Toda tau functions with quantum torus symmetries

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References

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1. Quantum torus algebra

1.1 Matrix realization

$\mathbf{Z} \times \mathbf{Z}$ matrices

$$\Lambda = \sum_{n \in \mathbf{Z}} E_{n-1,n} = \begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \end{pmatrix},$$

$$\Delta = \sum_{n \in \mathbf{Z}} n E_{nn} = \begin{pmatrix} \ddots & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}.$$

1.1 Matrix realization (cont'd)

Basis

$$v_m^{(k)} = q^{-km/2}\Lambda^m q^{k\Delta} = q^{-km/2}\sum_{n\in\mathbf{Z}} q^{kn}E_{n-m,n} \quad (k,m\in\mathbf{Z},\ |q|<1)$$

Commutation relations

$$[v_m^{(k)},\, v_n^{(l)}] = (q^{(lm-kn)/2} - q^{(kn-lm)/2})v_{m+n}^{(k+l)}$$

Remark: "Quantization" of classical torus algebra (Poisson brackets of functions on 2D torus)

1.2 Fermionic realization

Fermion creation/annihilation operators ψ_i, ψ_i^* $(i \in \mathbf{Z})$:

anti-commutation relations

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i+j,0}, \quad \psi_i \psi_j + \psi_j \psi_i = \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0$$

Vacuum states $\langle 0|$, $|0\rangle$

$$\psi_i|0\rangle = 0 \ (i \geq 0), \quad \psi_i^*|0\rangle = 0 \ (i \geq 1),$$
$$\langle 0|\psi_i = 0 \ (i \leq -1), \quad \langle 0|\psi_i^* = 0 \ (i \leq 0)$$

Excited states $\langle \lambda, s|$, $|\lambda, s\rangle$ labelled by partitions
$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n, 0, \cdots) \in \mathcal{P}$ and an integer (charge) $s \in \mathbf{Z}$

$$|\lambda, s\rangle = \psi_{-(s+\lambda_1-1)-1} \cdots \psi_{-(s+\lambda_n-n)-1} \psi^*_{-(s-n)+1} \cdots \psi^*_{-(s-1)+1} |s\rangle$$

1.2 Fermionic realization (cont'd)

Fermion bilinears (spanning $\widehat{\mathrm{gl}}(\infty)$ algebra)

$$E_{ij} \;\leftrightarrow\; :\psi_{-i}\psi_j^*: = \psi_{-i}\psi_j^* - \langle 0|\psi_{-i}\psi_j^*|0\rangle$$

$$v_m^{(k)} \;\leftrightarrow\; V_m^{(k)} = q^{-km/2}\sum_{n\in\mathbf{Z}} q^{kn}:\psi_{m-n}\psi_n^*:$$

$$= q^{k/2}\oint\frac{dz}{2\pi i}z^m:\psi(q^{k/2}z)\psi^*(q^{-k/2}z):$$

where

$$\psi(z) = \sum_{i\in\mathbf{Z}}\psi_i z^{-i-1}, \quad \psi^*(z) = \sum_{i\in\mathbf{Z}}\psi_i^* z^{-i}$$

1.2 Fermionic realization (cont'd)

Commutation relations

$$[V_m^{(k)}, V_n^{(l)}]$$

$$= (q^{(lm-kn)/2} - q^{(kn-lm)/2}) V_{m+n}^{(k+l)} - \frac{q^{(k+l)m/2} - q^{-(k+l)m/2}}{1-q^{k+l}} \delta_{m+n,0} q^{k+l}$$

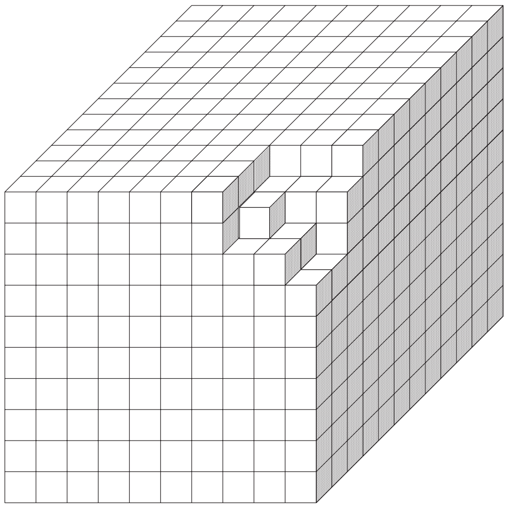
$$= (q^{-k(m+n)/2} - q^{k(m+n)/2}) V_{m+n}^{(0)} + m\delta_{m+n,0} \quad \text{if } k+l=0$$

$V_m^{(0)}$'s span a $\widehat{U(1)}$ (or Heisenberg) subalgebra:

$$V_m^{(0)} = J_m = \sum_{n \in \mathbf{Z}} {:}\psi_{m-n}\psi_n^*{:}, \quad [J_m, J_n] = m\delta_{m+n,0}$$

2. Melting crystal model

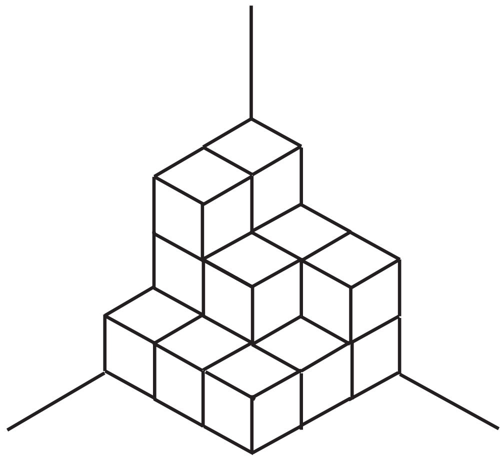
2.1 Melting crystal corner



melting crystal corner

complement

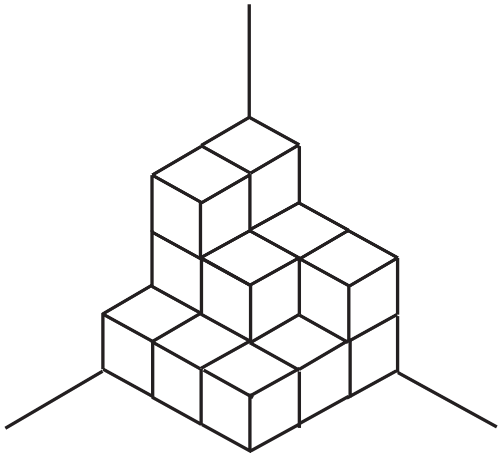
$\longleftrightarrow$



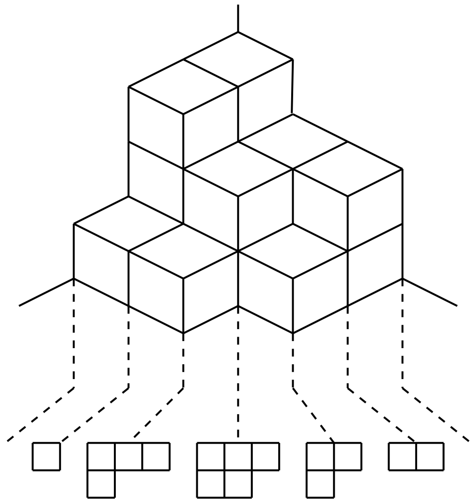
3D Young diagram

Ref: Okounkov, Reshetikhin & Vafa, “Quantum Calabi-Yau and classical crystals”

2.2 Plane partitions and diagonal slicing



plane partition π
(3D Young diagram)



diagonal slices $\pi(m)$ $(m \in \mathbf{Z})$
main diagonal slice $\lambda = \pi(0)$
(Young diagrams)

Generating function $Z = \sum_{\pi \in \mathcal{PP}} q^{|\pi|} = \prod_{n=1}^{\infty} (1 - q^n)^{-n}$ (MacMahon)

2.3 Generating operators of plane partitions

$$G_{\pm} = \exp\left(\sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1-q^k)} J_{\pm k}\right) \quad \text{(Okounkov \& Reshetikhin)}$$

generate a linear combination of (orthonormal) excited states weighted by special values $s_\lambda(q^\rho)$, $q^\rho = (q^{1/2}, q^{3/2}, q^{5/2}, \cdots)$, of Schur functions:

$$\langle 0|G_+ = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^\rho)\langle \lambda|, \quad G_-|0\rangle = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^\rho)|\lambda\rangle.$$

The scalar product gives the generating function of plane partitions:

$$\langle 0|G_+G_-|0\rangle = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^\rho)^2 = \prod_{n=1}^{\infty}(1-q^n)^{-n} = \sum_{\pi \in \mathcal{PP}} q^{|\pi|}$$

2.4 Deformed melting crystal model

$$Z(s, \boldsymbol{t}) = \langle s | G_+ e^{H(t)} G_- | s \rangle = \sum_{\lambda \in \mathcal{P}} e^{\Phi_\lambda(s,t)} s_\lambda(q^\rho)^2,$$

where s and $\boldsymbol{t} = (t_1, t_2, \cdots)$ are deformation variables, and

$$H(\boldsymbol{t}) = \sum_{k=1}^{\infty} t_k H_k, \quad H_k = V_0^{(k)} = \sum_{n \in \mathbf{Z}} q^{kn} {:}\psi_{-n}\psi_n^*{:}$$

$$\Phi_\lambda(s, \boldsymbol{t}) = \sum_{k=1}^{\infty} t_k \Phi_{\lambda,k}(s),$$

$$\Phi_{\lambda,k}(s) = \sum_{i=1}^{\infty} (q^{k(s+\lambda_i-i+1)} - q^{k(s-i+1)}) + q^k \frac{1-q^{sk}}{1-q^k},$$

$\langle s| = \langle \emptyset, s|, \quad |s\rangle = |\emptyset, s\rangle$ (ground states in charge-s sector)

3. Toda tau function from melting crystal model

3.1 Fermionic formula of tau function of 2D Toda hierarchy

$$\tau(s,\boldsymbol{T},\bar{\boldsymbol{T}}) = \langle s|\exp\left(\sum_{k=1}^{\infty} T_k J_k\right) g \exp\left(-\sum_{k=1}^{\infty} \bar{T}_k J_{-k}\right)|s\rangle$$
$$= \sum_{\lambda,\mu\in\mathcal{P}} \langle \lambda, s|g|\mu, s\rangle s_\lambda[\boldsymbol{T}] s_\mu[-\bar{\boldsymbol{T}}]$$

where $\boldsymbol{T} = (T_1, T_2, \cdots)$ and $\bar{\boldsymbol{T}} = (\bar{T}_1, \bar{T}_2, \cdots)$ are time variables of the 2D Toda hierarchy, $s_\lambda[\boldsymbol{T}]$ and $s_\mu[-\boldsymbol{T}]$ are the associated values of Schur functions (given by the Jacobi-Trudi formula) and g is an element of $\widehat{\mathrm{GL}}(\infty) \sim \exp\big(\widehat{\mathrm{gl}}(\infty)\big)$. $\langle \lambda, s|g|\mu, s\rangle$'s are semi-infinite determinants.

3.2 Partition function of deformed melting crystal model

Theorem The partition function

$$Z(s,\boldsymbol{t}) = \langle s|G_+ e^{H(\boldsymbol{t})} G_-|s\rangle$$

of the deformed melting crystal model coincides, up to simple functions, with the Toda tau function $\tau(s,\boldsymbol{T},\bar{\boldsymbol{T}})$ for the $\widehat{\mathrm{GL}}(\infty)$ element

$$g = q^{W_0/2}(G_-G_+)^2 q^{W_0/2}$$

where

$$W_0 = \sum_{n\in\mathbf{Z}} n^2{:}\psi_{-n}\psi_n^*{:} \quad \text{(zero-mode of } W^{(3)} \text{ algebra)}$$

3.3 Technical clue of proof: shift symmetries

First symmetry (shifting m)

$$G_-G_+\left(V_m^{(k)} - \delta_{m,0}\frac{q^k}{1-q^k}\right)(G_-G_+)^{-1} = (-1)^k\left(V_{m+k}^{(k)} - \delta_{m+k,0}\frac{q^k}{1-q^k}\right)$$

Second symmetry (shifting k)

$$q^{W_0/2}V_m^{(k)}q^{-W_0/2} = V_m^{(k-m)}$$

Remarks: 1. The second symmetry is a consequence of straightforward calculations. The first symmetry is far more technical.

2. W_0 is a fermionic realization of the cut-and-join operator.

3.4 To derive Toda tau function

Start from

$$Z(s,\boldsymbol{t}) = \langle s|G_+ e^{H(\boldsymbol{t})} G_-|s\rangle = \langle s|G_- G_+ e^{H(\boldsymbol{t})} G_- G_+|s\rangle$$

Step 1 Rewrite $G_- G_+ e^{H(\boldsymbol{t})}$, $H(\boldsymbol{t}) = \sum_{k=1}^{\infty} t_k V_0^{(k)}$, as

$$G_- G_+ e^{H(\boldsymbol{t})} = \exp\left(\sum_{k=1}^{\infty} t_k G_- G_+ V_0^{(k)} (G_- G_+)^{-1}\right) G_- G_+,$$

apply the shift symmetries as

$$\begin{aligned} G_- G_+ V_0^{(k)} (G_- G_+)^{-1} &= (-1)^k \left(\frac{q^k}{1-q^k} + V_k^{(k)}\right) \\ &= (-1)^k \left(\frac{q^k}{1-q^k} + q^{-W_0/2} V_k^{(0)} q^{W_0/2}\right), \end{aligned}$$

and note that $V_k^{(0)} = J_k$.

3.4 To derive Toda tau function (cont'd)

Step 2 Consequently,

$$G_-G_+e^{H(\boldsymbol{t})} = \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k q^k t_k}{1-q^k}\right) \times q^{-W_0/2}\exp\left(\sum_{k=1}^{\infty}(-1)^k t_k J_k\right) q^{W_0/2} G_-G_+$$

Substitution in $Z(s,\boldsymbol{t})$ yields

$$Z(s,\boldsymbol{t}) = \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k q^k t_k}{1-q^k}\right) \times \langle s|q^{-W_0/2}\exp\left(\sum_{k=1}^{\infty}(-1)^k t_k J_k\right) q^{W_0/2}(G_-G_+)^2|s\rangle$$

3.4 To derive Toda tau function (cont'd)

Step 3 Since

$$\langle s|q^{-W_0/2} = q^{-s(s+1)(2s+1)/12}\langle s|, \quad |s\rangle = q^{W_0/2}|s\rangle q^{-s(s+1)(2s+1)/12},$$

the last expression of $Z(s, \boldsymbol{t})$ can further be rewritten as

$$\begin{aligned} Z(s, \boldsymbol{t}) &= \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k q^k t_k}{1 - q^k}\right) q^{-s(s+1)(2s+1)/6} \\ &\quad \times \langle s| \exp\left(\sum_{k=1}^{\infty} (-1)^k t_k J_k\right) q^{W_0/2}(G_- G_+)^2 q^{W_0/2}|s\rangle \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k q^k t_k}{1 - q^k}\right) q^{-s(s+1)(2s+1)/6} \tau(s, \boldsymbol{T}, 0) \end{aligned}$$

where $T_k = (-1)^k t_k$. Thus the Toda tau function emerges.

3.4 To derive Toda tau function (cont'd)

Another expression Start from

$$Z(s,\boldsymbol{t}) = \langle s|G_-G_+e^{H(\boldsymbol{t})}G_-G_+|s\rangle$$

and rewrite $e^{H(\boldsymbol{t})}G_-G_+$ in the same way. This lead to

$$\begin{aligned} Z(s,\boldsymbol{t}) &= \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k q^k t_k}{1-q^k}\right) q^{-s(s+1)(2s+1)/6} \\ &\quad \times \langle s|q^{W_0/2}(G_-G_+)^2 q^{W_0/2}\exp\left(-\sum_{k=1}^{\infty}(-1)^k t_k J_{-k}\right)|s\rangle \\ &= \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k q^k t_k}{1-q^k}\right) q^{-s(s+1)(2s+1)/6}\tau(s,0,-\boldsymbol{T}) \end{aligned}$$

Many expressions $\longrightarrow$ Hidden symmetries

4. Quantum torus symmetries of tau function

4.1 Reduction to 1D Toda tau function

Theorem g satisfies the intertwining relations

$$J_k g = g J_{-k} \quad (k = 1, 2, \ldots).$$

Corollary The tau function satisfies the constraints

$$\left(\frac{\partial}{\partial T_k} + \frac{\partial}{\partial \bar{T}_k}\right) \tau(\boldsymbol{T}, \bar{\boldsymbol{T}}, s) = 0 \quad (k = 1, 2, \ldots)$$

and thereby becomes a function of $\boldsymbol{T} - \bar{\boldsymbol{T}}$:

$$\tau(\boldsymbol{T}, \bar{\boldsymbol{T}}, s) = \tau(\boldsymbol{T} - \bar{\boldsymbol{T}}, s).$$

The reduced tau function $\tau(\boldsymbol{T}, s)$ is a tau function of the 1D Toda hierarchy.

4.1 Reduction to 1D Toda tau function (cont'd)

Proof of intertwining relations: Use the shift symmetries repeatedly as

$$
\begin{aligned}
J_k g &= V_k^{(0)} q^{W_0/2} (G_- G_+)^2 q^{W_0/2} \\
&= q^{W_0/2} V_k^{(k)} (G_- G_+)(G_- G_+) q^{W_0/2} \\
&= q^{W_0/2} G_- G_+ (-1)^k (V_0^{(k)} - \frac{q^k}{1-q^k}) G_- G_+ q^{W_0/2} \\
&= q^{W_0/2} G_- G_+ G_- G_+ V_{-k}^{(k)} q^{W_0/2} \\
&= q^{W_0/2} G_- G_+ G_- G_+ q^{W_0/2} V_{-k}^{(0)} \\
&= g J_{-k}
\end{aligned}
$$

4.2 More symmetries

Theorem g satisfies more general intertwining relations

$$(V_m^{(k)} - \delta_{m,0}\frac{q^k}{1-q^k})g = g(V_{-2k-m}^{(-k)} - \delta_{2k+m,0}\frac{q^{-k}}{1-q^{-k}})$$

for $k = 1, 2, \cdots$, $m \in \mathbf{Z}$.

Theorem The Lax and Orlov-Schulman operators $L, M, \bar{L}, \bar{M}$ satisfy the constraints (generalized string equations)

$$L = \bar{L}^{-1}, \quad q^M = q^{-1}\bar{L}^{-2}q^{-\bar{M}}.$$

Remark: The first equation implies that

$$L = \bar{L}^{-1} = a(s)e^{\partial_s} + b(s) + a(s-1)e^{-\partial_s}$$

4.3 Classical (= thermodynamic) limit

The difference operators

$$L = ae^{\partial_s} + b + u_2 e^{-\partial_s} + \cdots , \quad \text{etc.}$$

are replaced by functions

$$\mathcal{L} = ap + b + u_2 p^{-1} + \cdots , \quad \text{etc.}$$

that satisfy the generalized string equations

$$\mathcal{L} = \bar{\mathcal{L}}^{-1}, \quad e^{-R\mathcal{M}} = \bar{\mathcal{L}}^{-2} e^{R\bar{\mathcal{M}}}$$

alongside the equations of the dispersionless Toda hierarchy. (R is a constant in the quasi-classical parametrization $q = e^{-R\hbar}$.)