

Integrable systems whose spectral curve is the graph of a function

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Abstract

For some integrable systems, such as the open Toda molecule, the spectral curve of the Lax representation becomes the graph $C = \{(\lambda, z) \mid z = A(\lambda)\}$ of a function $A(\lambda)$. Those integrable systems provide an interesting “toy model” of separation of variables. Examples of this type of integrable systems are presented along with generalizations for which $A(\lambda)$ lives on a cylinder, a torus or a Riemann surface of higher genus.

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1 Introduction

Separation of variables (SOV) is an old and new subject, which was born in the Hamilton-Jacobi theory of the nineteenth century [1], revived by Moser in the light of the modern theory of integrable systems [2], and extended by Sklyanin and his collaborators to a wide range of classical and quantum integrable systems [3]. Moreover, recent studies have revealed a number of new aspects of SOV in the context of bi-Hamiltonian structures [4, 5, 6], universal Hamiltonian structures [7], algebraic surfaces [8, 9], geometric Langlands duality [10], string theory [11], affine Jacobi varieties [12, 13], etc.

This article is a supplement to the joint work with Takebe [14] on SOV of integrable systems. This work was motivated by a rather naive question — what will be the simplest nontrivial examples of SOV of integrable systems? The first plan of this research was to use such systems as a “toy model” for better understanding of Sklyanin’s work on SOV of quantum integrable systems (in particular, Calogero-Moser and Ruijsenaars-Schneider systems [15]). Unfortunately, the program on quantum integrable systems has been unsuccessful due to technical difficulties. However, as we shall see below, searching for simplest examples of SOV is an interesting issue in itself

2 Separation of variables of integrable Hamiltonian systems

Let us consider a Hamiltonian system with canonical variables $q_1, \dots, q_n, p_1, \dots, p_n$ and a Hamiltonian $H = H(q_1, \dots, q_n, p_1, \dots, p_n)$. This system is said to be integrable (or completely integrable) if it has n functionally independent (i.e., $dH_1 \wedge \dots \wedge dH_n \neq 0$) first integrals H_1, \dots, H_n in involution (i.e., $\{H_j, H_k\} = 0$). The joint level set of the first integrals,

$$H_\ell(q_1, \dots, q_n, p_1, \dots, p_n) = E_\ell, \quad \ell = 1, \dots, n, \quad (1)$$

is a Lagrangian submanifold of the $2n$ -dimensional phase space, which (after canonical transformations of coordinates, if necessary) can be expressed as

$$p_j = \partial S / \partial q_j \quad (2)$$

by a function $S = S(q_1, \dots, q_n, E_1, \dots, E_n)$. In the traditional terminology of the Hamilton-Jacobi theory, S is a “complete solution” of the joint Hamilton-Jacobi equations

$$H_\ell(q_1, \dots, q_n, \partial S / \partial q_1, \dots, \partial S / \partial q_n) = E_\ell, \quad \ell = 1, \dots, n. \quad (3)$$

The system can be thereby transferred to an action-angle systems with the action variables E_ℓ and the angle variables $\phi_\ell = \partial S / \partial E_\ell$.

In this setup, SOV means that the equations of the joint level set takes the “separated” form

$$f_j(\lambda_j, \mu_j, E_1, \dots, E_n) = 0, \quad j = 1, \dots, n, \quad (4)$$

in another set of Darboux coordinates $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n$ (“separation variables”). One can then assume the generating function S in the separated form

$$S = \sum_{j=1}^n S_j(\lambda_j, E_1, \dots, E_n) \quad (5)$$

and obtain each term S_j by solving the ODE

$$f_j(\lambda_j, dS_j/d\lambda_j, E_1, \dots, E_n) = 0. \quad (6)$$

Note that the existence of first integrals itself tells nothing about separability; a prescription of SOV has to be sought elsewhere. Sklyanin’s “magic recipe” [3] is a prescription based on the Lax representation of integrable systems. The idea stems from the work of the Montreal group on generalized Moser systems [16], and is also closely related to the algebro-geometric method of the Moscow group [17]. Suppose that the integrable system has a Lax representation of the form

$$\dot{L}(z) = [L(z), M(z)] \quad (7)$$

with $N \times N$ matrices $L(z)$ and $M(z)$ that depends on a spectral parameter z . The characteristic polynomial

$$f(\lambda, z) = \det(\lambda I - L(z)) = \lambda^N + \sum_{\ell=1}^{N-1} p_\ell(z) \lambda^{N-\ell} \quad (8)$$

of $L(z)$ does not depend on the time variable. A set of first integrals in involution can be obtained from the coefficients of expansion of $p_\ell(z)$'s by a suitable basis of functions of z . The curve

$$C = \{(\lambda, z) \mid f(\lambda, z) = 0\} \quad (9)$$

defined by $f(\lambda, z)$ is called the ‘‘spectral curve’’ of the L -matrix $L(z)$. At each point (λ, z) of C is sitting the eigenspace of $L(z)$ with eigenvalue λ . For many integrable systems, this eigenspace is one-dimensional at almost all points of C , and the components of a suitably normalized eigenvector $\phi(\lambda, z)$ are meromorphic functions on C with a finite number of poles. It is the coordinates λ_j, z_j of these poles that play the role of separation variables in Sklyanin’s magic recipe. These variables are constrained by the equation

$$f(\lambda_j, z_j) = 0 \quad (10)$$

of the spectral curve, and these equations can be interpreted as the separated equations (4) of the joint level set of first integrals. Moreover, in most cases, the Poisson brackets of these separation variables take the separated form

$$\{\lambda_j, \lambda_k\} = \{z_j, z_k\} = 0, \quad \{\lambda_j, z_k\} = \delta_{jk} g(\lambda_k, z_k), \quad (11)$$

where $g(\lambda_k, z_k)$ is a common function of λ_k and z_k (typically, $1, \lambda_k, z_k$ or $\lambda_k z_k$). In other words, the symplectic form of the system can be written as

$$\Omega = \sum_{j=1}^N \frac{dz_j \wedge d\lambda_j}{g(\lambda_j, z_j)}. \quad (12)$$

3 If the spectral curve is the graph of a function

Our concern lies in the case where the spectral curve becomes the graph of a function $A(\lambda)$:

$$C = \{(\lambda, z) \mid z = A(\lambda)\}. \quad (13)$$

Here is a list of examples of integrable systems of this type:

1. Open Toda molecule
2. Open Ruijsenaars-Toda molecule
3. Rational and trigonometric Calogero-Moser systems

4. Rational and trigonometric Ruijsenaars-Schneider systems

These integrable systems are usually understood to be associated with no spectral curve, because the usual Lax representation does not contain a spectral parameter. The fact is that they do have a Lax representation with a spectral parameter, and that the spectral curve takes the form mentioned above.

SOV of these examples is almost parallel to more standard cases. The function $A(\lambda)$ is expanded to a linear combination of a suitable basis $f_0(\lambda), f_1(\lambda), \dots, f_N(\lambda)$ of functions as

$$A(\lambda) = f_0(\lambda) + \sum_{\ell=1}^N E_\ell f_\ell(\lambda), \quad (14)$$

and the separation variables comprise N pairs $(\lambda_1, z_1), \dots, (\lambda_N, z_N)$ of variables that satisfy the equations

$$z_j = A(\lambda_j), \quad j = 1, \dots, N. \quad (15)$$

These equations, in turn, determines the first integrals H_ℓ as a function $E_\ell = H_\ell(\lambda_1, \dots, \lambda_N, z_1, \dots, z_N)$ of the separation variables. More precisely, we assume that the non-degeneracy condition

$$\Delta = \det(a_k(\lambda_j) \mid j, k = 1, \dots, N) \neq 0 \quad (16)$$

is satisfied. One can then use Cramer's formula to solve the foregoing equations for the coefficients E_ℓ as

$$E_\ell = \Delta_\ell / \Delta, \quad (17)$$

where Δ_ℓ denotes the determinant in which the ℓ -th row of Δ is replaced as $a_\ell(\lambda_j) \rightarrow z_j - a_0(\lambda_j)$. This gives an explicit formula of the first integrals.

For illustration, we consider the case of the open Toda molecule in detail below.

4 Example: open Toda molecule

4.1 L -matrix and spectral curve

The open Toda molecule of length N has the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{j=2}^N e^{q_j - 1 - q_j}. \quad (18)$$

A Lax representation with a spectral parameter z is presented by Krichever and Vaninsky [18]. The L -matrix reads

$$L(z) = \sum_{j=2}^N c_j E_{j,j-1} + \sum_{j=1}^N b_j E_{j,j} + \sum_{j=2}^N E_{j-1,j} + z E_{N,1}, \quad (19)$$

where $E_{j,k}$ stands for the matrix unit with the (j,k) element equal to 1 and the others equal to 0; b_j and c_j are the Flaschka variables, $b_j = p_j$ and $c_j = c^{q_j-1-q_j}$. Note that $L = L(0)$ is the usual tri-diagonal L -matrix without spectral parameter.

This L -matrix can be derived from the L -matrix $L_{\text{cl}}(z)$ of the closed Toda molecule (namely, the Toda lattice with periodic boundary condition $q_{j+N} = q_j$, $p_{j+N} = p_j$) as follows. The Hamiltonian of the closed Toda molecule is the sum

$$H_{\text{cl}} = H + c_1, \quad c_1 = e^{q_N - q_1} \quad (20)$$

of the Hamiltonian of the open Toda molecule and the potential between q_N and q_1 . The associated L -matrix is also the sum of two pieces:

$$L_{\text{cl}}(z) = L(z) + c_1 z^{-1} E_{1,N}. \quad (21)$$

In the limit as the coupling between q_N and q_1 is turned off ($c_1 \rightarrow 0$), this L -matrix becomes the aforementioned matrix $L(z)$.

This correspondence carries over to the spectral curve as well. The characteristic polynomial of $L_{\text{cl}}(z)$ takes the well known form

$$\det(\lambda I - L_{\text{cl}}(z)) = z + c_1 \cdots c_N z^{-1} - P(\lambda), \quad (22)$$

where $P(\lambda)$ is a polynomial of degree N . The spectral curve defined by this polynomial is a hyperelliptic curve of genus $N - 1$. In the limit as $c_1 \rightarrow 0$, the characteristic polynomial reduces to

$$\det(\lambda I - L(z)) = z - A(\lambda), \quad (23)$$

where $A(\lambda)$ is nothing but the characteristic polynomial of $L = L(0)$:

$$A(\lambda) = \det(\lambda I - L). \quad (24)$$

The spectral curve of the open Toda molecule thus turns out to be the graph of $A(\lambda)$.

4.2 Separation variables

Having the L -matrix $L(z)$, one can obtain separation variables from poles of a suitably normalized eigenvector of $L(z)$. This is indeed achieved in the work of Krichever and Vaninsky [18]. One can derive the same set of separation variables from those of the periodic Toda molecule; all necessary data for the latter can be found in the work of Flaschka and McLaughlin [19]. We here present yet another approach based on Moser's classical method for solving the open Toda molecule [20].

Moser's method uses one of the diagonal elements of the resolvent $(\lambda I - L)^{-1}$:

$$R(\lambda) = {}^t \mathbf{e}_N (\lambda I - L)^{-1} \mathbf{e}_N, \quad {}^t \mathbf{e}_N = (0, \dots, 0, 1). \quad (25)$$

This is a rational function of the form

$$R(\lambda) = B(\lambda)/A(\lambda). \quad (26)$$

Since the numerator $B(\lambda)$ is a polynomial of degree $N-1$, $R(\lambda)$ can be expanded to partial fractions as

$$R(\lambda) = \sum_{j=1}^N \frac{\rho_j}{\lambda - \alpha_j}, \quad A(\lambda) = \prod_{j=1}^N (\lambda - \alpha_j). \quad (27)$$

Moser discovered that the new dynamical variables α_j and ρ_j obey the simple evolution equations

$$\dot{\alpha}_j = 0, \quad \dot{\rho}_j = \alpha_j \rho_j, \quad (28)$$

which can be readily solved as $\alpha_j(t) = \alpha_j(0)$, $\rho_j(t) = e^{\alpha_j(0)t} \rho_j$. The L -matrix can be reproduced from α_j and ρ_j by the Stieltjes method of continued fraction. This is the way how Moser solved the open Toda molecule.

From our point of view, the roots of $B(\lambda)$ are a half of separation variables:

$$B(\lambda) = \prod_{k=1}^{N-1} (\lambda - \lambda_k). \quad (29)$$

The other half z_1, \dots, z_{N-1} are given by the values $z_k = A(\lambda_k)$ of $A(\lambda)$ at the roots of $B(\lambda)$. The Poisson brackets of these variables turn out to take the form

$$\{\lambda_j, \lambda_k\} = \{z_j, z_k\} = 0, \quad \{\lambda_j, z_k\} = \delta_{jk} z_k. \quad (30)$$

One can derive these Poisson brackets in several different ways. The most direct way is to use the result of Faybusovich and Gekhtman [21] on the Poisson brackets of $A(\lambda)$ and $B(\lambda)$.

A comment on the number of variables will be in order. These $2N-2$ separation variables λ_k, z_k amount to $N-1$ of the N degrees of freedom of the open Toda molecule. The rest is the degree of freedom carried by the center of mass $Q = \sum_{j=1}^N q_j/N$ and the total momentum $P = \sum_{j=1}^N p_j$, which disappear in the center-of-mass frame.

4.3 Many faces of symplectic form

We now have at least three different sets of dynamical variables besides the canonical ones q_j, p_j — Moser's variables α_j, ρ_j , the separation variables λ_k, z_k , and the action-angle variables E_ℓ, ϕ_ℓ . Accordingly, the symplectic form Ω of the system has many different expressions. We present their explicit form below (in the center-of-mass frame, for simplicity).

Let us start from the expression

$$\Omega = \sum_{k=1}^{N-1} d \log A(\lambda_k) \wedge d\lambda_k \quad (31)$$

in the separation variables λ_k and $z_k = B(\lambda_k)$. Substituting $A(\lambda_k) = \prod_{j=1}^N (\lambda_k - \alpha_j)$ leads to an intermediate expression

$$\Omega = \sum_{k=1}^{N-1} \sum_{j=1}^N \frac{d\lambda_k \wedge d\alpha_j}{\lambda_k - \alpha_j}, \quad (32)$$

which can be reorganized to the expression

$$\Omega = \sum_{j=1}^N d \log B(\alpha_j) \wedge d\alpha_j \quad (33)$$

in the new variables α_j and $B(\alpha_j)$. As Faybusovich and Gekhtman point out [21] (see also Vaninsky's recent paper[22]), this is exactly the symplectic form that Atiyah and Hitchin [23] introduced on the moduli space of $SU(2)$ monopoles. In terms of Moser's variables α_j and $\rho_j = B(\alpha_j)/A'(\alpha_j)$, the last expression reads

$$\Omega = \sum_{j=1}^N d \log \rho_j \wedge d\alpha_j + \sum_{j \neq k} \frac{d\alpha_j \wedge d\alpha_k}{\alpha_j - \alpha_k}. \quad (34)$$

An expression in action-angle variables can be obtained as follows. Let E_ℓ be the coefficients of $A(\lambda)$:

$$A(\lambda) = \lambda^N + \sum_{\ell=2}^N E_\ell \lambda^{N-\ell}. \quad (35)$$

(E_1 is equal to the total momentum, hence absent in the center-of-mass frame.) The foregoing expression of Ω can now be rewritten as

$$\Omega = \sum_{\ell=2}^N dE_\ell \wedge d\phi_\ell, \quad (36)$$

where

$$\phi_\ell = \sum_{k=1}^{N-1} \int^{\lambda_k} \frac{\lambda^{N-\ell}}{A(\lambda)} d\lambda. \quad (37)$$

This shows that ϕ_ℓ is an angle variable conjugate to the action variable E_ℓ . It is easy to see that generating function S of the action-angle variables, too, has a similar simple expression:

$$S = \sum_{k=1}^{N-1} \int^{\lambda_k} \log A(\lambda) d\lambda. \quad (38)$$

5 Other examples

5.1 Open RT molecule

The open Ruijsenaars-Toda (RT) molecule may be thought of as a limit of the closed RT molecule (namely, the RT lattice with periodic boundary condition $q_{j+N} = q_j$ and $p_{j+N} = p_j$). The passage to the open molecule is parallel to the case of the Toda lattice. The closed molecule has the Hamiltonian

$$H_{\text{cl}} = \sum_{j=1}^N e^{p_j} (1 + e^{q_{j-1} - q_j})^{1/2} (1 + e^{q_j - q_{j+1}})^{1/2}, \quad (39)$$

which turns into the Hamiltonian H of the open molecule as the potential $e^{q_N - q_1}$ between q_1 and q_N is turned off.

SOV of this case is more delicate than the Toda molecules, because curves arising here are NOT a spectral curve in the usual sense. This is already the case for the closed molecule. The closed RT molecule was solved by Bruschi and Ragnisco [24] by an algebro-geometric method, but they had to use an unusual Lax representation with a matrix $L(\lambda, z)$ that depends on two parameters λ and z . Since $L(\lambda, z)$ amounts to $L(z) - \lambda I$ in the usual Lax representation, they considered the curve defined by the equation

$$f(\lambda, z) = \det L(\lambda, z) = 0, \quad (40)$$

and obtained a set of separation variables as Flaschka and McLaughlin [19] did for the closed Toda molecule. Therefore, bearing in mind the case of the Toda molecules, it is not difficult to see what appears in the limit to the open RT molecule.

In the limit to the open molecule, as expected, the hyperelliptic curve of Bruschi and Ragnisco degenerates to the graph of a polynomial $A(\lambda)$ of the form

$$A(\lambda) = \lambda^N + \sum_{\ell=1}^{N-1} E_\ell \lambda^{N-\ell} + 1. \quad (41)$$

Separation variables $\lambda_1, \dots, \lambda_{N-1}$ are the roots of a polynomial $B(\lambda)$ of degree $N - 1$, and their conjugate variables z_1, \dots, z_{N-1} are defined by $z_k = A(\lambda_k)$. The Poisson brackets are slightly different from those of the Toda molecules:

$$\{\lambda_j, \lambda_k\} = \{z_j, z_k\} = 0, \quad \{\lambda_j, z_k\} = \delta_{jk} \lambda_k z_k. \quad (42)$$

The symplectic form Ω of the system, too, is modified as

$$\begin{aligned} \Omega &= \sum_{k=1}^{N-1} d \log A(\lambda_k) \wedge d \log \lambda_k \\ &= \sum_{j=1}^N d \log B(\lambda_j) \wedge d \log \alpha_j \end{aligned}$$

$$= \sum_{\ell=1}^{N-1} dE_{\ell} \wedge d\phi_{\ell}, \quad (43)$$

where α_j 's are the roots of $A(\lambda)$, and ϕ_{ℓ} 's are given by

$$\phi_{\ell} = \sum_{k=1}^{N-1} \int^{\lambda_k} \frac{\lambda^{N-\ell}}{A(\lambda)} d \log \lambda. \quad (44)$$

5.2 Rational and trigonometric CM systems

The rational and trigonometric Calogero-Moser (CM) systems may be thought of as degeneration of the elliptic CM system. This allows us to find a Lax representation with a spectral parameter for these systems.

Let us recall the following L -matrix [25] for the elliptic CM system:

$$L(z) = \sum_{j=1}^N p_j E_{j,j} + \sum_{j \neq k} g_i \phi(q_j - q_k, z) E_{j,k}, \quad (45)$$

where $\phi(u, z)$ is written in terms of the Weierstrass σ function as

$$\phi(u, z) = \frac{\sigma(u+z)}{\sigma(u)\sigma(z)}. \quad (46)$$

As the elliptic curve degenerates to a rational curve, $\phi(u, z)$ turns into a trigonometric (hyperbolic) function

$$\phi(u, z) \rightarrow \frac{\sinh(u+z)}{\sinh(u)\sinh(z)} = \coth(u) + \coth(z)$$

or a rational function

$$\phi(u, z) \rightarrow \frac{u+z}{uz} = \frac{1}{u} + \frac{1}{z}.$$

Remarkably, the limit of $\phi(u, z)$ splits into two functions of a single variable. Accordingly, the limit of $L(z)$ becomes the sum of two matrices as

$$L(z) \rightarrow \begin{cases} L + g_i \coth(z) K & \text{(trigonometric)} \\ L + g_i z^{-1} K & \text{(rational)} \end{cases} \quad (47)$$

where

$$L = \sum_{j=1}^N p_j E_{j,j} + \sum_{j \neq k} g_i \coth(q_j - q_k) E_{j,k} \quad (48)$$

in the trigonometric limit,

$$L = \sum_{j=1}^N p_j E_{j,j} + \sum_{j \neq k} g_i (q_j - q_k)^{-1} E_{j,k} \quad (49)$$

in the rational limit, and

$$K = \sum_{j \neq k} E_{j,k} \quad (50)$$

for both cases. L is nothing but the usual L -matrix of the trigonometric and rational CM systems; K is the constant matrix that appears in the method of Hamiltonian reduction (or the “projection method”) for solving these systems [26, 27]. The linear combination of L and K in the limit (47) of $L(z)$ gives an L -matrix with a spectral parameter for the rational and trigonometric CM systems

We modify these rational and trigonometric L -matrices slightly. Note that K can be expressed as

$$K = \mathbf{e}^t \mathbf{e} - I, \quad {}^t \mathbf{e} = (1, \dots, 1). \quad (51)$$

Therefore the term proportional to K in (47) is a linear combination of a rank-one matrix and a scalar matrix; the latter does not affect to the Lax equation. We can thus drop the scalar term and *redefine* the L -matrix for the degenerate CM systems as

$$L(z) = L + gi \coth(z) \mathbf{e}^t \mathbf{e} \quad (52)$$

for the trigonometric case and

$$L(z) = L + giz^{-1} \mathbf{e}^t \mathbf{e} \quad (53)$$

for the rational case.

We now apply the so called Weinstein-Aronszajn formula

$$\det(M + \mathbf{x}^t \mathbf{y}) = (1 + {}^t \mathbf{y} M^{-1} \mathbf{x}) \det M$$

to the characteristic polynomial of $L(z)$. This yields the expression

$$\det(\lambda I - L(z)) = \begin{cases} P_0(\lambda) - gi \coth(z) P_1(\lambda) & \text{(trigonometric)} \\ P_1(\lambda) - giz^{-1} P_0(\lambda) & \text{(rational)} \end{cases} \quad (54)$$

where

$$P_0(\lambda) = \det(\lambda I - L), \quad P_1(\lambda) = {}^t \mathbf{e}(\lambda \widetilde{M} - L) \mathbf{e}. \quad (55)$$

(\widetilde{M} denote the cofactor matrix of M .) The equation of the spectral curve of $L(z)$ thus turns out to take the form

$$z = \frac{P_1(\lambda)}{P_0(\lambda)} \quad (56)$$

for the rational case and

$$e^{2z} = \frac{P_0(\lambda) + P_1(\lambda)}{P_0(\lambda) - P_1(\lambda)} \quad (57)$$

for the trigonometric form. This is what we have sought.

It should be mentioned that the same equation of the spectral curve for the trigonometric case was also derived by Vaninsky [28] from a different point of view. We can proceed further to the construction of separation coordinates, though we omit details here.

5.3 Rational and trigonometric RS systems

The rational and trigonometric Ruijsenaars-Schneider (RS) systems can be treated in the same way as the CM systems. We again start from the elliptic RS system, and consider the limit as the elliptic curve degenerates to a rational curve. It is convenient to use the following L -matrix of the elliptic RS system (which is gauge equivalent to the L -matrix of Bruschi and Calogero [29]):

$$L(z) = \sum_{j,k=1}^N (h_j h_k)^{1/2} \phi(q_j - q_k + \gamma, z) E_{j,k}, \quad (58)$$

where γ is a constant and h_j 's are fairly complicated functions of the canonical variables. In the limit to the trigonometric or rational models, the L -matrix again splits into the sum of two matrices, one being the L -matrix L without spectral parameter, and the other a rank-one matrix. One can thus use the Weinstein-Aronszajn formula again to show that the spectral curve is the graph of a function $A(\lambda)$. The same curve was derived by Braden and Marshakov [30] in the context of supersymmetric gauge theories.

6 Variants and generalizations

The goal of this research is to search for new integrable systems from the point of view of SOV. Let us show a few results below.

6.1 Trigonometric and elliptic analogues

Apart from the CM and RS systems, the foregoing examples are associated with a pair of polynomials $A(\lambda)$ and $B(\lambda)$. The main result of the joint work with Takebe [14] is to construct the following analogues defined on a cylinder and a torus:

1. Trigonometric version

$$A(\lambda) = \prod_{j=1}^N \sinh(\lambda - \alpha_j), \quad B(\lambda) = \prod_{j=1}^N \sinh(\lambda - \lambda_j). \quad (59)$$

2. Elliptic version

$$A(\lambda) = \prod_{j=1}^N \sigma(\lambda - \alpha_j), \quad B(\lambda) = \prod_{j=1}^N \sigma(\lambda - \lambda_j). \quad (60)$$

The construction can be further extended to the case where $A(\lambda)$ and $B(\lambda)$ have a different number of zeroes.

6.2 Elliptic fibration

Another possible elliptic analogue is to modify the z direction of the (λ, z) space to an elliptic curve. A naive way will be to replace

$$\Omega = \sum_{j=1}^N d \log z_j \wedge d\lambda_j \rightarrow \Omega = \sum_{j=1}^N \frac{dz_j}{y_j} \wedge d\lambda_j \quad (61)$$

where y_j and z_j are constrained by the equation

$$y_j^2 = 4z_j^3 + g_2 z_j + g_3$$

of an elliptic curve. Actually, one can go further by allowing the coefficients g_2 and g_3 to depend on λ_j as

$$y_j^2 = 4z_j^3 + g_2(\lambda_j)z_j + g_3(\lambda_j), \quad (62)$$

where $g_2(\lambda)$ and $g_3(\lambda)$ are functions (e.g., polynomials) of λ . Geometrically, this is the structure called an ‘‘elliptic fibration’’. Integrable systems of this type, too, are considered in the joint work with Takebe [14].

6.3 Higher genera

Generalization to a complex algebraic curve C_0 of genus $g > 1$ (and a Riemann surface of more general type) will be achieved in several different ways. Let us propose an idea below.

The idea is to choose the (A, B) pair as nonzero elements of the vector space $\mathcal{L}(R_1 + \dots + R_N)$ of meromorphic functions $f(P)$ ($P \in C_0$) on C_0 with at most simple poles at N fixed points R_1, \dots, R_N . If $N > 2g - 2$ (or if $N > g$ and R_1, \dots, R_N are in general position), the Riemann-Roch theorem shows that this vector space is spanned by $N - g + 1$ linearly independent elements $f_0(P), f_1(P), \dots, f_{N-g}(P)$. We write $A(P)$ and $B(P)$ by this basis as

$$A(P) = \sum_{\ell=0}^{N-g} u_\ell f_\ell(P), \quad B(P) = \sum_{\ell=0}^{N-g} v_\ell f_\ell(P) \quad (63)$$

and view $(u_\ell)_{\ell=0}^{N-g}$ and $(v_\ell)_{\ell=0}^{N-g}$ as homogeneous coordinates on two copies of the projective space \mathbf{P}^{N-g} . The (A, B) pair is thus parametrized by $2N - 2g$ ‘‘moduli’’, which play the role of dynamical variables. Now let us choose a holomorphic differential $d\lambda$ on C_0 , and consider its (multi-valued) primitive function

$$\lambda(P) = \int^P d\lambda. \quad (64)$$

In this quite general setup, one can use a trick based on the residue theorem [14] to derive the identity

$$\sum_{j=1}^N d \log B(P_j) \wedge d\lambda(P_j) = \sum_{j=1}^N d \log A(Q_j) \wedge d\lambda(Q_j), \quad (65)$$

where P_j and Q_j are zeroes of $A(P)$ and $B(P)$, respectively; d is understood to be the total differential in the $2N - 2g$ moduli of the (A, B) pair. This defines a symplectic form Ω on the moduli space. The inhomogeneous coordinates $H_\ell = u_\ell/u_0$ give a set of functions in involution with respect to Ω , thereby defining an integrable system. The role of separation variables is played by

$$z_j = A(Q_j), \quad \lambda_j = \lambda(Q_j), \quad (66)$$

though they are not independent variables.

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References

- [1] P. Stäckel, Über die Integration der Hamilton-Jacobischen Differentialgleichung mittelst Separation der Variablen (Habilitationsschrift, Univ. Halle-Wittenberg, 1981).
- [2] J. Moser, Geometry of quadrics and spectral theory, in *The Chern Symposium 1979* (Springer-Verlag, 1981), pp. 147–188.
- [3] E.K. Sklyanin, Separation of variables – new trends –, Prog. Theor. Phys. Suppl. **118** (1995), 35–60.
- [4] C. Morosi and G. Tondo, Quasi-bi-Hamiltonian systems and separability, J. Phys. A., Math. Gen., **30** (1997), 2799–2806; On a class of dynamical systems both quasi-bi-Hamiltonian and bi-Hamiltonian, Phys. Lett. **A247** (1998), 59–64.
- [5] M. Błaszak, On separability of bi-Hamiltonian chain with degenerate Poisson structures, J. Math. Phys. **39** (1998), 3213–3235.
- [6] F. Magri G. Falqui and M. Pedroni, The method of Poisson pairs in the theory of nonlinear PDEs, arXiv:nlin.SI/0002009; The bihamiltonian approach

to separation of variables in mechanics, in *Non linearity, Integrability, and all that. Twenty years after NEEDS '79* (World Scientific, Singapore, 2000), pp. 258–266.

- [7] I.M. Krichever and D.H. Phong, On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories, *J. Diff. Geom.* **45** (1998), 349–389; Symplectic forms in the theory of solitons, in *Surveys in Differential Geometry: Integrable Systems*, vol. 4 (International Press, Boston, 1999), pp. 239–313.
- [8] J.C. Hurtubise, Integrable systems and algebraic surfaces, *Duke Math. J.* **83** (1996), 19–50.
- [9] F. Bottacin, Poisson structure on Hilbert schemes of points of a surface and integrable systems, *Manuscripta Math.* **97** (1998), 517–527.
- [10] E. Frenkel, Affine algebras, Langlands duality and Bethe Ansatz, In *XIth International Congress of Mathematical Physics* (International Press, Boston, 1995), pp. 606–642.
- [11] A. Gorsky, N. Nekrasov and V. Rubtsov, Hilbert schemes, separated variables, and D-branes, *Commun. Math. Phys.* **222** (2001), 299–318.
- [12] A. Nakayashiki and F.A. Smirnov, Cohomology of affine Jacobi varieties and integrable systems, arXiv:math-ph/0001017; Euler characteristics of theta divisors of Jacobians for spectral curves, arXiv:math.AG/0012251.
- [13] F.A. Smirnov and V. Zeitlin, Affine Jacobians of spectral curves and integrable models, arXiv:math-ph/0203037.
- [14] K. Takasaki and T. Takebe, An integrable system on the moduli space of rational functions and its variants, arXiv:nlin.SI/0202042.
- [15] V.B. Kuznetsov and E.K. Sklyanin, Separation of variables in the A_2 type Jack polynomials, solv-int/9508002; Separation of variables for A_2 Ruijsenaars model and new integral representation for A_2 Macdonald polynomials, *J. Phys.* **A29** (1996), 2779–2804.
- [16] M.R. Adams, J. Harnad and J. Hurtubise, Darboux coordinates and Liouville-Arnold integration in loop algebras, *Commun. Math. Phys.* **155** (1993), 385–413.
- [17] B.A. Dubrovin, I.M. Krichever and S.P. Novikov, Integrable systems I, in *Encyclopedia of Mathematics, Dynamical System IV* (Springer-Verlag, Berlin, 1990).
- [18] I. Krichever and K. Vaninsky, The periodic and open Toda lattice, arXiv:hep-th/0010184.

- [19] H. Flaschka and D.W. McLaughlin, Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions, *Prog. Theor. Phys.* **55** (1976), 438–456.
- [20] J. Moser, Finitely many mass points on the line under the influence of an exponential potential, in *Dynamical Systems, Theory and Applications*, Lect. Notes. Phys. vol. 38 (Springer-Verlag, Berlin, 1975), pp. 467–497.
- [21] L. Faybusovich, M. Gekhtman, Poisson brackets on rational functions and multi-Hamiltonian structure for integrable lattices, *Phys. Lett. A* **272** (2000), 236–244.
- [22] K.L. Vaninsky, The Atiyah-Hitchin bracket and the open Toda lattice, arXiv:math-ph/0202047.
- [23] M. Atiyah and N. Hitchin, The geometry and dynamics of magnetic monopoles M. B. Porter Lectures. (Princeton University Press, Princeton, 1988).
- [24] O. Ragnisco and M. Bruschi, The periodic relativistic Toda lattice: direct and inverse problem, *Inverse Problems* **5** (1989), 389–405.
- [25] I.M. Krichever, Elliptic solutions of Kadomtsev-Petviashvili equation and integrable systems of particle, *Funct. Anal. Appl.* **14** (1980), 282–290.
- [26] M.A. Olshanetsky and A.M. Perelomov, Explicit solutions of the Calogero models in the classical case and geodesic flows on symmetric spaces of zero curvature, *Lett. Nuovo Cim.* **16** (1976), 333–339; Explicit solutions of some completely integrable systems, *Lett. Nuovo Cim.* **17** (1976), 97–101.
- [27] D. Kazhdan, B. Kostant and S. Sternberg, Hamiltonian group actions and dynamical systems of Calogero type, *Comm. Pure. Appl. Math.* **31** (1978), 481–507.
- [28] K.L. Vaninsky, On explicit parametrisation of spectral curves for Moser-Calogero particles and its applications, *Internat. Math. Res. Notices* **10** (1999), 509–529.
- [29] M. Bruschi and F. Calogero, The Lax representation for an integrable class of relativistic dynamical systems, *Commun. Math. Phys.* **109** (1987), 481–492.
- [30] H.W. Braden and A. Marshakov, Singular Phases of Seiberg-Witten Integrable Systems: Weak and Strong Coupling, *Nucl. Phys.* **B595** (2001), 417–466.