Integrable systems whose spectral curve is the graph of a function

— partly based on collaboration with T. Takebe (nlin/0202042)

I. SOV of integrable systems

II. Spectral curve of the form
$C = \{(\lambda, z) \mid z = A(\lambda)\}$

III. Example: open Toda molecule
(and its relativistic (Ruijsenaars) version)

IV. Some other examples (CM, RS)

V. Variants and generalizations

I. SOV of integrable systems

$\{H_\ell, H_m\} = 0$

$$H_\ell(q_1, \cdots, q_n, \underset{\frac{\partial S}{\partial q_1}}{\underset{\parallel}{p_1}}, \cdots, \underset{\frac{\partial S}{\partial q_n}}{\underset{\parallel}{p_n}}) = E_\ell \quad (\ell = 1 \cdots n)$$

$\downarrow$

canonical transformation

$(\underline{q}, \underline{p}) \mapsto (\underline{\lambda}, \underline{\mu})$

$$S = \sum_{j=1}^{n} S_j(\lambda_j, E_1, \cdots, E_n)$$

$$f_\ell(\lambda_j, \mu_j; E_1, \cdots, E_n) = 0 \quad (\ell = 1, \cdots, n)$$

separated equation

(of level set = Liouville torus)

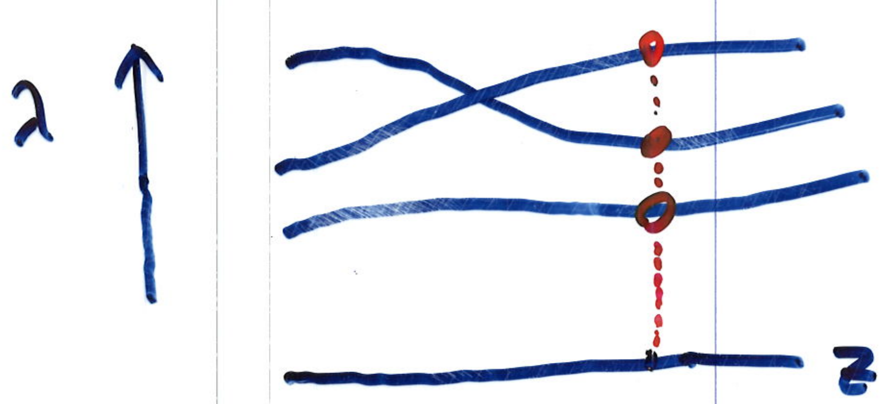
– for integrable system with a Lax representation ($\dot{L}(z) = [L(z), M(z)]$),

$$f_L(\lambda, \mu) = f(\lambda, \mu) = \det(\lambda I - L(z))$$

Spectral curve

$$C : \ f(\lambda, z) = \det(\lambda I - L(z)) = 0$$

r-fold covering (of shere, cylinder, torus, ⋯) if L(z) is an r×r matrix



$L(z) \mapsto (\lambda_j, z_j)$, $j = 1, 2, \ldots$

$\phi = \phi(\lambda, z)$: normalized eigenvector

$$L(z)\phi = \lambda\phi, \quad {}^t n \phi = 1$$

n : normalization vector (optional)

→ ϕ has poles $(\lambda_1, z_1), (\lambda_2, z_2), \ldots$ on the spectral curve C.

$\boxed{(\lambda_j, z_j),\ j = 1, \ldots, g, \text{ are separation variables}}$

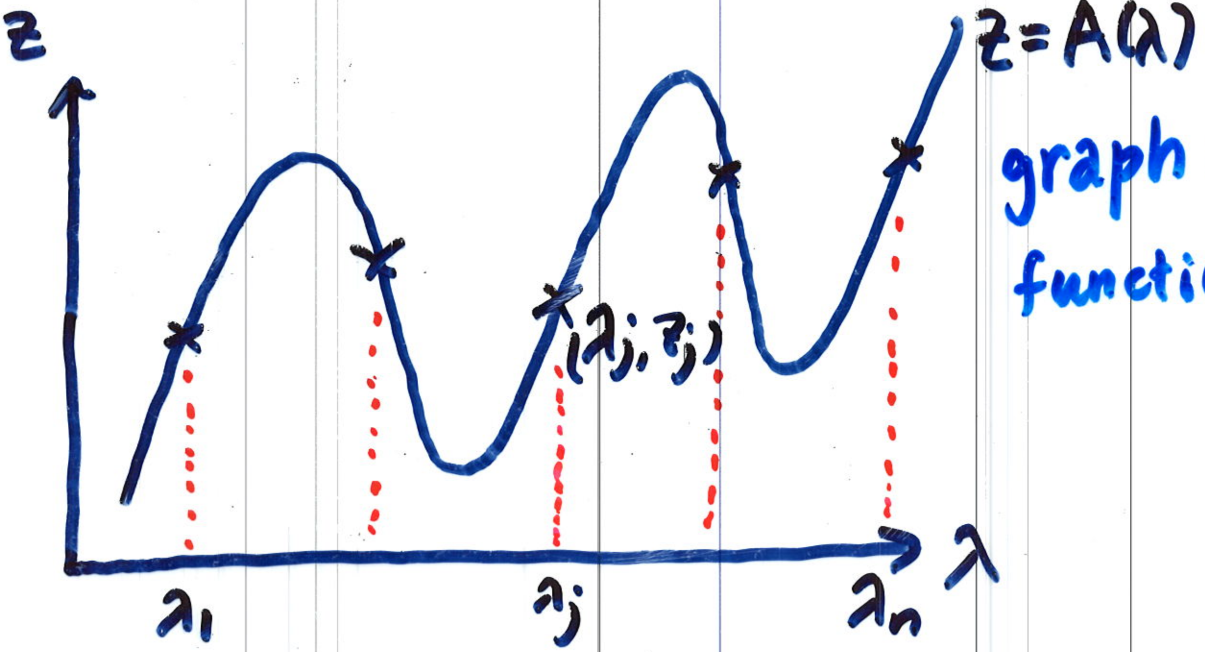
(Adams-Harnad-Hurtubise, …
… , Sklyanin, …)

$$\{\lambda_j, \lambda_k\} = \{z_j, z_k\} = 0,$$

$$\{\lambda_j, z_k\} = \underline{g(\lambda_j, z_j)}\,\delta_{jk}$$

$1, \lambda_j, z_j, \lambda_j z_j$, etc

II. Spectral curve of the form $C = \{(\lambda, z) \mid z = A(\lambda)\}$



graph of function $A(\lambda)$

pair of functions $A(\lambda), B(\lambda)$

$$\begin{cases} B(\lambda) = 0 \text{ at } \lambda = \lambda_j \\ z_j = A(\lambda_j) \end{cases}$$

Symplectic form

$$\Omega = \sum_{j=1}^{n} \frac{dz_j \wedge d\lambda_j}{g(\lambda_j, z_j)}$$

Hamiltonians in involution

parameters (moduli) of $A(\lambda)$

$$A(\lambda) = A(\lambda; E_1, E_2, \cdots)$$

$$z_j = A(\lambda_j; E_1, \cdots, E_n) \quad (j=1,\cdots,n)$$

separated equations

$\Big\downarrow$ solving for E_ℓ's :

$$E_\ell = H_\ell(\lambda_1, \cdots, \lambda_n, z_1, \cdots, z_n)$$

expression of Hamiltonians in separation variables

(cf. Stäckel Hamiltonians)

Geometric interpretation :
(cf. Beauville, Hurtubise)

$$\mathcal{X} = S^{(n)} = S^n / \mathfrak{S}_n \text{ (roughly!)}$$

$\pi \downarrow$ n-fold symmetric product of a surface S with a symplectic form $\omega = \dfrac{dz \wedge dx}{g(\lambda, z)}$

$\mathcal{U}$

$$\pi((\lambda_j, z_j)_{j=1}^n) = (H_1, \cdots, H_n)$$

Lagrangian fibration (or foliation)

— This gives an interesting
TOY MODEL of SOV.

Examples

- open Toda molecule
  our main example
- its relativistic version
  (Ruijsenaars-Toda molecule)
- rational/trigonometric
  Calogero-Moser systems
- rational/trigonometric
  Ruijsenaars-Schneider systems
- ...

Curiosity

- What do they look like
  in this picture?
- Possible generalizations?

III. Example : open Toda molecule

Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N-1} e^{q_j - q_{j+1}}$$

L-matrix (with spectral parameter)

$$L(z) = \begin{pmatrix} b_1 & 1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ z & & c_{N-1} & b_N \end{pmatrix}$$

$b_j = p_j, \quad c_j = e^{q_j - q_{j+1}}$ (Krichever-Vaninsky)

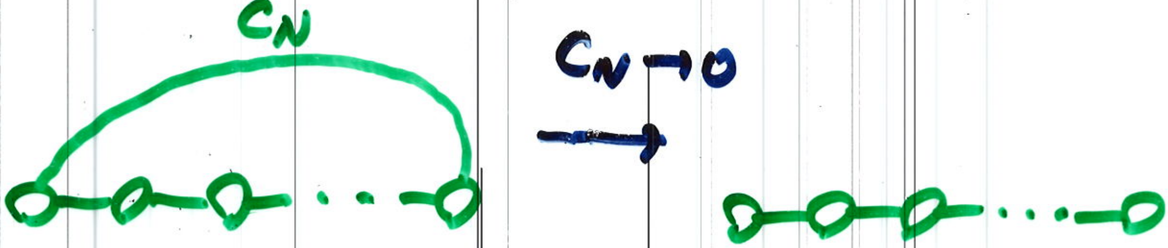
Why?

closed Toda molecule

$$H = \frac{1}{2}\sum p_j^2 + \sum_{j=1}^{N-1} e^{q_j - q_{j+1}} + \underbrace{e^{q_N - q_1}}_{= c_N}$$

$$L(z) = \begin{pmatrix} \ddots & \ddots & & c_N z^{-1} \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ z & & c_{N-1} & b_N \end{pmatrix}$$

turning off the coupling of q_N and q_1:



fate of spectral curve:

$$\det(\lambda I - \underset{\text{closed}}{L(z)}) = z + c_1 \cdots \underline{c_N}\, z^{-1} - (A(\lambda) + \underbrace{D(\lambda)}_{= c_N \cdot (\cdots)})$$

$$\Big\downarrow \; c_N \to 0$$

$$\det(\lambda I - \underset{\text{open}}{L}(z)) = z - A(\lambda)$$

where

$$A(\lambda) = \det\Bigg(\lambda I - \underbrace{\begin{pmatrix} b_1 & 1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & c_{N-1} & b_N \end{pmatrix}}_{\substack{\| \\ L(0) = L:}}\Bigg)$$

L-matrix without spectral parameter

$\therefore$ sectral curve is graph of $A(\lambda)$:

$$z = A(\lambda)$$

Moser's work

$$e_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$ {}^t e_N (\lambda I - L)^{-1} e_N = \frac{B(\lambda)}{A(\lambda)} = \sum_{j=1}^{N} \frac{\rho_j}{\lambda - \alpha_j}$$

where $B(\lambda) = \det\left(\lambda I - \underbrace{\begin{pmatrix} b_1 & 1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & c_{N-2} & b_{N-1} \end{pmatrix}}_{(N-1)\times(N-1)}\right)$

Moser's observation :

$$\boxed{\dot{\alpha}_j = 0, \quad \dot{\rho}_j = \alpha_j \rho_j}$$

$\left(\begin{array}{l}\text{higher flows :} \quad \dot{\alpha}_j = 0, \ \dot{\rho}_j = \alpha_j^{\ell} \rho_j \\ \quad \ell = 2, 3, \cdots \end{array}\right)$

flows are linearized in (α_j, ρ_j)

$$\begin{cases} \alpha_j(t) = \alpha_j(0), \\ \rho_j(t) = e^{\alpha_j(0) t} \rho_j(0) \end{cases}$$

$(\alpha_j, \rho_j) \mapsto L$: continued fraction

– This appears to imply that α_j and $\log \rho_j$ are Darboux coordinates (i.e. canonically conjugate). This is wrong.

A correct answer is that α_j and $\log B(\alpha_j)$ are Darboux coordinates : $\rho_j = \frac{B(\alpha_j)}{A'(\alpha_j)}$

$$\Omega = \sum_{j=1}^{N} d\log B(\alpha_j) \wedge d\alpha_j \quad (*)$$

(Faybusovich & Gekhtman)

(*) Atiyah-Hitchin symplectic (or Poisson) structure on the moduli space of rational functions $\frac{B(\lambda)}{A(\lambda)} = \sum_{j=1}^{N} \frac{\rho_j}{\lambda - \alpha_j}$

Three faces of Ω

$$A(\lambda) = \lambda^N + \sum_{\ell=1}^{N} u_\ell \lambda^{N-\ell} = \prod_{j=1}^{N} (\lambda - \alpha_j)$$

$$B(\lambda) = \rho \prod_{j=1}^{N-1} (\lambda - \lambda_k), \quad \rho = \sum_{j=1}^{N} \rho_j = 1$$

$$u_1 \sim P = \sum p_j, \quad u_2 \sim H$$

① $$\Omega = \sum_{j=1}^{N} d\log B(\alpha_j) \wedge d\alpha_j$$

$$\{\alpha_j, \alpha_k\} = 0 \quad (\Rightarrow \{u_\ell, u_m\} = 0)$$

② $$\Omega = \sum_{k=1}^{N-1} d\log A(\lambda_k) \wedge d\lambda_k \left(+\underbrace{d\log \rho}_{=0} \wedge du_1\right)$$

$$= \sum_{k=1}^{N-1} d\log z_k \wedge d\lambda_k \qquad u_1 : \text{Casimir}$$

$$\{\lambda_j, \lambda_k\} = \{z_j, z_k\} = 0,$$
$$\{\lambda_j, z_k\} = \delta_{jk} z_k$$

③ $$\Omega = \sum_{\ell=2}^{N} du_\ell \wedge d\phi_\ell, \quad \text{where}$$

$$\phi_\ell = \sum_{k=1}^{N} \int^{\lambda_k} \frac{\lambda^{N-\ell} d\lambda}{A(\lambda)} \qquad \text{(angle variables)}$$

Remark For the closed Toda molecule,

$$\phi_\ell = \sum_{k=1}^{N} \int^{\lambda_k} \frac{\lambda^{N-\ell}\, d\lambda}{\sqrt{P(\lambda)^2 - 4\underline{\underline{c}}}}$$

↑

holomorphic differentials on the spectral curve

$$z^2 - P(\lambda) z + \underline{\underline{c}} = 0$$

$(P(\lambda) = A(\lambda) + D(\lambda),\ c = c_1 \cdots c_N)$

Letting $c \to 0$ the previous results can be reproduced.

cf Krichever & Vaninsky: Baker-Akhiezer function on the singular rational curve

$$z^2 - A(\lambda) z = 0$$

open Ruijsenaars - Toda molecule

$$H = \sum_{j=1}^{N} e^{p_j} \left(1 + e^{q_j - q_{j+1}}\right)^{1/2} \left(1 + e^{q_j - q_{j-1}}\right)^{1/2}$$

sectral curve : $z = A(\lambda) = \prod_{j=1}^{N} (\lambda - \alpha_j)$

$$A(\lambda) = \lambda^N + u_1 \lambda^{N-1} + \cdots + u_{N-1} \lambda + 1$$

(or Casimir)

$$B(\lambda) = \prod_{k=1}^{N} (\lambda - \lambda_k)$$

symplectic form :

① $$\Omega = \sum_{j=1}^{N} d \log B(\alpha_j) \wedge d \log \alpha_j$$

modified Atiyah-Hitchin str.

② $$\Omega = \sum_{k=1}^{N-1} d \log A(\lambda_k) \wedge d \log \lambda_k$$

$$\omega = d \log z \wedge d \log \lambda$$

on the (λ, z) space

③ $$\Omega = \sum_{\ell=1}^{N-1} du_\ell \wedge d\phi_\ell ,$$

$$\phi_\ell = \sum_{k=1}^{N-1} \int^{\lambda_k} \frac{\lambda^{N-\ell}}{A(\lambda)} \, d \log \lambda$$

IV. Some other examples

* Calogero-Moser systems (A_{N-1} type) *

L-matrix for elliptic model :

$$L(z) = \sum_j p_j E_{jj} + \sum_{j\neq k} gi\,\phi(q_j - q_k, z) E_{jk},$$

$$\phi(u,z) = \frac{\sigma(u+z)}{\sigma(u)\sigma(z)} .$$

rational / hyperbolic limit :

$$\phi(u,z) = \frac{\sinh(u+z)}{\sinh(u)\sinh(z)}$$

$$= \frac{\cosh(u)}{\sinh(u)} + \frac{\sinh(z)}{\sinh(z)} .$$

$$\phi(u,z) = \frac{u+z}{u\,z}$$

$$= \frac{1}{u} + \frac{1}{z} .$$

factorization

$$\Rightarrow \quad L(z) = L + \frac{gi}{\tanh z} \sum_{j\neq k} E_{jk}$$

$$L(z) = L + \frac{gi}{z} \sum_{j\neq k} E_{jk}$$

rational case

$$L(z) = L + \frac{g_i}{z}\left(\mathbb{e}\,{}^t\mathbb{e} - I\right),$$

$$L = \sum_j p_j E_{jj} + \sum_{j \neq k} \frac{g_i}{q_j - q_k} E_{jk},$$

$$\mathbb{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

↖ usual L-matrix (Moser)

$$\dot{L}(z) = [L(z), M] = [L, M]$$

"$-I$" is irrelevant.

modify $L(z)$ as

$$L(z) = L + \underbrace{\frac{g_i}{z}\,\mathbb{e}\,{}^t\mathbb{e}}_{\text{rank } 1}$$

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cf. open Toda molecule

$$L(z) = L + \underbrace{z \begin{pmatrix} 0 & & \\ \vdots & & 0 \\ 0 & & \\ 1 & 0 \cdots & 0 \end{pmatrix}}_{\text{rank } 1}$$

characteristic polynomial of $L(z)$:

$$\det(\lambda I - L(z))$$
$$= \det(\lambda I - L - \frac{gi}{z} e\, {}^t e)$$
$$= \det(\lambda I - L) \det(I - \frac{gi}{z}(\lambda I - L)^{-1} e\, {}^t e)$$

formula: $\det(I - XY) = \det(I - YX)$

$$= \det(\lambda I - L)(1 - \frac{gi}{z}\, {}^t e (\lambda I - L)^{-1} e)$$
$$= P_0(\lambda) - \frac{gi}{z} P_1(\lambda),$$

where
$$P_0(\lambda) = \det(\lambda I - L),$$
$$P_1(\lambda) = \det(\lambda I - L)\, {}^t e (\lambda I - L)^{-1} e$$
$$= {}^t e (\lambda I - L)^{\sim} e$$

$\therefore$ spectral curve takes the form

$$z = gi \frac{P_1(\lambda)}{P_0(\lambda)}$$

Remark In fact, $P_1(\lambda) = P_0'(\lambda)$,

so that $$z = gi \sum_{j=1}^{N} \frac{1}{\lambda - \alpha_j}$$ (α_j: roots of $P_0(\lambda)$)

Similar calculations can be done for rational RS and rational CM/RS:

Results :

- rational CM/RS

$$z = \frac{P_1(\lambda)}{P_0(\lambda)} \quad (P_0(\lambda), P_1(\lambda) : \text{polynom})$$

- hyperbolic CM/RS

$$\tanh z = \frac{P_1(\lambda)}{P_0(\lambda)} \quad (\quad " \quad)$$

$$\left(\leftrightarrow \ e^{2z} = \frac{P_0(\lambda) + P_1(\lambda)}{P_0(\lambda) - P_1(\lambda)} \right)$$

<u>Remark</u> 1. The equations of the curves for the hyperbolic CM/RS systems are already known (Vaninski ; Braden & Marshakov).

2. "Three faces of Ω" become more exotic in this case.

V. Variants/generalizations

1° trigonometric/elliptic analogues

(1) $A(\lambda) = \prod_j \sinh(\lambda - \alpha_j)$, $B(\lambda) = \prod_k \sinh(\lambda - \lambda_k)$

(2) $A(\lambda) = \prod_j \sigma(\lambda - \alpha_j)$, $B(\lambda) = \prod_k \sigma(\lambda - \lambda_k)$

2° elliptic fibration

$$\omega = \frac{dz \wedge d\lambda}{z} \longrightarrow \omega = \frac{dz \wedge d\lambda}{\sqrt{z^3 + g_2(\lambda) z + g_3(\lambda)}}$$

S : ambient surface with symplectic form (K3, etc)

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3° higher genus? open problem

$A(\lambda)$ and $B(\lambda)$ on a Riemann surface of genus > 1

4° quantization?? open problem

$$z_j = A(\lambda_j) \longrightarrow \frac{\partial}{\partial \lambda_j} \psi_j = A(\lambda_j) \psi_j \ ?$$

quantum correction?? $\quad e^{\partial/\partial\lambda_j} \psi_j = A(\lambda_j) \psi_j$?

(cf. Sklyanin et al ... Babelon & Talon)

Summary

- Examples of integrable systems with spectral curve of the form $C = \{(\lambda, z) \mid z = A(\lambda)\}$ are presented.
- For open Toda and Ruijsenaars-Toda molecules, $A(\lambda)$ is a polynomial. The symplectic form Ω has three different "faces".
- For rational and hyperbolic CM/RS systems, $A(\lambda)$ is a rational function. Faces of Ω are more exotic.
- These results show a new interpretation of previously known integrable systems, and, hopefully, lead to new integrable systems (in connection with a non-zero-genus Riemann surface). A few new examples have been obtained.