Direct linearization and Sato Grassmannian

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Reference

K.T., Direct linearization, Cauchy matrix and Sato Grassmannian, arXiv:2608.24538 (62pp).

1. Fu and Nijhoff's DL scheme

Direct linearization (DL)

— Direct approach to integrable systems, in particular, discrete systems, without using Lax equations, inverse scattering, Riemann-Hilbert problem, etc.

- Using singular integral equations

  Fokas & Ablowitz 1981, 1983, Nijhoff, van der Linden, Quispel, Capel & Velthuizen 1982, 1984, Nijhoff, Quispel & Capel 1983

- Using infinite matrices

  Fu & Nijhoff 2017, 2018, Fu 2018

— Closely related to Cauchy matrix approach

Nijhoff, Atkinson & Hietarinta 2009, D.J. Zhang & S.L. Zhao 2013, W. Feng and S.L. Zhao 2013, D.D. Xu, D.J. Zhang & S.L. Zhao 2014, W. Feng & S. Zhao 2022.

Infinite matrix algebra

- Special $\mathbb{Z} \times \mathbb{Z}$ matrices

$$\Lambda^k = \sum_{i \in \mathbb{Z}} E_{i,i+k}, \quad \mathcal{O}_k = \sum_{i=0}^{k-1} E_{i,k-i-1},$$
$$\Omega = \sum_{i=0}^{\infty} E_{i,-i-1}, \quad E_{kl} = (\delta_{ik}\delta_{jl})_{i,j\in\mathbb{Z}}.$$

and algebraic relation $\mathcal{O}_k = \Omega\Lambda^k - \Lambda^{-k}\Omega$.

- Relation to Cauchy kernel

$$\sum_{i,j\in\mathbb{Z}} \Omega_{ij} w^i z^j = \sum_{i=0}^{\infty} \frac{w^i}{z^{i+1}} = \frac{1}{z-w} \quad \text{for } |z| > |w|.$$

1. Fu and Nijhoff's DL scheme

U-system (Fu & Nijhoff 2017, 2018)

Fu and Nijhoff employ the evolution equations

$$\frac{\partial U}{\partial t_k} = \Lambda^k U - U\Lambda^{-k} - U\mathcal{O}_k U, \quad k = 1, 2, \dots,$$

of a $\mathbb{Z} \times \mathbb{Z}$ matrix $U = (u_{ij})_{i,j\in\mathbb{Z}}$ for direct linearization of the KP hierarchy and its reductions.

• This system has quadratic nonlinearity in the last term $U\mathcal{O}_k U$.

• This system can be transformed, at least formally, to the linear system

$$\frac{\partial C}{\partial t_k} = \Lambda^k C - C\Lambda^{-k}, \quad k = 1, 2, \cdots$$

by the transformation $U = C(E + \Omega C)^{-1}$ (E is the unit matrix).

2. KP hierarchy as subsystem of U-sysstem

Infinite matrices and Sato Grassmannian (Sato & Sato 1982)

The Sato Grassmannian can be described as a coset space:

$$\begin{gathered}
\mathrm{Gr} \simeq \mathrm{Fr}(\mathbb{Z}, \mathbb{Z}_{<0})/\mathrm{GL}(\mathbb{Z}_{<0}) \simeq \mathrm{GL}(\mathbb{Z}_{\geq 0})\backslash \mathrm{Fr}(\mathbb{Z}_{\geq 0}, \mathbb{Z}), \\
\mathrm{Fr}(\mathbb{Z}, \mathbb{Z}_{<0}) = \{\xi \mid \mathbb{Z} \times \mathbb{Z}_{<0} \text{ matrix of full rank}\}, \\
\mathrm{Fr}(\mathbb{Z}_{\geq 0}, \mathbb{Z}) = \{\eta \mid \mathbb{Z}_{\geq 0} \times \mathbb{Z} \text{ matrix of full rank}\}, \\
\mathbb{Z}_{<0} = \{-1, -2, \ldots\}, \quad \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}.
\end{gathered}$$

Points of the top cell of Gr can be represented by the following matrices:

$$\begin{gathered}
\xi = \left(\frac{\delta_{ij}}{w_{i,-j-1}} \right) \in \mathrm{Fr}(\mathbb{Z}, \mathbb{Z}_{<0}), \\
\eta = \left(\; -w_{i,-j-1} \; \middle| \; \delta_{ij} \; \right) \in \mathrm{Fr}(\mathbb{Z}_{\geq 0}, \mathbb{Z}).
\end{gathered}$$

$\{w_{ij}\}_{i,j=0}^{\infty}$ are affine coordinates of the top cell.

Evolution equations for affine coordinates w_{ij}'s (T. 1985, 1989)

The KP hierarchy can be viewed as a dynamical system on Gr (Sato & Sato 82). The motion of a point of the top cell of Gr can be described by the evolution equations

$$\frac{\partial w_{ij}}{\partial t_k} = w_{i+k,j} - w_{i,j+k} - \sum_{l=0}^{k-1} w_{il} w_{k-l-1,j}, \quad i,j \geq 0.$$

for the affine coordinates w_{ij}. Note that these equations have quadratic nonlinear terms.

Remark: This system can be cast into the matrix forms

$$\frac{\partial \xi}{\partial t_k} = \Lambda^k \xi - \xi \mathcal{B}_k, \quad \frac{\partial \eta}{\partial t_k} = \mathcal{C}_k \eta - \eta \Lambda^k.$$

2. KP hierarchy as subsystem of U-sysstem

Key observation

The evolution equations of u_{ij}'s for $i, j \geq 0$ coincide with the evolution equations of w_{ij}'s.

$$\frac{\partial u_{ij}}{\partial t_k} = u_{i+k,j} - u_{i,j+k} - \sum_{l=0}^{k-1} u_{il} u_{k-l-1,j} \quad \text{for } i, j \in \mathbb{Z}$$

$$\frac{\partial w_{ij}}{\partial t_k} = w_{i+k,j} - w_{i,j+k} - \sum_{l=0}^{k-1} w_{il} w_{k-l-1,j} \quad \text{for } i, j \geq 0.$$

Interpretation

The KP hierarchy may be thought of as a subsystem of the U-system by identifying $w_{ij} = u_{ij}$ for $i, j \geq 0$.

Another description of KP hierarchy in U-system

Gr allows for another coset space representation

$$\mathrm{Gr} \simeq \mathcal{P}\backslash\mathrm{GL}(\mathbb{Z}),$$

where $\mathcal{P}$ is the subgroup of $\mathrm{GL}(\mathbb{Z})$ consisting of invertible matrices of the form

$$p = \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$$

The infinite matrix (playing a role in Fu and Nijhoff's work)

$$g = E - U\Omega = \left(\begin{array}{c|c} \delta_{ij} - u_{i,-j-1} & 0 \\ \hline -u_{i,-j-1} & \delta_{ij} \end{array}\right)$$

represents the same point as η in the previous coset space representation of $\mathrm{Gr} \simeq \mathrm{GL}(\mathbb{Z}_{\geq 0})\backslash\mathrm{Fr}(\mathbb{Z}_{\geq 0}, \mathbb{Z})$.

The motion of this point can be captured by the following differential equations:

- g satisfies the differential equations

$$\frac{\partial g}{\partial t_k} = (\Lambda^k - U\mathcal{O}_k)g - g\Lambda^k, \quad k = 1, 2, \cdots .$$

- The matrix $\Lambda^k - U\mathcal{O}_k$ has the block-triangular structure

$$\Lambda^k - U\mathcal{O}_k = \left(\begin{array}{c|c} * & * \\ \hline 0 & \mathcal{C}_k \end{array}\right).$$

The block-triangular structure of $\Lambda^k - U\mathcal{O}_k$ implies that these differential equations are consistent with the differential equations $\partial\eta/\partial t_k = \mathcal{C}_k\eta - \eta\Lambda^k$ of η.

Extension of U-system by negative flows

The U-system can be extended by the negative flows

$$\frac{\partial U}{\partial t_{-k}} = \Lambda^{-k} U - U\Lambda^{k} - U\mathcal{O}_{-k} U, \quad k = 1, 2, \ldots,$$

where $\mathcal{O}_{-k} = -\sum_{l=1}^{k} E_{-l,l-k-1}$. A new Grassmannian structure is hidden behind these flows. To describe it, we introduce the new infinite matrices

$$\bar{\Omega} = -\sum_{i=0}^{\infty} E_{-i-1,i},$$

$$\bar{g} = E - U\bar{\Omega} = \left(\begin{array}{c|c} \delta_{ij} & u_{i,-j-1} \\ \hline 0 & \delta_{ij} + u_{i,-j-1} \end{array} \right).$$

3. Extended U-system

$\bar{g}$ represents a point of the complementary Grassmannian

$$\overline{\mathrm{Gr}} \simeq \overline{\mathcal{P}} \backslash \mathrm{GL}(\mathbb{Z}).$$

The stabilizer $\overline{\mathcal{P}} \subset \mathrm{GL}(\mathbb{Z})$ consists of invertible matrices of the form

$$p = \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array} \right).$$

The negative flows of U defines a dynamical system of $\bar{g}$ on $\overline{\mathrm{Gr}}$. This is another copy of the KP hierarchy.

The extended U-system with two sets of time variables $\{t_k\}_{k=1}^{\infty}$ and $\{t_{-k}\}_{k=1}^{\infty}$ can be reformulated by the extended η matrix

$$\eta^{(2)} = \left(\begin{array}{c|c} g & \bar{g} \end{array} \right)$$

and turns out to be equivalent to the two-component KP hierarchy.

Conclusion

• Fu and Nijhoff's U-system literally contains the KP hierarchy as a subsystem. To explain this fact, part of the matrix elements of U are identified with affine coordinates of the top cell of the Sato Grassmannian.

• The U-system can be extended by negative flows. The negative flows are related to another Grassmannian. The extended U-system is substantially equivalent to the two-component KP hierarchy.