有理函数の空間の上の可積分系とその拡張

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- moduli space of rational ftn's
- rational model
- trigonometric, elliptic } models

M : moduli space of rational functions $f(\lambda)$ of degree N and $f(\infty)=0$

coordinates :

1. $u_1, \cdots, u_N, \ v_1, \cdots, v_N$

$$f(\lambda)=\frac{B(\lambda)}{A(\lambda)}, \qquad A(\lambda)=\lambda^N+u_1\lambda^{N-1}+\cdots+u_N$$

$$B(\lambda)=v_1\lambda^{N-1}+\cdots+v_N$$

2. $\alpha_1, \cdots, \alpha_N, \ \rho_1, \cdots, \rho_N$

$$f(\lambda)=\sum_{j=1}^{N}\frac{\rho_j}{\lambda-\alpha_j}. \qquad A(\lambda)=\prod_{j=1}^{N}(\lambda-\alpha_j)$$

$$\rho_j=\frac{B(\alpha_j)}{A'(\alpha_j)}$$

$$\left(\rho:=\sum_{j=1}^{N}\rho_j=v_1\right)$$

Moser's "inverse spectral method"

$$f(\lambda) = \left[(\lambda I - L)^{-1} \right]_{NN}$$

$$\begin{cases} u_1 = 0 \text{ (center-ofmass frame)} \\ \rho = 1 \end{cases}$$

finite nonperiodic Toda chain

$$\Updownarrow$$

$$\dot{\alpha}_j = 0, \quad \dot{\rho}_j = \alpha_j \rho_j \qquad \text{(linearized!)}$$

- Application to system control theory (Krishnaprasad, Nakamura, ...)
- higher flows

$$\dot{\alpha}_j = 0, \quad \dot{\rho}_j = \alpha_j^{\,l} \rho_j$$

Atiyah-Hitchin Poisson structure

$$\Omega = \sum_{j=1}^{N} d\log B(\alpha_j) \wedge d\alpha_j$$

$$\left(= \sum_{j=1}^{N} d\log \rho_j \wedge d\alpha_j - \sum_{j<k} \frac{d\alpha_j \wedge d\alpha_k}{\alpha_j - \alpha_k} \right)$$

— applied to twistor description of monopole moduli space (cf. Donaldson's work)

— $u_1 = 0$, $\rho = 1$: reduced moduli space

— α_j and $\log \rho_j$ are NOT canonical (Darboux) coordinates in this symplectic structure

Known Fact (Faybusovich & Gekhtman, ..)

> Moser's system is a Liouville-integrable system on $(M_{u_1=0,\,p=1}, \Omega)$

- $H \sim u_2$
- $\{u_m, u_n\} = 0 \quad (m, n = 2, \cdots, N)$

  ↳ This is OBVIOUS (because u_m is a function of α's only!)
- Poisson structure of $A(\lambda)$, $B(\lambda)$

  $\{A(\lambda), B(\mu)\} = \cdots$

New Interpretation

1. separation of variables (à la Sklyanin)
2. action-angle variables (à la Seiberg-Witten theory)

separation of variables

Hamilton-Jacobi eqs

$$\boxed{H_l\left(x_1,\cdots,x_N,\frac{\partial S}{\partial x_1},\cdots,\frac{\partial S}{\partial x_N}\right)=E_l}$$

$(l=1\cdots N)$

$\downarrow$ $(x,\xi)\to(\lambda,\mu)$ canonical

$$S=\sum_{k=1}^{N}S_k(\lambda_k,E_1,\cdots,E_N)$$

$$\boxed{f_k\left(\lambda_k,\frac{dS_k}{d\lambda_k},E_1,\cdots,E_N\right)=0}$$

$$\to\quad \frac{dS_k}{d\lambda_k}=\mu_k(\lambda_k,E_1,\cdots,E_N)$$

$$\to\quad S_k=\int^{\lambda_k}\mu_k(\lambda,E_1,\cdots,E_N)\,d\lambda$$

$$\to\quad S=\sum_{k=1}^{N}\int^{\lambda_k}\mu_k(\lambda,E_1,\cdots,E_N)\,d\lambda$$

$$\left(\to\quad \phi_l=\frac{\partial S}{\partial E_l}=\sum_{k=1}^{N}\int^{\lambda_k}\frac{\partial\mu_k}{\partial E_l}\,d\lambda\right)$$

"angle variables"

In the present case $(H, M_{u_1=1, P=0}, \Omega)$

variables for separation:

λ_k : zeroes of $B(\lambda)$

i.e. $B(\lambda) = \prod_{k=1}^{N-1} (\lambda - \lambda_k)$,

$\mu_k = \log z_k = \log A(\lambda_k)$.

separated Hamilton-Jacobi eqs:

$$\exp\left(\frac{dS_k}{d\lambda_k}\right) = A(\lambda_k) \qquad \Big(\underbrace{u_2, \cdots, u_N}_{E's} \Big)$$

→ solution:

$$S_k(\lambda_k) = \int^{\lambda_k} \log A(\lambda) d\lambda$$

→ angle variables: (conjugate of u_ℓ's)

$$\phi_\ell = \sum_{k=1}^{N-1} \int^{\lambda_k} \frac{\lambda^{N-\ell}}{A(\lambda)} d\lambda$$

...... analogue of Abel-Jacobi mapping for rational curve $z = A(\lambda)$ (spectral curve)

$$u_\ell = -\sum_{k=1}^{N} \frac{z_k - \lambda_k^N}{B'(\lambda_k)} \frac{\partial b_\ell}{\partial \lambda_k}$$

(In particular,

$$H = u_2 = \sum_{k=1}^{N} \frac{z_k - \lambda_k^N}{B'(\lambda_k)}$$)

$\uparrow$ solving for u_ℓ's (interpolation formula)

$$z_k = A(\lambda_k) = \lambda_k^N + u_2 \lambda_k^{N-2} + \cdots + u_N$$

$\downarrow$ Hamilton-Jacobi $\left(z_k = e^{p_k} \to \exp \frac{\partial S}{\partial \lambda_k} \right)$

$$\exp \frac{\partial S}{\partial \lambda_k} = A(\lambda_k)$$

cf.

- Morosi & Tondo

  Calogero's equation

$$\ddot{\lambda}_j = 2\sum_{k \neq j} \frac{\dot{\lambda}_j \dot{\lambda}_k}{\lambda_j - \lambda_k} + c\dot{\lambda}_j$$

→ SOV, Bi-Hamiltonian str.

$B(\lambda) = \prod(\lambda - \lambda_j)$, $\dot{B} = -A + cB$, $\dot{A} = 0$

- Krichever & Vaninsky

  open Toda lattice

→ rational spectral curve (singular)

universal symplectic structure

Cf. Seiberg-Witten integrable system

curve : $z^2 - A(\lambda) z + C = 0$

(spectral curve of finite periodic Toda chain, $C \sim$ coupling const)

$$\phi_\ell = \sum_{k=1}^{N-1} \int^{\lambda_k} \frac{\lambda^{N-\ell}}{\sqrt{A(\lambda)^2 - 4C}} d\lambda$$

$$\Big\downarrow \quad C \to 0 \text{ (singular limit)}$$

$$\phi_\ell = \sum_{k=1}^{N-1} \int^{\lambda_k} \frac{\lambda^{N-\ell}}{A(\lambda)} d\lambda$$

Variants

1. trigonometric (hyperbolic) analogue

$$A(\lambda) = \prod_{j=1}^{N} \sinh(\lambda - \alpha_j),$$

$$B(\lambda) = \prod_{j=1}^{N} \sinh(\lambda - \lambda_k).$$

$$\Omega = \sum_{j=1}^{N} d\log B(\alpha_j) \wedge d\alpha_j$$

$$= \sum_{j=1}^{N} d\log A(\lambda_k) \wedge d\lambda_k$$

$$= \sum_{\ell=1}^{N} du_\ell \wedge d\phi_\ell$$

Hamiltonians $u_0, \dots, u_N$ $(u_N = u_0^{-1})$:

$$A(\lambda) = 2^{-N}\big(u_0 x^N - u_1 x^{N-2} + \dots$$

$$\dots + (-1)^{N-1} u_{N-1} x^{2-N} + (-1)^N u_N x^{-N}\big),$$

$(x = e^\lambda)$

2. elliptic analogue

$$A(\lambda) = \prod_{j=1}^{N} \sigma(\lambda - \alpha_j),$$

$$B(\lambda) = \prod_{k=1}^{N} \sigma(\lambda - \lambda_k),$$

$$\Omega = \sum_{j=1}^{N} d\log B(\alpha_j) \wedge d\alpha_j$$

$$= \sum_{k=1}^{N} d\log A(\lambda_k) \wedge d\lambda_k$$

$$= \sum_{\ell=1}^{N} du_\ell \wedge d\phi_\ell ,$$

Hamiltonians $u_1, \cdots, u_N$ $\left(u_0 = \sum_{j=1}^{N} \alpha_j\right)$

$$A(\lambda) = \sum_{\ell=1}^{N} u_\ell f_\ell(\lambda, u_0)$$

$\uparrow$

a basis of sections of
a line bundle over
the elliptic curve

3. higher genus ?

Conclusion

- moduli space $M_{u_1=0,\, p=1}$ of rational functions has a poisson (symplectic) structure and a maximal number of conserved quantities (Liouville integrability)
- This integrable system is separable, and separation of variable leads to a Seiberg-Witten-like description
- Trigonometric and elliptic analogues can be constructed. new.
- These are presumably the simplest models of separation of variables and Seiberg-Witten integrable systems.