有理型函数対の空間の上の可積分系

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1. Integrable system in Atiyah-Hitchin structure (review)

- $\mathcal{M} = \{ (A, B) \mid$ pair of polynomials $\}$

$$A = \lambda^N + u_2 \lambda^{N-2} + \cdots + u_N.$$

$$B = \lambda^{N-1} + v_2 \lambda^{N-2} + \cdots + v_N.$$

(with genericity condition)

$$A = \prod_{j=1}^{N} (\lambda - \alpha_j) \ , \ \sum_{j=1}^{N} \alpha_j = 0.$$

$$B = \prod_{k=1}^{N-1} (\lambda - \lambda_k).$$

$\dim \mathcal{M} = 2N - 2$ $\qquad \left.\begin{matrix} (u_\ell, v_\ell) \\ (\alpha_j, \lambda_j) \end{matrix}\right\}$ coordinates on $\mathcal{M}$

- (quasi) symplectic structure

$$\Omega \underset{\text{def}}{=} \sum_{j=1}^{N} d\log B(\alpha_j) \wedge d\alpha_j$$

$$= \sum_{k=1}^{N-1} d\log A(\lambda_k) \wedge d\lambda_k$$

$$= \sum_{\ell=2}^{N} du_\ell \wedge d\phi_\ell \text{ , where}$$

$$\phi_\ell = \sum_{k=1}^{N-1} \int^{\lambda_k} \frac{\lambda^{N-\ell}}{A(\lambda)} d\lambda .$$

$\therefore \{u_\ell, u_m\} = 0$

- integrable Hamiltonian system

  $\begin{cases} \text{Hamiltonians } u_2, \cdots, u_N \\ \text{angle variables } \phi_2, \cdots, \phi_N \end{cases}$

- separable in (λ_k, z_k)

  $z_k = A(\lambda_k)$

- equivalent to open Toda chain

2. Analogous construction on algebraic curve of arbitrary genus

- C_0 : complex (non-singular) algebraic curve of genus g

- $R_1, \cdots, R_N \in C_0$: $N > 2g-2$
  (or $N > g$ and in general position)

$\Downarrow$

$$\dim \mathcal{L}(R_1 + \cdots + R_N) = N - g + 1$$

$\mathcal{L}$: vector space of merom. functions on C_0 with
$$(f) \geq -R_1 - \cdots - R_N$$
(poles of at most first order at $R_1, \cdots, R_N$)

$$\therefore \mathbb{P}\mathcal{L}(R_1 + \cdots + R_N) \simeq \mathbb{P}^{N-g}$$
$$\Psi$$
$$[f], \quad f \in \mathcal{L}(R_1 + \cdots + R_N)$$

- $\mathcal{M} = \{([A],[B]) \mid A, B \in \mathcal{L}(R_1 + \cdots + R_N) \setminus \{0\}$
  $+$ genericity cond. $\}$

Let $f_0, \cdots, f_{N-g}$ be a basis of $\mathcal{L}(R_1 + \cdots + R_N)$ and express A and B as

$$A = \sum_{l=0}^{N-g} a_l f_l ,$$

$$B = \sum_{l=0}^{N-g} b_l f_l .$$

$$u_l = \frac{a_l}{a_0}, \quad v_l = \frac{b_l}{b_0} \quad (l = 1, \cdots, N)$$

are affine coordinates of $\mathcal{M}$.

$$\dim \mathcal{M} = 2N - 2g .$$

- $P_1, \cdots, P_N$: zeros of A
  $Q_1, \cdots, Q_N$: zeros of B

- choose a non-zero holom. form $d\lambda$ and define the primitive function

$$\lambda(P) = \int_{P_0}^{P} d\lambda$$

- (quasi) symplectic structure

$$\Omega \underset{\mathrm{def}}{=} \sum_{j=1}^{N} d\log B(P_j) \wedge d\lambda(P_j)$$

$$= \sum_{j=1}^{N} d\log A(Q_j) \wedge d\lambda(Q_j)$$

$$= \sum_{\ell=1}^{N-g} du_\ell \wedge d\phi_\ell, \text{ where}$$

$$\phi_\ell = \sum_{j=1}^{N} \int_{P_0}^{Q_j} \frac{f_\ell(P)\, d\lambda(P)}{A(P)/a_0}$$

$$\therefore \{u_\ell, u_m\} = 0$$

- integrable Hamiltonian system on $\mathcal{M}$ with Hamiltonians $u_1, \cdots, u_{N-g}$.

$$A(P)/a_0 = f_0(P) + \sum_{\ell=1}^{N-g} u_\ell f_\ell(P).$$

- "separable" in λ_j, z_j ($j = 1, \cdots, N$)

$$\lambda_j = \lambda(Q_j)$$

$$z_j = A(Q_j)/a_0$$

▲ difficulty: $\lambda_1, \cdots, \lambda_N$, $z_1, \cdots, z_N$ are NOT independent

$(\dim \mathcal{M} = 2N - 2g)$