Dispersionless Hirota equations and reduction of universal Whitham hierarchy

(joint work with Takashi Takebe)

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K.T. and T. Takebe, arXiv:0808.1441 [nlin.SI]

1. Introduction

Finite-variable reductions of dispersionless integrable hierarchies

Example: dispersionless KP (dKP) hierarchy

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\} = \frac{\partial \mathcal{B}_n}{\partial p}\frac{\partial \mathcal{L}}{\partial x} - \frac{\partial \mathcal{B}_n}{\partial x}\frac{\partial \mathcal{L}}{\partial p}, \ \mathcal{L} = p + \sum_{j=2}^{\infty} u_j p^{-j+1},$$

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}$$

$\downarrow$

$\downarrow$ $\mathcal{L} = \mathcal{L}(p, \boldsymbol{\lambda})$, $\boldsymbol{\lambda} = (\lambda_1(\boldsymbol{t}), \ldots, \lambda_M(\boldsymbol{t}))$ (M-variable ansatz)

$\downarrow$

chordal Löwner equations $\frac{\partial \mathcal{L}}{\partial \lambda_j} = \frac{1}{p - U_j(\boldsymbol{\lambda})}\frac{\partial \mathcal{L}}{\partial p}\frac{\partial a(\boldsymbol{\lambda})}{\partial \lambda_j}$

$+$

diagonal hydrodynamic equations $\frac{\partial \lambda_j}{\partial t_n} = \chi_{nj}(\boldsymbol{\lambda})\frac{\partial \lambda_j}{\partial x}$

Remarks:

- The Löwner equations describe reduction of degrees of freedom (kinematical part). $U_j(\boldsymbol{\lambda})$ and $a(\boldsymbol{\lambda})(= u_2)$ are functional data (compatibility conditions are required if $M > 1$).

- The hydrodynamic equations describe dynamics of the reduced dynamical variables $\boldsymbol{\lambda}$ (dynamical part). The characteristic speeds $\chi_{nj}(\boldsymbol{\lambda})$ are determined by the system as $\chi_{nj}(\boldsymbol{\lambda}) = \frac{\partial \mathcal{B}_n}{\partial p}\Big|_{p=U_j(\boldsymbol{\lambda})}$.

- The Löwner equations are understood in a generalized (or loose) sense, not necessarily assuming the existence of a family of slit domains.

- Derivation of the Löwner equations in the literature is not fully deductive.

Traditional approach

first developed for reductions of Benney hierarchy ($\subset$ reductions of dKP hierarchy)

— Gibbons & Tsarev, Phys. Lett. **A211** (1996), 19–24; Phys. Lett. **A258** (1999), 263–271
— Yu & Gibbons, Inverse Problems **16** (2000), 605–618

further progress along the same lines

— Baldwin & Gibbons, J. Phys. A: Math. Gen. **36** (2003), 8393–8417; J. Phys. A: Math. Gen. **27** (2004), 5341–5354
— Mañas, Martíinez Alonso & Medina, J. Phys. **A35** (2002), 401–417
— Guil, Mañas & Martínez Alonso, J. Phys. **A36** (2003), 4047–4062
— Mañas, J. Phys. **A37** (2004), 11191–11221

Perspective in Hirota equations (cf. Takebe's talk)

Takebe, Teo & Zabrodin, J. Phys. A: Math. Gen. **39** (2006), 11479–11501 (dKP and dToda hierarchies)

• Direct (deductive) derivation of Löwner equations and hydrodynamic equations via dispersionless Hirota equations. Successful only for one-variable reductions (by technical reasons).

• Proof of converse (Löwner eqs + hydrodynamic equations $\Rightarrow$ dispersionless Hirota equations). Applicable to multi-variable reductions as well.

• New tools: generating functions of Faber polynomials and Grunsky coefficients (borrowed from complex analysis).

Goal of this talk

Generalization of results/methods of Takebe-Teo-Zabrodin to universal Whitham (UW) hierarchy of genus zero with arbitrary number of marked points

Remark: UW hierarchy is a master equation of many other dispersionless integrable systems.

- dKP = UW with one marked point (fixed at ∞)
- dToda = UW with two marked points
- UW with $N+1$ marked points is a quasi-classical limit of the $N+1$-component KP hierarchy (T & Takebe, Physica **D235** (2007), 109–125)

Motivations:

- Reductions of the UW hierarchy by Löwner-type equations have been studied by Guil, Mañas and Martínez Alonso (J. Phys. **A36** (2003), 4047–4062) from a different (rather traditional) approach.

- The UW hierarchy has several $\mathcal{L}$'s. What do the "Löwner equations" look like in this case? (An answer is known for dToda that has two $\mathcal{L}$'s.)

- Generalizations to nonzero genera (in particular, genus one) ?

2. Universal Whitham hierarchy (of genus zero)

Krichever, Comm. Pure. Appl. Math. **47** (1994), 437–475

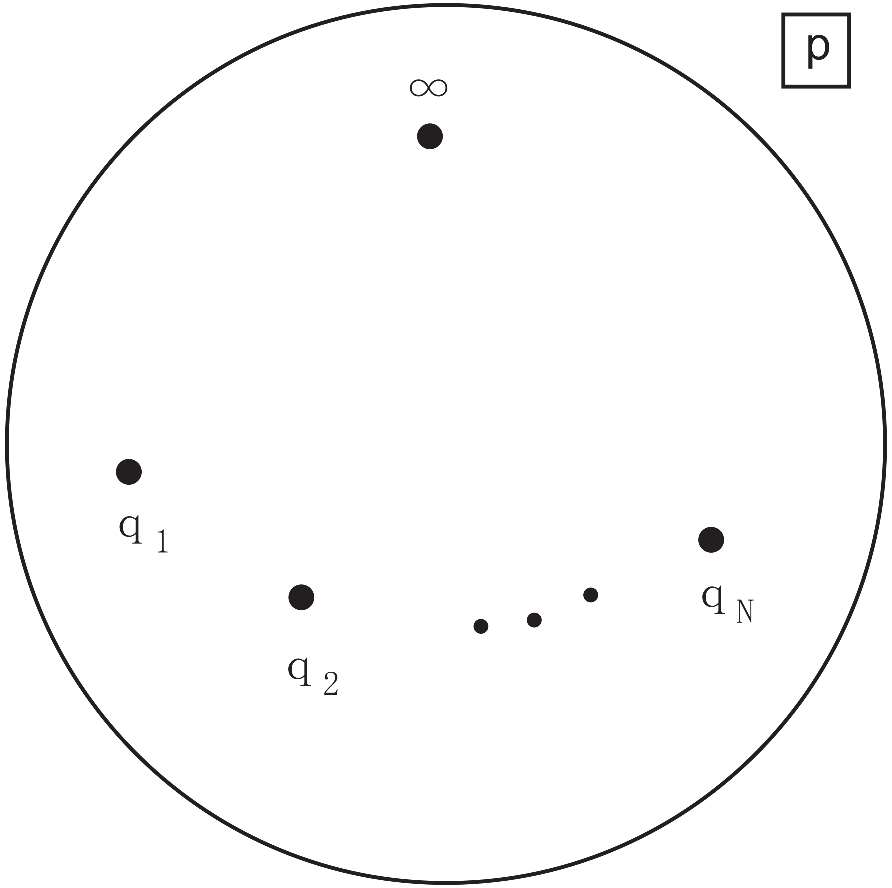
Dynamical variables

marked points ∞,q_1, ..., q_N

Laurent seires $z_0(p)$,$z_1(p)$, ..., $z_N(p)$ ($z_0(p) \sim \mathcal{L} = \mathcal{L}(p)$)

$$\begin{aligned} z_0(p) &= p + \sum_{j=2}^{\infty} u_{0j} p^{-j+1}, \\ z_\alpha(p) &= \frac{r_\alpha}{p - q_\alpha} + \sum_{j=1}^{\infty} u_{\alpha j} (p - q_\alpha)^{j-1} \end{aligned}$$

(might be formal Laurent series)


p
∞
q_1
q_2
q_N

Spacetime variables

$$\boldsymbol{t} = (\boldsymbol{t}_0, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_N), \quad \boldsymbol{t}_0 = (t_{0n})_{n=1}^{\infty}, \quad \boldsymbol{t}_\alpha = (t_{\alpha n})_{n=0}^{\infty}\ (\alpha = 1, \ldots, N)$$

$t_{01} \sim x$ (spatial variable of dKP hierarchy)

$t_{\alpha 0} \sim s$ (spatial variable of dToda hierarchy)

UW hierarchy contains $N+1$ copies of dKP hierarchy and N copies of dToda hierarchy.

Lax equations

$$\frac{\partial z_\beta(p)}{\partial t_{\alpha n}} = \{\Omega_{\alpha n}(p), z_\beta(p)\} = \frac{\partial \Omega_{\alpha n}(p)}{\partial p}\frac{\partial z_\beta(p)}{\partial t_{01}} - \frac{\partial \Omega_{\alpha n}(p)}{\partial t_{01}}\frac{\partial z_\beta(p)}{\partial p}$$

$$\text{for } \alpha, \beta = 0, 1, \ldots, N,\ n = \begin{cases} 1, 2, \ldots & (\beta = 0) \\ 0, 1, \ldots & (\beta \neq 0) \end{cases}$$

$\Omega_{0n}(p), \Omega_{\alpha n}(p)$ for $n \neq 0$

They are polynomials in p and $(p - q_\alpha)^{-1}$ of the form

$$\Omega_{0n}(p) = p^n + a_{0n2}p^{n-2} + \cdots + a_{0nn},$$

$$\Omega_{\alpha n}(p) = \frac{a_{\alpha n0}}{(p - q_\alpha)^n} + \frac{a_{\alpha n1}}{(p - q_\alpha)^{n-1}} + \cdots + \frac{a_{\alpha nn-1}}{(p - q_\alpha)},$$

and given by the singular part of Laurent expansion of $z_0(p)^n$ and $z_\alpha(p)^n$ (including the constant term for the former):

$$z_0(p)^n = \Omega_{0n}(p) + O(p^{-1}) \quad (p \to \infty),$$

$$z_\alpha(p)^n = \Omega_{\alpha n}(p) + O(1) \quad (p \to q_\alpha).$$

The first few of them read

$$\Omega_{01}(p) = p, \quad \Omega_{02}(p) = p^2 + 2u_{02}, \quad \ldots,$$

$$\Omega_{\alpha 1}(p) = \frac{r_\alpha}{p - q_\alpha}, \quad \Omega_{\alpha 2}(p) = \frac{r_\alpha^2}{(p - q_\alpha)^2} + \frac{2r_\alpha u_{\alpha 1}}{p - q_\alpha}, \quad \ldots.$$

$\Omega_{10}(p), \ldots, \Omega_{N0}(p)$

They are exceptional and given by logarithmic functions

$$\Omega_{\alpha 0}(p) = -\log(p - q_\alpha).$$

Generating functions (as Farber polynomials)

$$\log \frac{p_0(z) - q}{z} = -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Omega_{0n}(q),$$

$$\log \frac{q - p_\alpha(z)}{q - q_\alpha} = -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Omega_{\alpha n}(q).$$

where $p_0(z)$ and $p_\alpha(z)$ are inverse functions of $z = z_0(p)$ and $z = z_\alpha(p)$:

$$\begin{aligned} p_0(z) &= z - u_{02} z^{-1} + O(z^{-2}), \\ p_\alpha(z) &= q_\alpha + r_\alpha z^{-1} + O(z^{-2}). \end{aligned}$$

Rational reduction (algebraic orbit)

$$z_0(p) = E(p)^{1/k_0} \qquad \text{(Laurent expansion at } p = \infty\text{)}$$

$$z_\alpha(p) = E(p)^{1/k_\alpha} \qquad \text{(Laurent expansion at } p = q_\alpha\text{)}$$

where $E(p)$ is a rational function of the form

$$E(p) = p^{k_0} + \sum_{n=2}^{k_0} a_{0n} p^{k_0-n} + \sum_{\alpha=1}^{N} \sum_{n=1}^{k_\alpha} \frac{a_{\alpha n}}{(p-q_\alpha)^n} + \text{some other rational terms}$$

The Lax equations for the z-functions are reduced to the Lax equations

$$\frac{\partial E(p)}{\partial t_{\alpha n}} = \{\Omega_{\alpha n}(p), E(p)\}$$

for $E(p)$.

Remark: This is a special case of finite variable reductions.

Riemann invariants are given by $\lambda_j = E(U_j)$, $E'(U_j) = 0$.

3. Dispersionless Hirota equations

Passage to $\mathcal{F}$-function ($\mathcal{F} = \log \tau_{\text{quasi-classical}}$)

$z_0(p), \ldots, z_N(p)$ + Lax equations

$\downarrow$

Orlov-Schulman functions $\zeta_0(p), \ldots, \zeta_N(p)$ + extended Lax eqs

$\downarrow$

S-functions $S_0(z), \ldots, S_N(z)$ + Hamilton-Jacobi equations

$\downarrow$

$\mathcal{F}$-function $\mathcal{F}$ + dispersionless Hirota equations

(T & Takebe, Physica **D235** (2007), 109–125).

Dispersionless Hirota equations

$$e^{\hat{D}_0(z)\hat{D}_0(w)\mathcal{F}} = 1 - \frac{\partial_{01}(\hat{D}_0(z) - \hat{D}_0(w))\mathcal{F}}{z - w},$$

$$ze^{\hat{D}_0(z)\hat{D}_\alpha(w)\mathcal{F}} = z - \partial_{01}(\hat{D}_0(z) - \hat{D}_\alpha(w))\mathcal{F},$$

$$e^{\hat{D}_\alpha(z)\hat{D}_\alpha(w)\mathcal{F}} = -\frac{zw\partial_{01}(\hat{D}_\alpha(z) - \hat{D}_\alpha(w))\mathcal{F}}{z - w},$$

$$\epsilon_{\alpha\beta}e^{\hat{D}_\alpha(z)\hat{D}_\beta(w)\mathcal{F}} = -\partial_{01}(\hat{D}_\alpha(z) - \hat{D}_\beta(w))\mathcal{F} \quad (\alpha \neq \beta)$$

$$(\alpha, \beta = 1, \ldots, N)$$

where $\epsilon_{\alpha\beta}$ is a signature factor and

$$\hat{D}_0(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\partial_{0n} \quad (\partial_{0n} = \frac{\partial}{\partial t_{0n}}),$$

$$\hat{D}_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\partial_{\alpha n} + \partial_{\alpha 0} \quad (\partial_{\alpha n} = \frac{\partial}{\partial t_{\alpha n}})$$

Remark: This system is equivalent to the UW hierarchy itself.

Some technical formulae

- To express $p_0(z)$ and $p_\alpha(z)$ with $\mathcal{F}$-function:

$$p_0(z) = \partial_{01} S_0(z) = z - \partial_{01} \hat{D}_0(z)\mathcal{F},$$
$$p_\alpha(z) = \partial_{01} S_\alpha(x) = -\partial_{01} \hat{D}_\alpha(z)\mathcal{F}$$

- To express u_{02}, q_α and r_α with $\mathcal{F}$-function:

$$u_{02} = \partial_{01}^2 \mathcal{F}, \quad q_\alpha = -\partial_{01}\partial_{\alpha 0}\mathcal{F},$$
$$r_\alpha = -\partial_{01}\partial_{\alpha 1}\mathcal{F} = -e^{\partial_{\alpha 0}^2 \mathcal{F}}$$

Remark: $u_{02} \sim$ KdV/KP potential, $\partial_{\alpha 0}\mathcal{F} \sim$ Toda fields. The Boyer-Finley equation (dispersionless 2D Toda equation) shows up in the last line of the formulae above.

Second derivatives of $\mathcal{F}$ as Grunsky coefficients

- The dispersionless Hirota equations can be rewritten as

$$\hat{D}_0(z)\hat{D}_0(w)\mathcal{F} = \log \frac{p_0(z) - p_0(w)}{z - w},$$

$$\hat{D}_0(z)\hat{D}_0(w)\mathcal{F} = \log \frac{p_0(z) - p_\alpha(w)}{z},$$

$$\hat{D}_\alpha(z)\hat{D}_\alpha(w)\mathcal{F} = \log \frac{zw(p_\alpha(z) - p_\alpha(w))}{z - w},$$

$$\hat{D}_\alpha(z)\hat{D}_\beta(w)\mathcal{F} = \log \frac{p_\alpha(z) - p_\beta(w)}{\epsilon_{\alpha\beta}} \quad (\alpha \neq \beta)$$

$$(\alpha, \beta = 1, \ldots, N)$$

The right hand side are generating functions of generalized Grunsky coefficients $b_{\alpha m \beta n}$ of $p_0(z), p_1(z), \ldots, p_N(z)$.

• Generating functions of the generalized Grunsky coefficients $b_{\alpha m\beta n}$:

$$\log \frac{p_0(z) - p_0(w)}{z - w} = -\sum_{m,n=1}^{\infty} z^{-m} w^{-n} b_{0m0n},$$

$$\log \frac{p_0(z) - p_\alpha(w)}{z} = -\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} z^{-m} w^{-n} b_{0m\alpha n},$$

$$\log \frac{zw(p_\alpha(z) - p_\alpha(w))}{z - w} = -\sum_{m,n=0}^{\infty} z^{-m} w^{-n} b_{\alpha m\alpha n},$$

$$\log \frac{p_\alpha(z) - p_\beta(w)}{\epsilon_{\alpha\beta}} = -\sum_{m,n=0}^{\infty} z^{-m} w^{-n} b_{\alpha m\beta n} \quad (\alpha \neq \beta).$$

• Consequently, the second derivatives of $\mathcal{F}$ coincide with the Grunsky coefficients (up to numerical factors):

$$\hat{\partial}_{\alpha m}\hat{\partial}_{\beta n}\mathcal{F} = -b_{\alpha m \beta n} \quad (\alpha, \beta = 0, 1, \ldots, N),$$

where

$$\hat{\partial}_{\alpha n} = \begin{cases} \frac{1}{n}\partial_{\alpha n} & (n \neq 0), \\ \partial_{\alpha 0} & (n = 0). \end{cases}$$

- Conversely, one can define the $\mathcal{F}$-function by these equations

$$\hat{\partial}_{\alpha m}\hat{\partial}_{\beta n}\mathcal{F} = -b_{\alpha m \beta n} \quad (\alpha, \beta = 0, 1, \dots, N).$$

In that case, $b_{\alpha m \beta n}$ have to satisfy the integrability conditions

$$\hat{\partial}_{\gamma n} b_{\alpha l \beta m} = \hat{\partial}_{\alpha l} b_{\gamma n \beta m}.$$

This is a clue of the second half (Löwner equations + hydrodynamic equations $\Longrightarrow$ dispersionless Hirota equations) of the Takebe-Teo-Zabrodin method.

4. One-variable reduction

One-variable ansatz

Suppose that all fundamental dynamical variables of the UW hierarchy depend on $\boldsymbol{t}$ through a single reduced variable $\lambda = \lambda(\boldsymbol{t})$ as

$$u_{\alpha n} = u_{\alpha n}(\lambda(\boldsymbol{t})), \quad q_\alpha = q_\alpha(\lambda(\boldsymbol{t})).$$

Consequently, $z_\alpha(p)$'s are functions of p and $\lambda(\boldsymbol{t})$:

$$z_\alpha(p) = z_\alpha(p, \lambda(\boldsymbol{t})).$$

Results

Theorem 1 If $u_{\alpha n}$ and q_α are functions of a single variable $\lambda = \lambda(\boldsymbol{t})$, then there is a function $U = U(\lambda)$ of λ such that $z_\alpha(p) = z_\alpha(p, \lambda)$, $\alpha = 0, 1, \ldots, N$, satisfy the Löwner-type equations

$$\frac{\partial z_\alpha(p)}{\partial \lambda} = \frac{1}{p - U} \frac{\partial z_\alpha(p)}{\partial p} \frac{\partial u_{02}}{\partial \lambda}.$$

Moreover, $\lambda = \lambda(\boldsymbol{t})$ satisfies the hydrodynamic equations

$$\partial_{\alpha n} \lambda = \chi_{\alpha n}(\lambda) \partial_{01} \lambda$$

with characteristic speeds

$$\chi_{\alpha n}(\lambda) = \left. \frac{\partial \Omega_{\alpha n}(p)}{\partial p} \right|_{p = U(\lambda)}.$$

Remarks:

• As a consequence of the Löwner-type equations, $z_\alpha(p)$ are mutually functionally dependent. Namely, there are functions $f_\alpha(z)$ of a single variable such that

$$z_\alpha(p) = f_\alpha(z_0(p)), \quad \alpha = 1, \ldots, N$$

• The Löwner-type equations for $z_\alpha(p)$ are equivalent to the dual equations

$$\frac{\partial p_\alpha(z)}{\partial \lambda} = \frac{1}{U - p_\alpha(z)} \frac{\partial u_{02}}{\partial \lambda}$$

for the inverse functions $p_\alpha(z)$. It is these equations that are primarily derived in the proof of the theorem.

• The proof is done by straightforward calculations (cf. Takebe's talk). It seems difficult to generalize this proof to multi-variable reductions.

Theorem 2 Let $F(\lambda)$ be an arbitrary function of λ, and $\lambda = \lambda(\boldsymbol{t})$ a function that satisfies the hodograph equation

$$\sum_{n=1}^{\infty} t_{0n}\chi_{0n}(\lambda) + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} t_{\alpha n}\chi_{\alpha n}(\lambda) = F(\lambda).$$

Further assume that the regularity condition

$$\sum_{n=1}^{\infty} t_{0n}\frac{\partial\chi_{0n}(\lambda)}{\partial\lambda} + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} t_{\alpha n}\frac{\partial\chi_{\alpha n}(\lambda)}{\partial\lambda} \neq \frac{\partial F(\lambda)}{\partial\lambda}$$

holds for $\lambda = \lambda(\boldsymbol{t})$. Then $\lambda = \lambda(\boldsymbol{t})$ satisfies the hydrodynamic equations

$$\partial_{\alpha n}\lambda = \chi_{\alpha n}(\lambda)\partial_{01}\lambda.$$

Theorem 3 If $z_\alpha(p) = z_\alpha(p, \lambda)$, $\alpha = 0, 1, \ldots, N$, satisfy the Löwer-type equations and $\lambda = \lambda(\boldsymbol{t})$ satisfy the hydrodynamic equations, then the integrability conditions for the equations

$$\hat{\partial}_{\alpha m}\hat{\partial}_{\beta n}\mathcal{F} = -b_{\alpha m\beta n} \quad (\alpha, \beta = 0, 1, \ldots, N).$$

are satisfied. The $\mathcal{F}$-function $\mathcal{F} = \mathcal{F}(\boldsymbol{t})$ thus defined satisfies the dispersionless Hirota equations.

4. Concluding remarks

• All Löwner-type equations for $z_\alpha(p)$'s turn out to have the same "driving force" U. This is in accord with the result of Takebe-Teo-Zabrodin for the case of the dToda hierarchy.

This turns out to be the case for M-variable reductions as well. The Löwner-type equations therein read

$$\frac{\partial z_\alpha(p)}{\partial \lambda_j} = \frac{1}{p - U_j} \frac{\partial z_\alpha(p)}{\partial p} \frac{\partial u_{02}}{\partial \lambda_j}, \quad j = 1, \ldots, M.$$

• Generalization to the universal Whitham hierarchies of nonzero genera is still an open problem (even for the case of genus one).