Isomonodromic deformations as modulation of isospectral deformations

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In the Spring Meeting last year, I reported a few results on the integrable structure of the Seiberg-Witten solution to the $N = 2$ supersymmetric Yang-Mills theory, which includes an interpretation in terms of isomonodromic problems. The isomonodromic problems arising there are a generalization of the 3rd Painlevé equation, and the main observation is an asymptotic description of the the isomonodromic problem as an isospectral problem with slowly varying constants of motion. The isospectral problem can be solved by the Abel-Jacobi theory on a hyperelliptic Riemann surface. The slow dynamics, meanwhile, is governed by the so called Whitham equations.

The isomonodromic problem can be thus asymptotically reduced to "modulation" of a quasi-periodic isospectral problem. This is a modernized version of the classical Boutroux analysis. Boutroux pointed out that Painlevé transcendents look asymptotically like a modulated elliptic function.

In this talk, I consider the Schlesinger equation from the same point of view. (In fact, this is a rather old issue, first studied by Garnier early in this century, and continued by Flaschka and Newell in their papers on isomonodromic deformations. The slow dynamics, however, was overlooked therein.) Unfortunately, my recipe to this issue, as in the case of the Seiberg-Witten solution, is still heuristic. A more rigorous justification, as well as a numerical test, has to be done in near future.

To make the problem more tractable, I first propose to modify the Schlesinger equation into the following an ϵ-dependent form:

$$\epsilon\frac{\partial A_j}{\partial T_k} + \frac{[A_j, A_k]}{T_k - T_j} = 0 \quad (j \neq k),$$

$$\epsilon\frac{\partial A_j}{\partial T_j} + \sum_{k\neq j}\frac{[A_j, A_k]}{T_j - T_k} = 0.$$

ϵ is a “small” parameter and we are concerned with the asymptotics as $\epsilon \to 0$. I write the deformation variables T_j in order to stress that they are now “slow” variables. (I owe this idea to V. Verecshagin, who treated all the six Painlevé equations in this modified form. His papers are stored in the hep-th e-print archive.)

My recipe is to apply the concept of “multiscale analysis” to this problem. Namely, I suppose that all quantities are functions of the “slow” variables T_j and “fast” variables $t_j = T_j/\epsilon$. The latter are used to describe the quasi-periodic motion of an underlying isospectral problem. The isospectral problem is written

$$\frac{\partial A_j}{\partial t_k} + \frac{[A_j, A_k]}{T_k - T_j} = 0 \quad (j \neq k),$$
$$\frac{\partial A_j}{\partial t_j} + \sum_{k \neq j} \frac{[A_j, A_k]}{T_j - T_k} = 0.$$

These equations have a Lax representation of the form $\partial L(\lambda)/\partial t_j = [P_j(\lambda), L(\lambda)] = 0$, hence give an isospectral problem. Note that the slow variables T_j enter as parameters in this isospectral problem. Constants of motion, in particular the spectral curve $\det(\mu - L(\lambda)) = 0$, are thereby T-dependent. Differential equations describing the slow dynamics can be derived by multiscale analysis of the linear problem of the Schlesinger equation. This eventually boils down to equations of the Flaschka-Forest-McLaughlin form:

$$\frac{\partial}{\partial T_j} d\Omega_k = \frac{\partial}{\partial T_k} d\Omega_j,$$

where $d\Omega_j$ are abelian differentials on the spectral curv. They are used in the construction of a Baker-Akhiezer function $\psi(\lambda)$ of the isospectral problem: $\psi(\lambda) = \phi(\lambda) \exp(\sum t_j \int^\lambda d\Omega_j)$.

The same recipe can be (at least formally) applied to other isospectral problems on the Riemann sphere, such as the so called Garnier system (a multi-time analogue of the 6th Painlevé equation). A particularly interesting issue is to extend these results to isomonodromic problems on Riemann surfaces.