98-01 ??

~~Three~~ Four types of isomonodromic systems on $\Sigma = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$

0. Scalar type (Okamoto, Iwasaki; Kawai)
1. Gaudin type (new!)
2. Calogero-Gaudin type (Korotkin - Samtleben, Levin - Olshanetsky)
3. Calogero-Moser type (Levin - Olshanetsky)

Result: Explicit construction based on r-matrix and Hamiltonian structure

Concerns:

relation to CFT, KZ equations, spin systems, integrable systems, Whitham equations, Seiberg-Witten theory, ...

Isomonodromic System of Gaudin Type

(Lett. Math. Phys. 44 (1998) 143-156)

$$L(z) = \sum_{j=1}^{N} \sum_{(ab)} w_{ab}(z-t_j) J_{ab} A_j^{ab}.$$

$$A_j(z) = \operatorname*{res}_{w=t_j} \operatorname{tr}_2 (r(z-w)\, I \otimes L(w))$$

$$= \sum_{(ab)} w_{ab}(z-t_j) J_{ab} A_j^{ab}.$$

$A_j^{ab} = A_j^{ab}(t_1, \ldots, \tau)$

$$A_\tau(z) = -\frac{1}{2\pi\sqrt{-1}} \sum_{j=1}^{N} \sum_{ab} w_{ab}(z) \times$$

$$\times \left(\frac{\theta_{[ab]}'(z)}{\theta_{[ab]}(z)} - \frac{\theta_{[ab]}'(0)}{\theta_{[ab]}(0)} \right) J_{ab} A_j^{ab}.$$

$$r(z-w) = \sum_{(ab)} w_{ab}(z-w) J_{ab} \otimes J^{ab}$$

(Belavin's $\mathbb{Z}_n$-symmetric r-matrix)

- SU(n) only
- Related to CFT & spin system

  twisted WZW model (Kuroki & Takebe)

  elliptic KZ equation (Etingof)

  elliptic SU(n) Gaudin model (K&T)

Isomonodromic System of Calogero-Gaudin type

$$L(z) = \sum_{j=1}^{N} \sum_{a \neq b} \sigma_{q_a - q_b}(z - t_j)\, E_{ab} A_j^{ab} + \sum_{j=1}^{N} \sum_{a} \rho(z - t_j)\, E_{aa} A_j^{aa} + \sum_a p_a E_{aa} \qquad \left(\sum_{j=1}^{N} A_j^{aa} = 0 \right).$$

$$A_j(z) = \operatorname*{res}_{w = t_j} \mathrm{tr}_2 \left(r(z-w)\, I \otimes L(w) \right) = \sum_{a \neq b} \sigma_{q_a - q_b}(z - t_j)\, E_{ab} A_j^{ba} + \sum_a \rho(z - t_j)\, E_{aa} A_j^{aa}.$$

$$A_\tau(z) = -\frac{1}{2\pi\sqrt{-1}} \sum_{j=1}^{N} \sum_{a \neq b} \sigma_{q_a - q_b}(z - t_j)\, \times$$

$A_j^{ab} = A_j^{ab}(t_1, \cdots, \tau)$

$q_j = q_j(t_1, \cdots, \tau)$

$p_j = p_j(t_1, \cdots, \tau)$

$$\times \left(\rho(z - t_j - q_a + q_b) + \rho(q_a - q_b) \right) E_{ab} A_j^{ab} - \frac{1}{4\pi\sqrt{-1}} \sum_{j=1}^{N} \sum_a \left(\rho(z - t_j)^2 - \wp(z - t_j) \right) E_{aa} A_j^{aa}.$$

$$r(z - w) = \sum_{a \neq b} \sigma_{q_a - q_b}(z - w)\, E_{ab} \otimes E_{ba} + \sum_a \rho(z - w)\, E_{aa} \otimes E_{aa}$$

(Felder and Wieczerkowski's dynamical r-matrix)

q_a, p_a : Calogero-Moser particles

- general simple Lie algebra $\mathfrak{g}$:

$$q_a - q_b \to \alpha \cdot q$$
$$E_{ab} \to E_\alpha \quad (\alpha \in \Delta)$$

- Related CFT, Spin system

WZW model

KZB equation (Felder & Wieczerkowski)

Calogero - Gaudin system

(Nekrasov, Enriquez & Rubtsov)

and integrable system

generalized Hitchin system (Markman)

Korotkin & Samtleben

SU(2), explicit formula of

$L(z)$, $A_j(z)$ and $A_\tau(z)$

Levin & Olshanetsky

general $\mathfrak{g}$, general Σ,

$t_1, \cdots, t_N, \tau_1, \cdots, \tau_{3g-3}$

explicit form of $L(z)$ $(g=1)$

no explicit formula for $A_j(z)$ and $A_\tau(z)$

Isomonodromic System of Calogero-Moser Type

no spin variables (A's) (Levin & Olshanetsky problematical?)
no variable puncture (t's)

$$2\pi\sqrt{-1}\,\frac{\partial L(z)}{\partial \tau} - \frac{\partial M(z)}{\partial z} + [L(z), M(z)] = 0 \quad (*)$$

$$L(z) = \sum_{i=1}^{n} p_i E_{ii} + m \sum_{i\neq j} \sigma_{q_i - q_j}(z) E_{ij},$$

$$M(z) = m \sum_{i \neq j} \wp(q_i - q_j) E_{ii} +$$

$$+ m \sum_{i\neq j} \left(\rho(z - q_i + q_j) + \rho(q_i - q_j) \right) \sigma_{q_i - q_j}(z) E_{ij}.$$

$q_i = q_i(\tau)$, $p_i = p_i(\tau)$ (m : constant)

- isospectral version = elliptic Calogero-Moser
  $\tau \to \tau_0 = \text{const}$, $2\pi\sqrt{-1}\frac{\partial}{\partial \tau} \to \frac{d}{dt}$, $\frac{\partial M}{\partial z} \to 0$
  $q_i(\tau) \to q_i(t)$, $p_i(\tau) \to p_i(t)$

  $(*) \to$ gauge transformation of Krichever's Lax pair for elliptic CM system

- gauge equivalent to 1-point ($N=1$) Calogero-Gaudin (with special A_j's) ($SU(n)$ only?)

- generalizations to $SO(n)$, $Sp(n)$, etc (seemingly NOT equivalent to 1-point Calogero-Gaudin) and "twisted" versions (à la D'Hoker & Phong)

Hamiltonian Structure

Kostant-Kirillov bracket for A's

canonical Poisson bracket for p's and q's

$$\frac{\partial}{\partial t_j} = \{\,\cdot\,, H_j\},$$

$$\frac{\partial}{\partial z} = \{\,\cdot\,, H_z\}$$

(non-autonomous Hamiltonian system)

$$H_j = \operatorname*{res}_{z=t_j} \operatorname{tr} \frac{L(z)^2}{2}$$

H_z : depends on the type of the system

(explicit formulae for the three cases)

E.g. Gaudin type

$$\operatorname{tr} \frac{L(z)^2}{2} = \sum_{j=1}^{N} C_j \wp(z-t_j) + \sum_{j=1}^{N} H_j \zeta(z-t_j) + H_0$$

C_j : quadratic Casimir of $A_j^{ab} J_{ab}$

---

geometric origin of Hamiltonian structure

— Poincaré-Lefschetz duality of de Rham cohomology (Iwasaki, Kawai)