m).$$

(Here we used the same notations as those of [19], § 14.3.) Then paths corresponding to $\delta_j(\xi)$ $(1\leq j\leq m)$ by (5.11) are what we need, for $\alpha_{\sigma,j}$ satisfies the former half of (5.10).

Thus we obtained

Proposition 5.4. There exists a holomorphic matrix solution $W_\sigma(t,x,\xi')$ which is holomorphic in $(\mathbb{C}-0)\times X\times\Omega'$ and satisfies (5.9). $\square$

Set

$$(5.13)\quad V_\sigma(t,x,\xi') = T(t^{q+1})W_\sigma(t,x,\xi').$$

Then V_σ is a holomorphic solution of (5.1) defined in $(\mathbb{C}-0)\times X\times\Omega'$ and invertible. Hence it is continued to a holomorphic solution defined in Z'_σ which we denote by the same notation V_σ. Immediately follows

Proposition 5.5. There exists a positive constant C_1 such that, if $(t,x,\xi')\in Z'_\sigma$,

$$(5.14)\quad \left.\begin{array}{l} |T(d(t,1)^{-q})V_\sigma e^{-\sqrt{-1}\Psi} d(t,1)^{-M(x,\xi')}| \\ |d(t,1)^{M(x,\xi')} e^{\sqrt{-1}\Psi} V_\sigma^{-1} T(d(t,1)^q)| \end{array}\right\} \le C_1,$$

where C_1 is a positive constant and

$$(5.15)\quad M(x,\xi') = \mathrm{Diag}[\mu_1(x,1,\xi'),\dots,\mu_m(x,1,\xi')]. \ \Box$$

Remark 5.6. Here we used the inequalities

$$\begin{cases} k_0^{-1} d(t,1)^q \le |t|^q \le k_0 d(t,1)^q, \\ k_0^{-1} d(t,1)^{\mu_j(x,1,\xi')} \le |t^{\pi_j(x,1,\xi')}| \le k_0 d(t,1)^{\mu_j(x,1,\xi')}, \end{cases}$$

$$(|t| \ge 1,\ (t,x,\xi') \in Z'_\sigma),$$

where k_0 is a positive constant. □

(5.14) may be false outside of Z'_σ, for "Stokes' phenomena" may occur. (See [19], § 15.) We need another set of holomorphic and invertible matrix solutions of (5.1) for the later use.

Let us take a family of sectors

$$\mathcal{S} = \{e^{\pm 2\sqrt{-1}\varepsilon'}\omega^k S'_\sigma,\ \omega^k S'_\sigma\ ;\ k = 0,1,\dots,2q+1\},$$

where $\varepsilon' = \varepsilon - (q+1)^{-1} b$ and $\omega = e^{\sqrt{-1}\pi/(q+1)}$. $\mathcal{S}$ covers the whole plane and containes S'_σ. Then we have

Proposition 5.7. If ε and b are sufficiently small, there exist $\alpha_{S,j}, \alpha_{S,j,k} \in S$ ($j \ne k$, $S \in \mathcal{S}$) such that $\alpha_{S,j}{}^{q+1} = \alpha_{\sigma,j}{}^{q+1}$, $\alpha_{S,j,k}{}^{q+1} = \alpha_{\sigma,j,k}{}^{q+1}$. In particular the same assertion as (5.10) holds for each $S \in \mathcal{S}$. □

Hence arguments so far are also valid for each $S \in \mathcal{S}$. Thus we obtain

Proposition 5.8. For each $S \in \mathcal{S}$ there exists a holomorphic matrix solution V_S of (5.1) defined in Z' and invertible such that, if $(t,x,\xi') \in Z'_S$,

$$(5.16)\quad \left.\begin{array}{l} |T(d(t,1)^{-q})V_S e^{-\sqrt{-1}\Psi} d(t,1)^{-M(x,\xi')}| \\ |d(t,1)^{M(x,\xi')} e^{\sqrt{-1}\Psi} V_S^{-1} T(d(t,1)^q)| \end{array}\right\} \le C_1,$$

where C_1 is a positive constant and we set

$$Z'_S = S \times X \times \Omega'. \ \Box$$

Now we proceed to construct solutions of 0-th transport equations with the growth conditions and the initial conditions. Set, for $1 \le j,k \le m$,

$$(5.17)\quad U^{(0)}_{\sigma,j}(t,x,\xi') = V_\sigma(t,x,\xi')\, E_j\, V_\sigma(0,x,\xi')^{-1},$$

$$(5.18)\quad U^{(0)}_{\sigma,j;S,k}(t,x,\xi') = V_S(t,x,\xi') E_k C_{S,\sigma}(x,\xi') E_j \times V_\sigma(0,x,\xi')^{-1},$$

where

$$(5.19)\quad C_{S,\sigma}(x,\xi') = V_S(0,x,\xi')^{-1} V_\sigma(0,x,\xi'),$$

$$(5.20)\quad E_j = \mathrm{Diag}[0,\dots,0,\underset{(j)}{1},0,\dots,0].$$

Then from Propositions 5.5 and 5.8 immediately follows

Proposition 5.9. (i) For each $S \in \mathcal{S}$, $U^{(0)}_{\sigma,j;S,k}$ satisfies $(4.5)_{\xi_1=1}$ for $\nu = 0$ and

$$(5.21)\quad U^{(0)}_{\sigma,j} = \sum_{k=1}^{m} U^{(0)}_{\sigma,j;S,k},$$

$$(5.22)\quad |T(d(t,1)^{-q}) e^{-\sqrt{-1}\psi_k} U^{(0)}_{\sigma,j;S,k} |d(t,1)^{-\mu_k} \le C/m,$$

for $(t,x,\xi')\in Z_S'$, where C is a positive constant.

(ii) $U^{(0)}_{\sigma,j}$ $(1\le j\le m)$ satisfy $(4.5)_{\xi_1=1}$, $(4.6)_{\xi_1=1}$ and $(4.7)_{\xi_1=1}$ for $\nu=0$. □

Remark 5.10. $C_{S,\sigma}$ is the "Stokes' multiplier" for sectors S and S_σ' with respect to V_S and V_σ. One can show from (5.14) and (5.16) that $C_{S,\sigma}$ is bounded in $X\times\Omega'$.

§6. Solutions of higher transport equations.

In this section we construct solutions of ν-th transport equations by the method of variations of constants, using the same paths as those of Nishimoto [13] , and estimate them by the induction on ν.

By some technical reasons we shall establish, for certain constants $B\ge 1$, $L>1$, the following inequalities for any sufficiently small $\delta>0$, instead of $(4.7)_{\xi_1=1}$:

$$(6.1)\quad |T(d(t,1)^{-q})e^{-\sqrt{-1}\psi_j}U^{(\nu)}_{\sigma,j}|\le C\{\delta^{-L/(q+1)}h\, d(t,(B\nu+1)^{-1})\}^{\nu}\times$$

$$\times\begin{cases} d(t,1)^{\mu_j(x,1,\xi')} & \text{if } (t,x,\xi')\in Z'_{\sigma,\delta},\\ d(t,1)^{\mu(x,1,\xi')}e^{r_j(t,1,\xi')} & \text{if } (t,x,\xi')\in Z'_\delta.\end{cases}$$

Moreover in order to establish the latter half of (6.1) it is sufficient to construct, for each $S\in\mathcal{S}$, matrix functions $U^{(\nu)}_{\sigma,j;S,k}$ $(1\le j,k\le m,\ \sigma=\pm 1)$ defined in Z' such that they satisfy transport equations of the same form as $(4.5)_{\xi_1=1}$ and the following conditions

$$(6.2)\qquad U^{(\nu)}_{\sigma,j}=\sum_{k=1}^{m}U^{(\nu)}_{\sigma,j;S,k},$$

$$(6.3)\quad |T(d(t,1)^{-q})e^{-\sqrt{-1}\psi_k}U^{(\nu)}_{\sigma,j;S,k}|\le (C/m)(\delta^{-L/(q+1)}h)^{\nu}\times$$

$$\times\ d(t,(B\nu+1)^{-1})^{\nu}d(t,1)^{\mu_k(t,x,1,\xi')}\quad \text{if } (t,x,\xi')\in Z'_{S,\delta}.$$

Here we set

$Z'_{\sigma,\delta}=S'_\sigma\times X_\delta\times\Omega'$, $Z'_{S,\delta}=S\times X_\delta\times\Omega'$, $Z'_\delta=\mathbb{C}\times X_\delta\times\Omega'$,

$X_\delta=\{x\in X;\ \mathrm{dist}(x,\partial X)>\delta\}$.

6.a. Hereafter we proceed to construct and estimate $U^{(\nu)}_{\sigma,j}$ and $U^{(\nu)}_{\sigma,j;S,k}$ by the induction on ν.

Now we assume, as the assumption of the induction, that $U^{(\lambda)}_{\sigma,j}$ and $U^{(\lambda)}_{\sigma,j;S,k}$ $(\lambda<\nu)$ are already constructed and (6.1), (6.2) and (6.3) are fulfilled. If $U^{(\nu)}_{\sigma,j}$ and $U^{(\nu)}_{\sigma,j;S,k}$ are constructed and satisfy (6.1), (6.2) and (6.3), our proof of Theorems 1 and 2 is completed, for $U^{(0)}_{\sigma,j}$ and $U^{(0)}_{\sigma,j;S,k}$ are obtained in §5.

In the following arguments we often omit to write the variable (x,ξ) for the simplicity of notations.

6.b. At first we construct $U^{(\nu)}_{\sigma,j}$ and estimate it in the domain $Z'_{\sigma,\delta}$:

Set

$$(6.4)\qquad F^{(\nu)}_{\sigma,j}=-\Sigma^*\frac{1}{\alpha!}(\partial_\xi^\alpha A^{(\kappa)})(D_x^\alpha U^{(\lambda)}_{\sigma,j}),$$

$$(6.5)\quad V^{(\nu)}_{\sigma,j}(t)=\sum_{i=1}^{m}\int_{\gamma_{\sigma,i,j}(t)}V_\sigma(t)E_iV_\sigma(s)^{-1}F^{(\nu)}_{\sigma,j}(s)ds,$$

$$(6.6)\quad U^{(\nu)}_{\sigma,j}(t)=V^{(\nu)}_{\sigma,j}(t)-U^{(0)}_{\sigma,j}(t)\sum_{k=1}^{m}V^{(\nu)}_{\sigma,k}(0),$$

where $\gamma_{\sigma,i,j}(t)$ is a path starting at $s=\infty$ and ending in $s=t$, which we choose later.

If infinite integrals in (6.5) make sense, the matrix functions $U^{(\nu)}_{\sigma,j}$ $(1\leq j\leq m)$ actually satisfy $(4.5)_{\xi_1=1}$ and $(4.6)_{\xi_1=1}$. The problem is how to estimate them in the domain $Z'_{\sigma,\delta}$.

At first we obtain

Proposition 6.1. If $B\geq 1$, $L>1$, $\delta^{L/(q+1)}h_0+n^{1/(q+1)}\leq h/2$, $e^L\delta^{L-1}\leq 1$ and $(t,x,\xi')\in Z'_{\sigma,\delta}$, then

$$(6.7)\quad |T(d(t,1)^{-q})e^{-\sqrt{-1}\psi_j}F^{(\nu)}_{\sigma,j}|d(t,1)^{-\mu_j}$$
$$\leq C_2(\delta^{-L/(q+1)}h)^{\nu}d(t,(B\nu+1)^{-1})^{\nu-1},$$

where $C_2=2m(\delta^{L/(q+1)}h_0+n^{1/(q+1)})C_0C/h$. ($C_0$ and h_0 are those which appeared in Proposition 3.2.) □

In order to prove this proposition we prepare

Lemma 6.2. Under the same assumptions as above,

$$(6.8)\quad |T(d(t,1)^{-q})e^{-\sqrt{-1}\psi_j}D^{\alpha}_{x}U^{(\lambda)}_{\sigma,j}|d(t,1)^{-\mu_j}$$
$$\leq C\delta^{-L(|\alpha|+\lambda/(q+1))}h^{\lambda}d(t,(|\alpha|+B\lambda+1)^{-1})^{\lambda+(q+1)|\alpha|}\quad(\lambda<\nu).$$

Proof of Lemma 6.2. We prove (6.8) by the induction on $|\alpha|$. In the case $|\alpha|=0$, (6.8) is nothing but what we assumed as the assumption of the induction on ν for the construction of $U^{(\nu)}_{\sigma,j}$ and $U^{(\nu)}_{\sigma,j;S,k}$.

Suppose that (6.8) holds for any muti-index β with $|\beta|<|\alpha|$. We may assume that $\alpha_k>0$ for certain k.

Set $\beta=\alpha-(0,\dots,0,\underset{(k)}{1},0,\dots,0)$ and apply the assumption of the induction to β and the domain $Z'_{\sigma,\{1-(|\alpha|+\lambda/(q+1))^{-1}\}\delta}$. Then for $(t,x,\xi')\in Z'_{\sigma,\{1-(|\alpha|+\lambda/(q+1))^{-1}\}\delta}$,

$$|T(d(t,1)^{-q})e^{-\sqrt{-1}\psi_j}D^{\alpha}_{x}U^{(\lambda)}_{\sigma,j}|d(t,1)^{-\mu_j}$$
$$\leq Ch^{\lambda}\{(1-\frac{1}{|\alpha|+\lambda/(q+1)})\delta\}^{-L(|\alpha|-1+\lambda/(q+1))}d(t,(|\alpha|+B\lambda)^{-1})^{\lambda+(q+1)(|\alpha|-1)}.$$

Hence Cauchy's inequality implies that, if $(t,x,\xi')\in Z'_{\sigma,\delta}$,

$$|T(d(t,1)^{-q})e^{-\sqrt{-1}\psi_j}D^{\alpha}_{x}U^{(\nu)}_{\sigma,j}|d(t,1)^{-\mu_j}$$
$$\leq C(\frac{|\alpha|+\lambda/(q+1)}{\delta})\{(1-\frac{1}{|\alpha|+\lambda/(q+1)})\delta\}^{-L(|\alpha|-1+\lambda/(q+1))}h^{\lambda}$$
$$\times\ d(t,(|\alpha|+B\lambda)^{-1})^{\lambda+(q+1)(|\alpha|-1)}$$
$$\leq e^{L}\delta^{L-1}C\delta^{-L(|\alpha|+\lambda/(q+1)}h^{\lambda}d(t,(|\alpha|+1+B\lambda)^{-q})^{\lambda+(q+1)|\alpha|}.$$

Here we used the following inequalities

$$(1-\frac{1}{|\alpha|+\lambda/(q+1)})^{-L(|\alpha|-1+\lambda/(q+1))}\leq e^{L},$$
$$(|\alpha|+\lambda/(q+1))d(t,(|\alpha|+B\lambda)^{-1})^{\lambda+(q+1)(|\alpha|-1)}$$
$$\leq d(t,(|\alpha|+1+B\lambda)^{-1})^{\lambda+(q+1)|\alpha|}.$$

This proves Lemma 6.2. Q.E.D.

Proof of Proposition 6.1. (6.8) and (3.7) implies that, for $(t,x,\xi')\in Z'_{\sigma,\delta}$,

$$|T(d(t,1)^{-q})e^{-\sqrt{-1}\psi_j}F^{(\nu)}_{\sigma,j}|d(t,1)^{-\mu_j}$$
$$\leq mC_0C\quad\Sigma^{*}h_0^{\kappa}h^{\lambda}\delta^{-L(|\alpha|+\lambda/(q+1))}d(t,1)^{\kappa-1}$$
$$\times\ d(t,(|\alpha|+B\lambda+1)^{-1})^{\lambda+(q+1)|\alpha|}$$

$$\leq mC_0C\ \Sigma^* h_0^{\kappa} h^{\lambda} \delta^{-L(|\alpha|+\lambda/(q+1))} d(t,(B\nu+1)^{-1})^{\nu}.$$

Here we used the inequality

$$d(t,1)^{\kappa-1} d(t,(|\alpha|+B\lambda+1)^{-1})^{\lambda+(q+1)|\alpha|} \leq d(t,(B\nu+1)^{-1})^{\nu-1}.$$

Since

$$\Sigma^* h_0^{\kappa} h^{\lambda} \delta^{-L(|\alpha|+\lambda/(q+1))}$$

$$\leq (\delta^{-L/(q+1)} h)^{\nu} \sum_{\lambda=0}^{\nu-1} [(\delta^{L/(q+1)} h_0 + n^{1/(q+1)})/h]^{\nu-\lambda}$$

$$\leq 2(\delta^{-L/(q+1)} h)^{\nu} (\delta^{L/(q+1)} h_0 + n^{1/(q+1)})/h,$$

(6.7) follows. Q.E.D.

Now we choose the path $\gamma_{\sigma,i,j}(t)$ as follows:

Change variables as follows:

(6.9) $z = t^{q+1}$, $\zeta = s^{q+1}$.

Denote by Σ'_σ the sector in the ζ-plane corresponding to the sector S'_σ in the s-plane. Let $\gamma_{\sigma,i,i}(t)$ be the segment from t to 0 in the s-plane and $\gamma_{\sigma,i,j}(t)$ $(i \neq j)$ be, if $t \in S'_\sigma$, the path in S'_σ corresponding to the path in the ζ-plane

$$\delta_{\sigma,i,j}(z) = \{ z + \alpha_{\sigma,i,j}^{q+1} r ;\ r \geq 0\}.$$

(See Proposition 5.3 .) If $t \notin S'_\sigma$, we choose $\gamma_{\sigma,i,j}(t)$ $(i \neq j)$ to be the union of the segment from 0 to t and $\gamma_{\sigma,i,j}(0)$.

We can rewrite (6.5) in the form

$$(6.10)\quad V^{(\nu)}_{\sigma,j}(t) e^{-\sqrt{-1}\psi_j(t)} = \sum_{i=1}^{m} \int_{\gamma_{\sigma,i,j}(t)} (V_\sigma(t) e^{-\sqrt{-1}\Psi(t)} E_i) \times$$

$$\times (V_\sigma(s) e^{-\sqrt{-1}\Psi(s)} E_i)^{-1} (F^{(\nu)}_{\sigma,j}(s) e^{-\sqrt{-1}\psi_j(s)}) e^{\sqrt{-1}\psi_{i,j}(t,s)} ds,$$

where we set

$$\psi_{i,j}(t,s) = \{\psi_i(t) - \psi_j(t)\} - \{\psi_i(s) - \psi_j(s)\}.$$

(5.10) implies that, for $i \neq j$, $e^{\sqrt{-1}\psi_{i,j}(t,s)}$ deacreases exponentially at $s = \infty$ along $\gamma_{\sigma,i,j}(t)$, while (5.14) and (6.7) imply that other factors in the integrals of (6.10) increase at most at the rate of polynomial growth. Hence the integrals in (6.10) actually converge and make sense.

Next, let us consider the estimation of these integrals and $U^{(\nu)}_{\sigma,j}$:

(5.14) and (6.7) imply that, if $(t,x,\xi') \in Z'_{\sigma,\delta}$,

$$|V^{(\nu)}_{\sigma,j} e^{-\sqrt{-1}\psi_j}| \leq m^3 C_2 (\delta^{-L/(q+1)} h)^{\nu} \sum_{i=1}^{m} \int_{\gamma_{\sigma,i,j}(t)} d(t,(B\nu+1)^{-1})^{\nu-1} \times$$

$$\times \left(\frac{d(t,1)}{d(s,1)}\right)^{\mu_{i,j}} e^{\mathrm{Re}[\sqrt{-1}\psi_{i,j}(t,s)]} |ds| ,$$

where we set

$$\mu_{i,j} = \mu_i(x,1,\xi') - \mu_j(x,1,\xi').$$

As for the estimation of these integrals, we have

Proposition 6.3. If $B \geq 2(q+1)^{-1}\beta^{-1}$ and $t \in S'_\sigma$,

$$(6.11)\quad \int_{\gamma_{\sigma,i,j}(t)} e^{\mathrm{Re}[\sqrt{-1}\psi_{i,j}(t,s)]} \left(\frac{d(t,1)}{d(s,1)}\right)^{\mu_{i,j}} d(t,(B\nu+1)^{-1})^{\nu-1} ds$$

$$\leq C_3 d(t,(B\nu+1)^{-1})^{\nu},$$

where C_3 is a positive constant and

$$\beta = \min_{i \neq j} \inf_{\xi' \in \Omega'} \mathrm{Re}[\sqrt{-1}(\lambda_j(1,\xi') - \lambda_j(1,\xi'))].$$

Proof. In the case $i = j$, (6.11) follows immediately if we take $C_3 \geq 1$.

Let us consider the case $i \neq j$. (5.14) and the definition of β imply that, if $\zeta = z + \alpha_{\sigma,i,j}{}^{q+1}r$ ($r \geq 0$) and s and t are the same as (6.9),

$$e^{\mathrm{Re}[\sqrt{-1}\psi_{i,j}(t,s)]}\left(\frac{d(s,(B\nu+1)^{-1})^{\nu-1}}{d(t,(B\nu+1)^{-1})^{\nu}}\right)$$
$$\leq e^{-\beta r}\left(\frac{d(s,(B\nu+1)^{-1})}{d(t,(B\nu+1)^{-1})}\right)^{\nu}$$
$$\leq e^{-\beta r}\left(1+\frac{r}{|z|+B+1}\right)^{\nu/(q+1)}.$$

One can show by simple calculations that, if $B \geq \frac{2}{(q+1)\beta}$,

$$e^{\frac{-\beta r}{2}}\left(1+\frac{1}{|z|+B\nu+1}\right)^{\nu/(q+1)} \leq 1 \quad \text{for } r \geq 0.$$

On the other hand, noting that $ds = \frac{d\zeta}{(q+1)\zeta^{q/(q+1)}}$, one can show easily that

$$\int_{\gamma_{\sigma,i,j}(t)} e^{\frac{-\beta r}{2}}\left(\frac{d(t,1)}{d(s,1)}\right)^{\mu_{i,j}} ds \leq \sup_{\substack{t\in S'_\sigma \\ s\in\gamma_{\sigma,i,j}(t)}}\left[\left(\frac{d(t,1)}{d(s,1)}\right)^{\mu_{i,j}}\left|\frac{z-\zeta}{\zeta}\right|^{\frac{q}{q+1}}\right] \times \frac{1}{q+1}\int_0^\infty e^{\frac{-\beta r}{2}}\frac{dr}{r^{q/(q+1)}},$$

and each factor in the righthand side of this inequality is a finite number which is independent of t.

This proves Proposition 6.3. Q.E.D.

Summing up; if $B \geq \max[1, 2(q+1)^{-1}\beta^{-1}]$, $e^{L}\delta^{L-1} \leq 1$, $\delta^{L/(q+1)}h_0 + n^{1/(q+1)} \leq h/2$ and $(t,x,\xi') \in Z'_{\sigma,\delta}$, then we have the following inequality

$$|V^{(\nu)}_{\sigma,j}e^{-\sqrt{-1}\psi_j}|\,d(t,1)^{-\mu_j} \leq 2m^4C_0C_3C(\delta^{-L/(q+1)}h)^{\nu}\times$$
$$\times\,(\delta^{L/(q+1)}h_0+n^{1/(q+1)})h^{-1}d(t,(B\nu+1)^{-1})^{\nu},$$

and hence the following estimation

$$(6.12)\quad |T(d(t,1)^{-q})e^{-\sqrt{-\psi}_j}\,U^{(\nu)}_{\sigma,j}|\,d(t,1)^{-\mu_j}$$
$$\leq 2m^4C_0C_3(1+mC)(\delta^{L/(q+1)}h_0+n^{1/(q+1)})h^{-1}$$
$$\times\; C[\delta^{-L/(q+1)}h\,d(t,(B\nu+1)^{-1})]^{\nu}.$$

Hence, if h is chosen sufficiently large for any constants $L>1$ and $B \geq \max[1,2(q+1)^{-1}\beta^{-1}]$, $U^{(\nu)}_{\sigma,j}$ satisfies the former half of (6.1) for sufficiently small $\delta > 0$.

This completes the induction on ν for the construction and the estimation of $U^{(\nu)}_{\sigma,j}$.

6.c. Next we construct and estimate $U^{(\nu)}_{\sigma,j;S,k}$:

We proceed once more by the induction on ν, and repeat the same arguments as 6.b.

We obtain $U^{(\nu)}_{\sigma,j;S,k}$ in the form

$$(6.13)\quad U^{(\nu)}_{\sigma,j;S,k}(t) = V^{(\nu)}_{\sigma,j;S,k}(t) - U^{(0)}_{S,k}(t)\times$$
$$\times\,[(\sum_{k=1}^{m}V^{(\nu)}_{\sigma,j;S,k}(0)) - U^{(\nu)}_{\sigma,j}(0)],$$

where we set

$$(6.14)\quad V^{(\nu)}_{\sigma,j;S,k}(t) = \sum_{i=1}^{m}\int_{\gamma_{S,i,j}(t)}V_S(t)E_iV_S(s)^{-1}F^{(\nu)}_{\sigma,j;S,k}(s)ds,$$

$$(6.15)\quad F^{(\nu)}_{\sigma,j;S,k} = -\Sigma^*\frac{1}{\alpha!}(\partial^{\alpha}_{\xi}A^{(\kappa)})(D^{\alpha}_xU^{(\lambda)}_{\sigma,j;S,k}),$$

$$(6.16)\quad U^{(0)}_{S,k}(t) = V_S(t)E_kV_S(0)^{-1}.$$

We choose $\gamma_{S,i,j}(t)$ to be the path defined in the same way as $\gamma_{\sigma,i,j}(t)$, using $\alpha_{S,i,j}$ instead of $\alpha_{\sigma,i,j}$.

(See Proposition 5.3 and 5.7.) Then the integrals in (6.14) make sense, and (6.2) is fulfilled.

Moreover we can show by the same arguments as 6.b that, if h is sufficiently large, the condition (6.3) is fulfilled for any sufficiently small $\delta > 0$. Since the estimation is the same as 6.b, we omit details.

Thus we have proved Theorems 1 and 2.

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ELEMENTARY SOLUTIONS AND PROPAGATION OF SINGULARITIES FOR HYPERBOLIC OPERATORS WITH INVOLUTIVE CHARACTERISTICS

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We consider the non-characteristic Cauchy problem for a hyperbolic differential operator with real analytic coefficients. It is well-posed in the category of hyperfunctions (J. M. Bony et P. Schapira [2]) and so is in the category of ultradifferentiable functions or ultradistributions of class (s) with $s \leq d/(d-1)$ (of class {s} with $s<d/(d-1)$) where d is the maximum multiplicity of the characteristics (V. Ja. Ivrii [5], J. M. Trepreau [13]).

The aim of this paper is to study the propagation of singularities of the solution to the Cauchy problem for an operator with involutive characteristics. To do this we construct elementary solutions using the method of T. Kawai and G. Nakamura [6] (in [6] the multiplicities were assumed to be at most double). So our results depend essentially on those of T. Kobayashi [7]. We also know by a characterization of ultradistributions as boundary values of holomorphic functions (H. Komatsu [9]) that the elementary solu-