05-03-27

変形KP階層のq-類似とその準古典極限

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(第1式)

ぬけている

$$\oint_{\lambda=\infty} \tau(s', t' - [\lambda^{-1}])\, \tau(s, t + [\lambda^{-1}])\, \lambda^{s'-s} e^{\xi(t',\lambda) - \xi(t,\lambda)} d\lambda = 0$$

(第3式)

$\leftarrow (1-q^2)$

$$[\alpha]_q^{(n)} = \left(\cdots, \frac{(1-q)^2 \alpha^2}{2(1-q^2)}, \cdots, \frac{(1-q)^k \alpha^k}{k(1-q^k)}, \cdots \right)$$

$n \to k$

(2ページ目, 第1式)

$$[qx]_q^{(n)} = [x]_q^{(n)} - \sum_{m=0}^{n-1} [e^{2\pi i m/n} (1-q)^{\frac{1}{n}} x^{\frac{1}{n}}]$$

$x \to$

(2ページ目, 第5式)

ぬけている

$$(1-q^n) D_{q^n}(x_n) \Psi_q = B_n \Psi_q, \quad B_n = \cdots$$

(2ページ目, 下段) $L_2 \to Li_2$

(2ページ目, 下から3行目) $\partial S/\partial s = p \to e^{\partial S/\partial s} = p$

Modified KP hierarchy

- tau function: $\tau(s, t)$

  $s \in \mathbb{Z}$, $t = (t_1, t_2, \cdots)$

- bilinear equations: For $s' \geq s$,

$$\oint_{\lambda=\infty} \tau(s', t' - [\lambda^{-1}])\, \tau(s, t + [\lambda^{-1}]) \times \lambda^{s'-s} e^{\xi(t', \lambda) - \xi(t, \lambda)} d\lambda = 0.$$

$\oint_{\lambda=\infty}$: $\circlearrowleft \cdot\infty$

$$[\alpha] = \left(\alpha, \frac{\alpha^2}{2}, \cdots, \frac{\alpha^k}{k}, \cdots\right)$$

$$\xi(t, \lambda) = \sum_{k=1}^{\infty} t_k \lambda^k$$

- wave function:

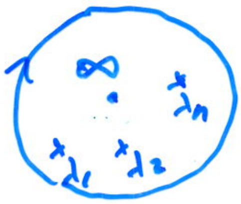
$$\Psi(s, t, \lambda) = \frac{\tau(s, t - [\lambda^{-1}])}{\tau(s, t)} \lambda^s e^{\xi(t, \lambda)},$$

$$\Psi^*(s, t, \lambda) = \frac{\tau(s, t + [\lambda^{-1}])}{\tau(s, t)} \lambda^{-s} e^{-\xi(t, \lambda)}.$$

$$\partial_{t_n} \Psi(s, t, \lambda) = (\underbrace{e^{n\partial_s} + a_{n,1} e^{(n-1)\partial_s} + \cdots + a_{n,n}}_{\text{difference operator}}) \Psi(s, t, \lambda)$$

- yet another "modification" (2nd modification)

$$\oint_{\lambda=\infty} \tau(s', t'-[\lambda_1^{-1}]-\cdots-[\lambda_n^{-1}]-[\lambda^{-1}]) \times \tau(s, t+[\lambda^{-1}])\, \lambda^{s'-s}\, e^{\xi(t',\lambda)-\xi(t,\lambda)} \times \prod_{j=1}^{n}(\lambda-\lambda_j)\, d\lambda = 0$$



($\because$) Substitute

$$t' \to t' - [\lambda_1^{-1}] - \cdots - [\lambda_n^{-1}]$$

in the previous bilinear equation.

The exponential factor $e^{\xi(t',\lambda)-\xi(t,\lambda)}$ gets multiplied by

$$\exp\Big(-\sum_{j=1}^{n}\xi([\lambda_j^{-1}],\lambda)\Big)$$

$$=\prod_{j=1}^{n}\exp\Big(-\sum_{k=1}^{\infty}\frac{\lambda^k}{k\lambda_j^{\,k}}\Big)$$

$$=\prod_{j=1}^{n}\Big(1-\frac{\lambda}{\lambda_j}\Big). \qquad \square$$

q-analogue of tau function

- new variables $x = (x_1, x_2, \cdots)$
  parameters $q = (q_1, q_2, \cdots)$ $(|q_n| < 1)$

$$\tau_q(s, t, x) = \tau\left(s, t + \sum_{n=1}^{\infty} [x_n]_{q_n}^{(n)}\right)$$

where

$$[\alpha]_q^{(n)} = \Big(\underbrace{0, \cdots, 0, \alpha}_{n}, \underbrace{0, \cdots, 0, \frac{(1-q)^2\alpha^2}{2(1-q^2)}}_{n}, \cdots$$

$$\cdots, 0, \cdots, 0, \frac{(1-q)^k \alpha^k}{k(1-q^k)}, \cdots\Big).$$

- Ref 1) For Toda hierarchy, (← Kajiwara & Satsuma)
  Mironov, Morozov & Vinet (1993)

  2) For KP (with x_1 only)
  Khesin et al
  Adler et al
  Iliev, Heine-Iliev
  } 1997~98

  3) Kajiwara-Noumi-Yamada 2002 (→ q-Painlevé)

- properties of $[\alpha]_q^{(n)}$

1) $[\alpha]_0^{(1)} = [\alpha] = \left(\alpha, \frac{\alpha^2}{2}, \cdots, \frac{\alpha^k}{k}, \cdots\right)$

2) $[\alpha]_q^{(1)} = [\alpha]_q = \left(\alpha, \frac{(1-q)^2\alpha^2}{2(1-q^2)}, \cdots, \frac{(1-q)^k\alpha^k}{k(1-q^k)}, \cdots\right)$

$$\begin{cases} [q\alpha]_q = [\alpha]_q - [(1-q)\alpha]. \\ [\alpha]_q = \sum_{m=0}^{\infty} [(1-q)\alpha q^m]. \end{cases}$$

3) $[\alpha]_0^{(n)} = (\overbrace{0, \cdots, 0, \alpha}^{n}, \overbrace{0, \cdots, 0, \frac{\alpha^2}{2}}^{n}, \cdots)$

$\parallel$

$[\alpha]^{(n)}$

$$= [e^{2\pi i/n}\alpha^{1/n}] + [(e^{2\pi i/n})^2\alpha^{1/n}] + \cdots + [\alpha^{1/n}]$$

4) $[q\alpha]_q^{(n)} = [\alpha]_q^{(n)} - [(1-q)\alpha]^{(n)}$

$$= [\alpha]_q^{(n)} - \sum_{j=1}^{n} [e^{2\pi i j/n}(1-q)^{1/n}\alpha^{1/n}]$$

Apply this identity to 2nd modified bilinear equations! $\left(t' - \sum_{j=1}^{n} [\lambda_j^{-1}] \right)$

- *bilinear equations for q-shifted tau function*

$$\tau_q(s,t,x) = \tau\Big(s,\ t + \sum_{n=1}^{\infty} [x_n]^{(n)}_{q_n}\Big),$$

$$[q_n x_n]^{(n)}_{q_n} = [x_n]^{(n)}_{q_n} - \sum_{j=1}^{n} \left[e^{2\pi i j/n}(1-q_n)^{1/n} x_n^{1/n}\right]$$

$\downarrow$ k-times iteration

$$[q_n^{k} x_n]^{(n)}_{q_n} = [x_n]^{(n)}_{q_n} - \sum_{j=1}^{n}\sum_{\ell=0}^{k-1} \left[e^{2\pi i j/n}(1-q_n)^{1/n} x_n^{1/n} q_n^{\ell/n}\right]$$

$\downarrow$

bilinear equation for $\tau_q(s',t',x')$ and $\tau_q(s,t,x)$

with $\underline{s' \geq s}$, $\underline{x_1' = q_1^{k_1} x_1}$, $\underline{x_2' = q_2^{k_2} x_2}$, $\cdots$:

$$\oint_{\lambda=\infty} \tau(s', t' - [\lambda^{-1}], q_1^{k_1}x_1, q_2^{k_2}x_2, \cdots)\, \tau(s, t + [\lambda^{-1}], x)$$

$$\times \lambda^{s'-s} e^{\xi(t',\lambda) - \xi(t,\lambda)}$$

$$\times \prod_{n=1}^{\infty}\prod_{\ell=0}^{k_n-1} \underline{\left(1 - (1-q_n) x_n q_n^{\ell} \lambda^{n}\right)}\, d\lambda = 0$$

$\nearrow$

$$\prod_{j=1}^{n}\left(\lambda - e^{-2\pi i j/n}(1-q_n)^{-1/n} x_n^{-1/n} q_n^{-\ell/n}\right)$$

$$= \lambda^{n} - (1-q_n)^{-1} x_n^{-1} q_n^{-\ell}$$

Linear equations of wave functions

- wave functions

$$\Psi_q(s,t,x,\lambda) = \frac{\tau_q(s,t-[\lambda^{-1}],x)}{\tau_q(s,t,x)}\lambda^s e^{\xi(t,\lambda)} e_q(x),$$

$$\Psi_q^*(s,t,x,\lambda) = \frac{\tau_q(s,t+[\lambda^{-1}],x)}{\tau_q(s,t,x)}\lambda^{-s} e^{-\xi(t,\lambda)} e_q(x)^{-1},$$

where

$$e_q(x) = \prod_{n=1}^{\infty} e_{q_n}^{x_n\lambda^n},$$

$$e_q^x = \exp\left(\sum_{k=1}^{\infty}\frac{(1-q)^k x^k}{k(1-q^k)}\right) = \prod_{m=0}^{\infty}(1-(1-q)xq^m)^{-1}$$

(q-exponential)

- bilinear equations

$$\oint \Psi_q(s',t',x',\lambda)\,\Psi_q^*(s,t,x,\lambda)\,d\lambda = 0$$

for $s' \geq s$, $\underset{(k_1\geq 0)}{x_1' = q_1^{k_1}x_1}$, $\underset{(k_2\geq 0)}{x_2' = q_2^{k_2}x_2}$, ……,

and arbitrary values of t' and t.

- linear equations

$$\partial_{t_n} \Psi_q(s,t,x,\lambda) = A_n(e^{\partial_s})\, \Psi_q(s,t,x,\lambda),$$

$$D_{x_n}(q_n)\, \Psi_q(s,t,x,\lambda) = B_n(e^{\partial_s})\, \Psi_q(s,t,x,\lambda),$$

where

$$D_{x_n}(q_n)\, f(x_n) = \frac{f(q_n x_n) - f(x_n)}{q_n x_n - x_n},$$

$$A_n(e^{\partial_s}) = e^{n\partial_s} + \sum_{m=1}^{n} a_{nm}\, e^{(n-m)\partial_s},$$

$$B_n(e^{\partial_s}) = e^{n\partial_s} + \sum_{m=1}^{n} b_{nm}\, e^{(n-m)\partial_s}.$$

- Remark

1) $D_{x_n}(q_n)\, e_q(x) = \lambda^n e_q(x)$

2) dressing operator $W(s) = 1 + \sum_{m=1}^{\infty} w_m e^{-m\partial_s}$:

$$\Psi_q(s,t,x,\lambda) = W(s)\, \lambda^s e^{\xi(t,\lambda)} e_q(x)$$

3) zero-curvature (Zakharov-Shabat) equations and Lax equations

Quasi-classical limit

- setup

1) $q_n = q^n = e^{-n\beta\hbar}$ $\quad (q = e^{-\beta\hbar},\ \beta:\text{constant})$

2) rescaling $\quad s \to \frac{s}{\hbar},\ t_n \to \frac{t_n}{\hbar},\ x_n \to \frac{x_n}{1-q^n}$

3) a_{nm} and b_{nm} can depend on $\hbar$

- what's new?

i) $\lambda^s e^{\xi(t,\lambda)} e_q(x) \longrightarrow \lambda^{s/\hbar} e^{\xi(t,\lambda)/\hbar} \prod_{n=1}^{\infty} \exp\left(\sum_{k=1}^{\infty} \frac{x_n \lambda^n}{k(1-q^{kn})}\right)$

$(1-q)^n$ disappeared

ii) linear equations

$$\hbar \partial_{t_n} \Psi_q = A_n(e^{\hbar\partial_s}) \Psi_q,$$

$$(1-q^n) D_{x_n}(q^n) \Psi_q = B_n(e^{\hbar\partial_s}) \Psi_q.$$

- WKB ansatz

$$\Psi_q \sim e^{S/\hbar}, \quad S = s\log\lambda + \sum_{n=1}^{\infty} t_n \lambda^n + \sum_{n=1}^{\infty} \frac{Li_2(x_n\lambda^n)}{n\beta},$$

$$a_{nm} = a_{nm}^{(0)} + O(\hbar),$$

$$b_{nm} = b_{nm}^{(0)} + O(\hbar).$$

$$Li_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

$$= -\int_0^x \frac{\log(1-t)}{t}\, dt$$

(dilogarithm)

- Hamilton-Jacobi (or Eikonal) equations

$$\frac{\partial S}{\partial t_n} = \mathcal{A}_n(e^{\partial S/\partial s}),$$

$$\frac{1-e^{-n\beta x_n \partial S/\partial x_n}}{x_n} = \mathcal{B}_n(e^{\partial S/\partial s}),$$

where

$$\left.\begin{aligned} \mathcal{A}_n(p) &= p^n + \sum_{m=1}^{n} a_{nm}^{(0)} p^{n-m} \\ \mathcal{B}_n(p) &= p^n + \sum_{m=1}^{n} b_{nm}^{(0)} p^{n-m} \end{aligned}\right\} \text{ classical limit of } A_n \text{ and } B_n$$

Remark Actually, $a_{nm}^{(0)} = b_{nm}^{(0)}$ (therefore $\mathcal{A}_n = \mathcal{B}_n$).

- "dispersionless" Lax formalism

$$\frac{\partial S}{\partial s} = p \longrightarrow \lambda = \mathcal{L}(s,t,x,p) = p + \sum_{m=0}^{\infty} u_m p^{-m},$$

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{ A_n(p), \mathcal{L} \},$$

$$x_n \frac{\partial \mathcal{L}}{\partial x_n} = -\frac{1}{n\beta}\{ \log(1 - x_n \mathcal{B}_n(p)), \mathcal{L} \}$$

where $\{f, g\} = p\dfrac{\partial f}{\partial p}\dfrac{\partial g}{\partial s} - \dfrac{\partial f}{\partial s} p \dfrac{\partial g}{\partial p}$.

q-analogue of Toda hierarchy

$\tau(s,t,\bar{t})$ $\quad t=(t_1,t_2,\cdots)$

$\bar{t}=(\bar{t}_1,\bar{t}_2,\cdots)$

$$\tau_q(s,t,\bar{t},x,\bar{x})=\tau\Big(s,\ t+\sum_{n=1}^{\infty}[x_n]^{(n)}_{q_n},\ \bar{t}+\sum_{n=1}^{\infty}[\bar{x}_n]^{(n)}_{\bar{q}_n}\Big)$$

$x=(x_1,x_2,\cdots)$

$\bar{x}=(\bar{x}_1,\bar{x}_2,\cdots)$

$q=(q_1,q_2,\cdots)$

$\bar{q}=(\bar{q}_1,\bar{q}_2,\cdots)$

(Mironov-Morozov-Vinet $\supset$ Kajiwara-Satsuma)

- Everything is parallel
- Application — instanton calculus ?
  random partition ?
- 2-comp. BKP ?