Landau-Lifshitz equation, elliptic AKNS hierarchy and Sato Grassmannian

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Lax equation (zero-curvature eq) } on algebraic curve

$$\left.\begin{array}{l}\partial_t A(P) = [B(P), A(P)] \\ [\partial_x - A(P), \partial_t - B(P)] = 0\end{array}\right\} \quad (r \times r)$$

$P \in \Gamma$: algebraic curve

$A(P), B(P)$: meromorphic on Γ

fixed poles $P_1, \cdots, P_K$

order $m_1, \cdots, m_K$

$g = \text{genus}(\Gamma)$

$g = 0 \longrightarrow$ # equation = # variables

$g > 0 \longrightarrow$ overdetermined if A and B take a general form

(Zakharov - Mikhailov)

Krichever (2002) —

- add extra "movable" poles $\gamma_1, \cdots, \gamma_{rg} \in \Gamma$ to A and B
- special structure of A, B at these poles

  $\rightarrow$ extra parameters

  $\alpha_1, \cdots, \alpha_{rg} \in \mathbb{P}^{r-1}$

γ_s, α_s $(s = 1, \cdots, rg)$: Tyurin parameters

$\boxed{\text{Krichever's result:}}$

Introducing these new parameters as dynamical variables leads to a consistent Lax/zero-curvature equation.

Examples by Krichever ($g=1$)

- elliptic Calogero-Moser system

  ($\gamma_1, \cdots, \gamma_r$ = coordinates of particles)

- its "field analogue"

  ($\gamma_s = \gamma_s(x,t), \cdots$)

(cf. "Krichever-Novikov equation"

— "rank 2" solution of KP eqn)

This talk

- yet another example ($g \geqq 1$)

  AKNS + Tyurin parameters

- interpretation in terms of Sato Grassmannian

- similar interpretation of Landau-Lifshitz eqn

Ordinary AKNS hierarchy

$$[\partial_x - A(\lambda),\ \partial_{t_n} - A_n(\lambda)] = 0 \quad (n = 2, 3, \cdots)$$

$$A(\lambda) = \begin{pmatrix} \lambda & u \\ v & -\lambda \end{pmatrix} = J\lambda + A^{(1)} \quad (\lambda \in \mathbb{P}^1)$$

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ A^{(1)} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

(homogeneous grading)

- $A_n(\lambda)$'s are described by a single generating function $U(\lambda) = I + \sum_{n=1}^{\infty} U_n \lambda^{-n}$ as

$$A_n(\lambda) = U_0 \lambda^n + U_1 \lambda^{n-1} + \cdots + U_n$$

- $U(\lambda)$ is determined by

$$\begin{cases} [\partial_x - A(\lambda),\ U(\lambda)] = 0 \\ U(\lambda)^2 = I \end{cases}$$

uniquely. ($\to$ recursion relation of U_n's)

- AKNS hierarchy can be formulated as a system of Lax equations:

$$[\partial_{t_n} - A_n(\lambda), U(\lambda)] = 0$$

$$\left(\longrightarrow \begin{array}{l} \partial_{t_n} u = f_n(u, u_x, \cdots, v, v_x, \cdots) \\ \partial_{t_n} v = g_n(u, u_x, \cdots, v, v_x, \cdots) \end{array} \right)$$

- zero curvature equations

$$[\partial_{t_m} - A_m(\lambda), \partial_{t_n} - A_n(\lambda)] = 0$$

are satisfied for any solution of this hierarchy (commutativity of flows)

- nonlinear Schrödinger equation in (x, t_2)

A-matrix on elliptic curve

$$\Gamma = \mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z}).$$

$A(z)$ $(z \in \Gamma)$: 2×2 matrix of meromorphic functions on Γ with the following properties —

1. $A(z)$ has poles at $z = 0, \gamma_1, \gamma_2$ and holomorphic elsewhere

2. At $z = 0$,
$$A(z) = J z^{-1} + A^{(1)} + O(z)$$

3. At $z = \gamma_s$ $(s = 1, 2)$
$$A(z) = \frac{\beta_s\, {}^t\alpha_s}{z - \gamma_s} + O(1)$$
where α_s and β_s are column vectors
$${}^t\alpha_s = (\alpha_s \;\; 1)$$

$\gamma_1, \gamma_2, \alpha_1, \alpha_2$ — Tyurin parameters

<u>Lemma</u> $A(z)$ is uniquely determined by these conditions, and can be written explicitly as:

$$A(z) = \sum_{s=1,2} \beta_s {}^t\alpha_s \left(\zeta(z-\gamma_s) + \zeta(\gamma_s) \right) + \begin{pmatrix} \zeta(z) & u \\ v & -\zeta(z) \end{pmatrix}.$$

$$\beta_1 = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} -1 \\ -\alpha_2 \end{pmatrix}, \quad \beta_2 = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix}.$$

<u>Lemma</u> Solutions of the auxiliary linear problem $(\partial_x - A(z))\psi(z) = 0$ is regular if the following equations are satisfied:

$$\partial_x \gamma_s + \mathrm{Tr}\, \beta_s {}^t\alpha_s = 0,$$

$$\partial_x {}^t\alpha_s + {}^t\alpha_s A^{(s,1)} = \kappa_s {}^t\alpha_s$$

$$\left(A(z) = \frac{\beta_s {}^t\alpha_s}{z-\gamma_s} + \underset{\sim}{A^{(s,1)}} + O(z-\gamma_s) \right)$$

- More explicitly,

$$\partial_x \gamma_1 = \frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2},$$

$$\partial_x \gamma_2 = -\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}.$$

$$\partial_x \alpha_1 = -2\alpha_1 \zeta_{12} - v - \alpha_1^2 u,$$

$$\partial_x \alpha_2 = 2\alpha_2 \zeta_{12} - v - \alpha_2^2 u.$$

$$(\quad \zeta_{12} := \zeta(\gamma_1) - \zeta(\gamma_2) - \zeta(\gamma_1 - \gamma_2))$$

Remark : These conditions are analogous to those of apparent singularities in ODE's.

cf. $\frac{d^2 y}{dz^2} + p(z) y = 0$ with 4 generic regular singular points and 1 apparent singular point $\to$ Painlevé VI

Construction of hierarchy

- generating function $U(z)$

$$U(z) = I + \sum_{n=1}^{\infty} U_n z^{-n},$$

$$[\partial_x - A(z), U(z)] = 0,$$

$$U(z)^2 = I$$

$\longrightarrow$ recursion relations

$$2JU_{n+1} = \partial_x U_n - \sum_{m=1}^{n+1} [A^{(m)}, U_{n+1-m}] - \sum_{m=1}^{n} U_m U_{n+1-m} ,$$

where

$$A(z) = Jz^{-1} + \sum_{m=1}^{\infty} A^{(m)} z^{m-1}$$

(Laurent expansion at $z=0$)

U_n's are "local" in u, v, γ_s, α_s $(s=1,2)$
(differential polynomials)

- generators of time evolutions

$A_n(z)$ $(z \in \Gamma)$: 2×2 matrix of meromorphic functions on Γ with the following properties:

1. poles at $z = 0, \gamma_1, \gamma_2$ and holomorphic elsewhere.

2. At $z = 0$,

$$A_n(z) = \underline{U(z)}\, z^{-n} + O(z)$$

3. At $z = \underline{\gamma_s}$ $(s = 1, 2)$,

$$A_n(z) = \frac{\beta_{n,s}\, {}^t\alpha_s}{z - \gamma_s} + O(1)$$

$\left({}^t\alpha_s = (\underline{\alpha_s}, 1) \quad \cdots\cdots \text{ common to all } A_n\text{'s and } A \right)$

— uniquely determined

— explicit formula in terms of zeta function

- evolution equations for Tyurin param.

$$\partial_{t_n}\gamma_s + \mathrm{Tr}\, \beta_{n,s}{}^t\alpha_s = 0,$$

$$\partial_{t_n}{}^t\alpha_s + {}^t\alpha_s A_n^{(s,1)} = \kappa_{n,s}{}^t\alpha_s .$$

where $A_n^{(s,1)}$ is the constant term of the Laurent expansion of $A_n(z)$:

$$A_n(z) = \frac{\beta_{n,s}{}^t\alpha_s}{z-\gamma_s} + A_n^{(s,1)} + O(z-\gamma_s)$$

(— ensures regularity of solutions of $\partial_{t_n}\psi(z) = A_n(z)\psi(z)$ at $z=\gamma_s$)

- elliptic AKNS hierarchy, by definition, consists of these equs for γ_s, α_s and the Lax equations

$$[\partial_{t_n} - A_n(z), U(z)] = 0$$

Lowest equations in (x, t_2)

$$\partial_{t_2} u = \frac{1}{2} u_{xx} - u\Big(uv + \wp(\gamma_1) + \wp(\gamma_2) - 2\,\frac{\alpha_2 \wp(\gamma_1) - \alpha_1 \wp(\gamma_2)}{\alpha_1 - \alpha_2}\Big) + \partial_x\Big(\frac{\wp(\gamma_1) - \wp(\gamma_2)}{\alpha_1 - \alpha_2}\Big) - \frac{\wp'(\gamma_1) - \wp'(\gamma_2)}{\alpha_1 - \alpha_2},$$

$$\partial_{t_2} v = -\frac{1}{2} v_{xx} + v\Big(uv + \wp(\gamma_1) - \wp(\gamma_2) + 2\,\frac{\alpha_1 \wp(\gamma_1) - \alpha_2 \wp(\gamma_2)}{\alpha_1 - \alpha_2}\Big) + \partial_x\Big(\frac{\alpha_1\alpha_2(\wp(\gamma_1) - \wp(\gamma_2))}{\alpha_1 - \alpha_2}\Big) + \frac{\alpha_1\alpha_2(\wp'(\gamma_1) - \wp'(\gamma_2))}{\alpha_1 - \alpha_2},$$

$$\partial_{t_2} \gamma_1 = -\frac{\alpha_1\alpha_2 u - v}{\alpha_1 - \alpha_2}, \quad \partial_{t_2} \gamma_2 = \frac{\alpha_1\alpha_2 u - v}{\alpha_1 - \alpha_2},$$

$$\partial_{t_2} \alpha_1 = (\alpha_1^2 u - v) S_{12} - 2\alpha_1 \wp(\gamma_1) + \alpha_1^2 w_1 - w_2 - 2\alpha_1 w_3,$$

$$\partial_{t_2} \alpha_2 = -(\alpha_2^2 u - v) S_{12} - 2\alpha_2 \wp(\gamma_2) + \alpha_2^2 w_1 - w_2 - 2\alpha_2 w_3,$$

$$U_2 = \begin{pmatrix} w_3 & w_1 \\ w_2 & -w_3 \end{pmatrix}$$

Riemann-Hilbert problem

- Two solutions to auxiliary linear system $\partial_{t_n}\psi(z) = A_n(z)\psi(z)$

1. local solution at $z=0$

$$\psi(z) = \phi(z)\exp\left(\sum_{n=1}^{\infty} t_n J z^{-n}\right) \quad (t_1 = x)$$

$$\phi(z) = I + \sum_{n=1}^{\infty} \phi_n z^n .$$

2. global solution $\psi(z) = \chi(z)$ $(\underline{z \in \Gamma})$
normalized as

$$\psi(z)\big|_{t=0} = I \qquad (\underline{t = (t_1, t_2, \dots)})$$

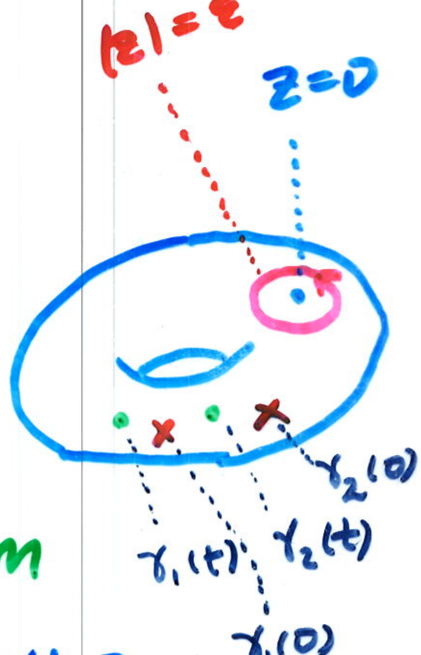
- essential singularity at $z=0$
- poles at $z = \gamma_1(0), \gamma_2(0)$
$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad$ $\| \quad \quad \quad \quad \quad \|$
$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad$ $\gamma_1|_{t=0} \quad \gamma_2|_{t=0}$
- At $z \to \gamma_s$,

$$\chi(z) = \frac{\beta_{x,s}(1+i\alpha_s^{+}(0))}{z - \gamma_s(0)} + O(1)$$

$\gamma_s(0), \alpha_s(0)$ — initial values at $t=0$

- algebraic relation connecting $\phi(t,z)$, $\chi(t,z)$ and the initial value $\phi(0,z)$:

$$\phi(0,z)\exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right) = \chi(t,z)^{-1}\phi(t,z)$$



- This is a kind of **Riemann-Hilbert problem**
  ("Given $\phi(0,z)$, find $\phi(t,z)$ and $\chi(t,z)$.")
- differs from the ordinary one in that $\chi(t,z)$ **degenerates** at $z=\gamma_1(t),\gamma_2(t)$ ($\det\chi=0$ at these points) (poles at $\gamma_1(0),\gamma_2(0)$)
- A similar RHP appears in Krichever's work (1978) on comm. ring of diff. ops.

Perspectives in Sato Grassmannian

- infinite Grassmannian

$$V = \left\{ \sum_{n=-\infty}^{\infty} a_n z^n \;\middle|\; a_n \in gl(2, \mathbb{C}) \right\}$$

+ convergence condition

$$V_+ = \left\{ \sum a_n z^n \in V \;\middle|\; a_n = 0 \text{ for } n \leq 0 \right\}$$

$$Gr = \{ W \subset V \mid \dim \mathrm{Ker}(W \to V/V_+) = \dim \mathrm{Coker}(W \to V/V_+) < \infty \}$$

(where $W \to V/V_+$ is the composition of $W \hookrightarrow V$ and the canonical projection $V \to V/V_+$.)

big cell $Gr^{\circ} = \{ W \subset V \mid W \simeq V/V_+ \}$

- **Vacuum** $W_0 \in Gr^\circ$

$$W_0 = \langle w_{n,ij}(z) \mid n \geq 0,\ i,j = 1,2 \rangle$$

$w_{n,ij}(z)$: 2×2 matrix of mero. functions on Γ with the following properties.

1. $w_{nij}(z)$ has poles at $z = 0$, $\gamma_1(0)$, $\gamma_2(0)$, and is holomorphic elsewhere.

2. At $z = 0$,
$$w_{n,ij}(z) = E_{ij} z^{-n} + O(z)$$

3. At $z = \gamma_s(0)$ $(s = 1, 2)$
$$w_{n,ij}(z) = \frac{\beta_{n,ij,s}\, {}^t\alpha_s(0)}{z - \gamma_s(0)} + O(1)$$

<u>Lemma</u> $\chi(t, z) \in W_0$

<u>Corollary</u> $W_0 \chi(t, z) = W_0$
(at least for small t's)

- Space of dressed vacua

$$\mathcal{M} = \mathcal{M}(\gamma_s(0), \alpha_s(0))$$

$$= \{ W \mid W = W_0 \phi(z),\ \phi(z) = I + \sum_{n=1}^{\infty} \phi_n z^{n},\ \phi_n \in gl(2, \mathbb{C}) \} \subset Gr^0$$

<u>Theorem</u> Let $W(t) = W_0 \phi(t, z)$.

Then $W(t) \in \mathcal{M}$ and

$$W(t) = W(0) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right). \quad (*)$$

Conversely, from the dynamical system on $\mathcal{M}$ with these exponential flows, one can obtain a solution of the elliptic AKNS hierarchy.

$\boxed{\text{elliptic AKNS hierarchy}} \longleftrightarrow \boxed{\text{dynamical system on } \mathcal{M} \subset Gr \text{, } \mathcal{M} = \mathcal{M}(\gamma_s(0), \alpha_s(0))}$ (*)

Geometric meaning

$$W = W_0 \, \phi(z)$$

W_0: data of holomorphic vector bundle $\mathcal{F}$ (with Tyurin parameters $\alpha_s(0), \gamma_s(0)$)

$\phi(z)$: choice of local trivialization of $\mathcal{F}$ at $z=0$

cf. geometric data for classification of commutative ring of differential operators

$$(\Gamma, P_0, z, \underline{\mathcal{F}}, \underline{\phi})$$

(Mulase, $\cdots$)

Landau - Lifshitz equation

$$[\partial_x - A(z), \partial_t - B(z)] = 0$$

$$A(z+2w_a) = \sigma_a A(z) \sigma_a \quad (a=1,2,3)$$

$\rightarrow$ rigid holomorphic $sl(2,\mathbb{C})$ bundle
(no parameter)

V, V_+ — the same

$$W_0 = \{ X(z) \in V \mid \text{pole at } z=0 \text{ or ess. sing.}, \text{ holomorphic elsewhere}, \; X(z+2w_a) = \sigma_a X(z) \sigma_a \; (a=1,2,3) \}$$

$$W(t) = W(0) \exp\left(-\sum t_n \sigma_3 z^{-n}\right)$$

for $W(t) = W_0 \phi(t,z)$

$\rightarrow$ unified point of view in Gr

Conclusion

Results :

- construction of a new hierarchy
- interpretation as dynamical system on a subset of infinite dimensional Grassmann variety
- similar interpretation of Landau-Lifshitz equation

Outlook:

- $g > 1$: possible (though not in an explicit form)
- symmetries ? (Krichever-Novikov algebras ?)
- principal grading ? (analogue of Drinfeld-Sokolov systems ?)
- systems of Toda type ? (cf. Krichever's plenary talk)