Construction of isomonodromic problems on torus

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Isomonodromic problems on a torus (complex elliptic curve) have been studied by K. Okamoto, K. Iwasaki and S. Kawai. They have mostly treated the case of the scalar second order equation $\frac{dy}{dz^2} + p(z)y = 0$, where $p(z)$ is an elliptic function on the torus $E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$. Those isomonodromic problems give a generalization of the Painlevé equations and the Garnier systems.

This talk deals with the problem of constructuting isomonodromic deformations of a first order matrix system $\frac{dY}{dz} = L(z)Y$ on the torus. The matrix linear system is assumed to have regular singularities at N points $z = t_1, \cdots, t_N$. The problem is to construct isomonodromic deformations that leave a set of monodromy data invariant while the position of the poles and the modulus τ being varied. As in the ordinary cases on the Riemann sphere, the deforamtion equations are expected to be written in the "Lax form",

$$
\begin{aligned}
\frac{\partial L(z)}{\partial t_j} &= [L(z), A_j(z)] - \frac{\partial A_j(z)}{\partial z}, \\
\frac{\partial L(z)}{\partial \tau} &= [L(z), A_\tau(z)] - \frac{\partial A_\tau(z)}{\partial z}.
\end{aligned}
$$

Such an isomonodromic system may be thought of as a generalization of the Schlesinger equations.

I shall present two types of such isomonodromic systems. Both have an isospectral partner (the elliptic Calogero-Gaudin system / the ordinary elliptic Gaudin system), and are also related to a conformal field theory (the ordinary WZW model / its "twisted" version by Kuroki and Takebe) on a torus. It is well known that these physical systems are all governed by an r-matrix structure. This r-matrix structure is also a clue in the construction of the isomonodromic systems.

- **Isomonodromic system of Calogero-Gaudin type**: The L-matrix is an SL(n) matrix of the form

$$L(z) = \sum_{j=1}^{N} \sum_{a \neq b} \sigma_{q_a - q_b}(z - t_j) E_{ab} A_{j,ba}$$
$$+ \sum_{j=1}^{N} \sum_{a} \rho(z - t_j) E_{aa} A_{j,aa} + \sum_{a} p_a E_{aa}.$$

Here $A_{j,ab}$ $(a, b = 1, \cdots, n)$ are classical SL(n) spin variables under the constraint $\sum_{j=1}^{N} A_{j,aa} = 0$. E_{ab} are the standard matrix units. q_a and p_a are the positions and momenta of Calogero-Moser particles. $\sigma_q(z)$ and $\rho(z)$ are the functions that appear in the dynamical r-matrix of Felder and Wieczerkowski:

$$r(z - w) = \sum_{a \neq b} \sigma_{q_a - q_b}(z - w) E_{ab} \otimes E_{ba} + \sum_{a} \rho(z - w) E_{aa} \otimes E_{aa}.$$

- **Isomonodromic system of Gaudin type**: The L-matrix is an SL(n) matrix of the form

$$L(z) = \sum_{j=1}^{N} \sum_{(\alpha,\beta) \neq (00)} w_{\alpha\beta}(z - t_j) J_{\alpha\beta} A_j^{\alpha\beta}.$$

Here $A_j^{\alpha\beta}$ $(\alpha, \beta \in \mathbf{Z}_n)$ are classical SL(n) spin variables in the different basis $J_{ab} = g^a h^b$ composed of "quantum torus" generators g and h $(gh = e^{2\pi i/n} hg)$. $w_{\alpha\beta}$ are the building blocks of the classical limit of Belavin's $\mathbf{Z}_n$-symmetric R-matrix:

$$r(z - w) = \sum_{(\alpha,\beta) \neq (0,0)} w_{\alpha\beta}(z - w) J_{\alpha\beta} \otimes J^{\alpha\beta}.$$

In both cases, The matrices $A_j(z)$ are given by the same simple formula:

$$A_j(z) = \operatorname*{Res}_{w=t_j} \operatorname{tr}_2\Big(r(z - w) I \otimes L(w)\Big).$$

Such a universal expression is not available for $A_\tau(z)$, and only a rather complicated expression can be found case-by-case.