Hyperelliptic integrable systems on elliptic K3 and rational surfaces

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After the work of Mukai and Beauville, it has come to be widely recognized that the moduli spaces of a certain class of sheaves on a K3 surface have the structure of algebraically integrable Hamiltonian systems (AIHS). The aim of this talk is to present a special example of those AIHS's on elliptic K3 (and rational) surfaces, along with an explicit description of "action-angle variables".

Surfaces. The elliptic K3 and rational surfaces S in our construction are written in the Weierstrass form

$$y^2 = x^3 + f(z)x + g(z)$$

for suitable affine coordinates x, y, z. $f(z)$ and $g(z)$ are assumed to be (generic) polynomials of $\deg f = 8,\ \deg g = 12$ for the K3 and $\deg f = 4,\ \deg g = 6$ for the elliptic rational surfaces (also called "$\frac{1}{2}$K3"). Since the construction is mostly parallel, the following consideration will be focussed on the case of K3.

Curves, Jacobians, and AIHS on K3. Our AIHS, as usual, is a holomorphic symplectic manifold $\mathcal{X}$ with a Lagrangian fibration (Hamiltonian map) $h : \mathcal{X} \to \mathcal{U}$. The fibers $\mathcal{X}_u = h^{-1}(u)$ are the Jacobi varieties $\mathrm{Jac}(C_u)$ of a family of curves C_u $(u \in \mathcal{U})$ embedded in S. These curves C_u are hyperelliptic curves of genus 5 defined by the equation

$$y^2 = p(z)^3 + f(z)p(z) + g(z),$$

where $p(z)$ is the polynomial

$$p(z) = u_1 z^4 + u_2 z^3 + u_3 z^2 + u_4 z + u_5$$

with five parameters $u = (u_1, \cdots, u_5)$. These hyperelliptic curves form a five-dimensional linear system of curves on S defined by the equation

$$x = p(z).$$

Accordingly, the five parameters $(u_1, \cdots, u_5)$ should be considered an affine coordinate system on the projective space $\mathbf{P}^5$; $\mathcal{U}$ can be chosen to be this projective space itself, or an open subset so as to avoid singular curves. Anyway, by Beauville's construction, $\mathcal{X}$ becomes an AIHS and u_j's are commuting Hamiltonians.

Explicit description. To make the above construction more explicit, we now use effective divisors (of degree 5) as a representative of the points of the Jacobi variety or, rather, the Picard variety $\mathrm{Pic}^5(C_u)$ of C_u. Let D be such a divisor on C_u: $D = \sum_{j=1}^5 P_j,\ P_j = (x_j, y_j, z_j)$. If the five points $P_1, \cdots, P_5$ are distinct, this divisor can be identified with a point of the 5-fold symmetric product $S^{(5)}$ of S. Note that the polynomial $p(z)$ (therefore the curve C_u as well) is uniquely determined in turn by these five points. The holomorphic symplectic form $\omega = dx \wedge dz/y$ on S induces a symplectic structure on $S^{(5)}$ with the symplectic form

$$\Omega = \sum_{j=1}^{5} \frac{dx_j \wedge dz_j}{y_j}.$$

Upon being mapped by the Abel-Jacobi map, this symplectic form can be rewritten

$$\Omega = \sum_{k=1}^{5} du_k \wedge d\psi_k,$$

where ψ_k's ("angle variables") are affine coordinates on $\mathrm{Jac}(C_u) \simeq \mathbf{C}^5/L$, which are now parameterized by the divisor D as

$$\psi_k = \sum_{j=1}^{5} \int_{P_0}^{P_j} \frac{z^{5-k} dz}{y}.$$

This result shows a remarkable similarity with "separation of variables" of more classical examples of hyperelliptic AIHS's such as the Toda chain, the Neumann system, etc.

AIHS on $\frac{1}{2}$K3. A similar construction of an AIHS can be done for the elliptic rational surfaces. The polynomial $p(z)$ in that case is linear, i.e., $p(z) = u_1 z + u_2$, and the hyperelliptic curves C_u are of genus 2.

Remark. For both cases, one can further define a "special coordinate system" $a = (a_1, \cdots, a_5)$ and an associated "prepotential" $\mathcal{F}(a)$ in a neighborhood of each point of $\mathcal{U}$.