1. Integrable system in Atiyah-Hitchin structure (review)

- \( M = \{ (A, B) \mid \text{pair of polynomials} \} \)
- \( A = \lambda^N + u_2 \lambda^{N-2} + \ldots + u_N. \)
- \( B = \lambda^{N-1} + u_2 \lambda^{N-2} + \ldots + u_N. \)
  (with genericity condition)
- \( A = \prod_{j=1}^{N} (\lambda - \alpha_j), \quad \sum_{j=1}^{N} \alpha_j = 0. \)
- \( B = \prod_{k=1}^{N-1} (\lambda - \alpha_k). \)
- \( \dim M = 2N - 2 \)
  \( (u_2, u_3) \) coordinates
\( (\text{quasi}) \text{ symplectic structure} \)

\[
\Omega = \sum_{j=1}^{N} \text{def} \, d \log B(x_j) \wedge dx_j
\]

\[
= \sum_{k=1}^{N-1} d \log A(\lambda_k) \wedge d \lambda_k
\]

\[
= \sum_{k=2}^{N} du_k \wedge d \phi_k, \text{ where}
\]

\[
\phi_k = \sum_{k=1}^{N-1} \int_{\lambda_k}^{\lambda} \frac{2^{N-2}}{A(\lambda)} d \lambda.
\]

\[
\therefore \{ u_2, u_m \} = 0
\]

- integrable Hamiltonian system
  - Hamiltonians \( u_2, \ldots, u_N \)
  - angle variables \( \phi_2, \ldots, \phi_N \)
- separable in \( (\lambda_k, z_k) \)
  - \( z_k = A(\lambda_k) \)
- equivalent to open Toda chain
2. Analogous construction on algebraic curve of arbitrary genus

- \( C_0 \): complex (non-singular) algebraic curve of genus \( g \)

- \( R_1, \ldots, R_N \in C_0 : N > 2g - 2 \) (or \( N > g \) and in general position)

\[ \dim L(R_1 + \cdots + R_N) = N - g + 1 \]

\( L \) vector space of meromorphic functions on \( C_0 \) with

\( (f) \geq -R_1 - \cdots - R_N \)

(poles of at most first order)

at \( R_1, \ldots, R_N \)

\[ \therefore \quad \text{Proj} L(R_1 + \cdots + R_N) \cong \mathbb{P}^{N-g} \]

\( \psi \frac{[f]}{[f]}, \ f \in L(R_1 + \cdots + R_N) \)
\[ M = \{ ([A], [B]) \mid A, B \in L(R_1 + \cdots + R_N) \setminus \{0\} \} \]

Let \( f_0, \ldots, f_{N-g} \) be a basis of \( L(R_1 + \cdots + R_N) \) and express \( A \) and \( B \) as

\[
A = \sum_{l=0}^{N-g} a_l f_l,
\]

\[
B = \sum_{l=0}^{N-g} b_l f_l.
\]

\[
u_l = \frac{a_l}{a_0}, \quad \nu_l = \frac{b_l}{b_0} \quad (l = 1, \ldots, N)
\]

are affine coordinates of \( M \).

\[
dim M = 2N - 2g.
\]

\( p_1, \ldots, p_N \) : zeros of \( A \)

\( q_1, \ldots, q_N \) : zeros of \( B \)
• choose a non-zero holomorphic form $d\lambda$ and define the primitive function

$$\lambda(p) = \int_{p_0}^p d\lambda$$

• (quasi) symplectic structure

$$\Omega = \sum_{j=1}^{n} d\log B(p_j) \wedge d\lambda(p_j)$$

$$= \sum_{j=1}^{n} d\log A(q_j) \wedge d\lambda(q_j)$$

$$= \sum_{j=1}^{n} d\mu_j \wedge d\phi_j, \text{ where}$$

$$\phi_j = \sum_{j=1}^{N} \int_{p_0}^{p_j} \frac{f_j(p) \, d\lambda(p)}{A(p)/a_0}$$

$$\therefore \{u_2, u_m\} = 0$$
• integrable Hamiltonian system on $\mathcal{M}$ with Hamiltonians $U_1, \ldots, U_{N-g}$.

$$A(p)/a_0 = f_0(p) + \sum_{q=1}^{N-g} u_q f_q(p).$$

• "separable" in $x_j, z_j \ (j = 1, \ldots, N)$

$$x_j = \lambda(Q_j)$$

$$z_j = A(Q_j)/a_0$$

A difficulty: $x_1, \ldots, x_N, z_1, \ldots, z_N$ are NOT independent

(dim $\mathcal{M} = 2N - 2g$)