Construction of isomonodromic problems on torus

K. Takasaki  Kyoto University

Isomonodromic problems on a torus (complex elliptic curve) have been studied by K. Okamoto, K. Iwasaki and S. Kawai. They have mostly treated the case of the scalar second order equation $\frac{dw}{dz^2} + p(z)y = 0$, where $p(z)$ is an elliptic function on the torus $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. Those isomonodromic problems give a generalization of the Painlevé equations and the Garnier systems.

This talk deals with the problem of constructing isomonodromic deformations of a first order matrix system $\frac{dY}{dz} = L(z)Y$ on the torus. The matrix linear system is assumed to have regular singularities at $N$ points $z = t_1, \cdots, t_N$. The problem is to construct isomonodromic deformations that leave a set of monodromy data invariant while the position of the poles and the modulus $\tau$ being varied. As in the ordinary cases on the Riemann sphere, the deformation equations are expected to be written in the “Lax form”,

$$\frac{\partial L(z)}{\partial t_j} = [L(z), A_j(z)] - \frac{\partial A_j(z)}{\partial z},$$

$$\frac{\partial L(z)}{\partial \tau} = [L(z), A_\tau(z)] - \frac{\partial A_\tau(z)}{\partial z}.$$

Such an isomonodromic system may be thought of as a generalization of the Schlesinger equations.

I shall present two types of such isomonodromic systems. Both have an isospectral partner (the elliptic Calogero-Gaudin system / the ordinary elliptic Gaudin system), and are also related to a conformal field theory (the ordinary WZW model / its “twisted” version by Kuroki and Takebe) on a torus. It is well known that these physical systems are all governed by an $r$-matrix structure. This $r$-matrix structure is also a clue in the construction of the isomonodromic systems.
Isomonodromic system of Calogero-Gaudin type: The $L$-matrix is an $\text{SL}(n)$ matrix of the form

$$L(z) = \sum_{j=1}^{N} \sum_{a \neq b} \sigma_{q_a - q_b}(z - t_j) E_{ab} A_{j,ba}$$

$$+ \sum_{j=1}^{N} \sum_{a} \rho(z - t_j) E_{aa} A_{j,aa} + \sum_{a} p_a E_{aa}.$$

Here $A_{j,ab} (a, b = 1, \ldots, n)$ are classical $\text{SL}(n)$ spin variables under the constraint $\sum_{j=1}^{N} A_{j,aa} = 0$. $E_{ab}$ are the standard matrix units. $q_a$ and $p_a$ are the positions and momenta of Calogero-Moser particles. $\sigma_q(z)$ and $\rho(z)$ are the functions that appear in the dynamical $r$-matrix of Felder and Wieczerkowski:

$$r(z - w) = \sum_{a \neq b} \sigma_{q_a - q_b}(z - w) E_{ab} \otimes E_{ba} + \sum_{a} \rho(z - w) E_{aa} \otimes E_{aa}.$$

Isomonodromic system of Gaudin type: The $L$-matrix is an $\text{SL}(n)$ matrix of the form

$$L(z) = \sum_{j=1}^{N} \sum_{(\alpha, \beta) \neq (0,0)} w_{\alpha \beta}(z - t_j) J_{\alpha \beta} A_{j,\alpha \beta}.$$

Here $A_{j,\alpha \beta} (\alpha, \beta \in \mathbb{Z}_n)$ are classical $\text{SL}(n)$ spin variables in the different basis $J_{ab} = g^a h^b$ composed of “quantum torus” generators $g$ and $h$ ($gh = e^{2\pi i/n} hg$). $w_{\alpha \beta}$ are the building blocks of the classical limit of Belavin’s $\mathbb{Z}_n$-symmetric $R$-matrix:

$$r(z - w) = \sum_{(\alpha, \beta) \neq (0,0)} w_{\alpha \beta}(z - w) J_{\alpha \beta} \otimes J^{\alpha \beta}.$$