Elliptic Calogero-Moser Systems and Isomonodromic Deformations

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The Elliptic Calogero-Moser systems are a one-dimensional many-body problem with elliptic two-body potentials. The most classical case is the $A_{\ell-1}$ model defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{\ell} p_j^2 + \frac{g^2}{2}\sum_{j\neq k} \wp(q_j - q_k),$$

where $q_1, \cdots, q_\ell$ and $p_1, \cdots, p_\ell$ are the canonical coordinates and momenta, g a coupling constant, and $\wp$ the Weierstrass $\wp$ function with the primitive periods normalized to be $1, \tau$. The equations of motion can thus be written

$$\frac{dq_j}{dt} = \{q_j, \mathcal{H}\}, \quad \frac{dp_j}{dt} = \{p_j, \mathcal{H}\}.$$

"$A_{\ell-1}$" stands for the $A_{\ell-1}$ root system that underlies this model. Similarly, an elliptic Calogero-Moser system can be defined for each irreducible (but not necessary reduced) root system. Furthermore, for non-simply laced root systems, a kind of variants called "twisted model" and "extended twisted models" are also known.

Those elliptic Calogero-Moser systems are known to possess an isospectral Lax pair $L(z)$ and $M(z)$. z is an "elliptic spectral parameter", i.e., a parameter that lives on the torus $E_\tau = \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$. The equations of motion can be written in the Lax form

$$\frac{\partial L(z)}{\partial t} = [L(z), M(z)].$$

This, in particular, implies the existence of a maximal number of first integrals in involution, hence the Liouville integrability.

We consider a non-autonomous analogue of those elliptic Calogero-Moser systems obtained by formally replacing $d/dt \to 2\pi i d/d\tau$, e.g.,

$$2\pi i\frac{dq_j}{d\tau} = \{q_j, \mathcal{H}\}, \quad 2\pi i\frac{dp_j}{d\tau} = \{p_j, \mathcal{H}\},$$

for the $A_{\ell-1}$ model. Thus it is the modulus τ that plays the role of time variable in this system. Note that the Hamiltonian now depends on τ explicitly through the τ-dependence of $\wp(u) = \wp(u \mid 1, \tau)$.

Our observations are the following:

- For "many" models of the elliptic Calogero-Moser systems, one can find a suitable Lax pair $L(z)$ and $M(z)$ for which the non-autonomous analogue, too, can be written in the Lax form

$$2\pi i \frac{\partial L(z)}{\partial \tau} + \frac{\partial M(z)}{\partial z} = [L(z), M(z)].$$

- This Lax equation is the Frobenius integrability condition of the linear system

$$\frac{\partial Y(z)}{\partial z} = L(z)Y(z), \quad 2\pi i \frac{\partial Y(z)}{\partial \tau} + M(z)Y(z) = 0.$$

  This fact, combined with complex analytic properties of $L(z)$ and $M(z)$, implies that the non-autonomous system gives isomonodromic deformations of the linear ordinary differential system $dY(z)/dz = L(z)Y(z)$ on the torus E_τ.

- This also applies to the Inozemtsev system defined by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{\ell} p_j^2 + \frac{g^2}{2}\sum_{\epsilon,\epsilon'=\pm 1}\sum_{j\neq k}\wp(\epsilon q_j + \epsilon' q_k) + \sum_{j=1}^{\ell}\sum_{n=0}^{3} g_n^2 \wp(q_j + \omega_n),$$

  where $(\omega_0, \omega_1, \omega_2, \omega_3) = (0, \frac{1}{2}, \frac{1}{2} + \frac{\tau}{2}, \frac{\tau}{2})$. In particular, Manin's elliptic ("μ-equation") form of the sixth Painlevé equation,

$$(2\pi i)^2 \frac{d^2 q}{d\tau^2} = \sum_{n=0}^{3} \alpha_n \wp'(q + \omega_n),$$

  turns out to be an isomonodromic system on the torus E_τ.

For details, see: K. Takasaki, "Elliptic Calogero-Moser Systems and Isomonodromic Deformations", e-print `math.QA/9905101`.