Volterra-type hierarchies for specialized hypergeometric tau functions

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Hypergeometric tau function

Hypergeometric tau function (A. Orlov)

$$\mathcal{T}(s, \mathbf{t}, \bar{\mathbf{t}}) = \sum_{\lambda \in \mathcal{P}} S_\lambda(\mathbf{t}) g_\lambda(s) S_\lambda(-\bar{\mathbf{t}})$$

- s is the lattice coordinate, and $\mathbf{t} = (t_k)_{k=1}^\infty$ and $\bar{\mathbf{t}} = (\bar{t}_k)_{k=1}^\infty$ are the time variables of the 2D Toda hierarchy.
- $\mathcal{P}$ is the set of all partitions $\lambda = (\lambda_i)_{i=1}^\infty$.
- $S_\lambda(\mathbf{t})$ and $S_\lambda(\bar{\mathbf{t}})$ are the Schur functions.
- $g_\lambda(s)$'s are determined by a diagonal matrix $g = \mathrm{diag}(g_i)_{i=-\infty}^\infty \in \mathrm{GL}(\infty)$.

Hypergeometric tau function (cont'd)

- The Schur functions are defined as (effectively finite-dimensional) determinants:

$$S_\lambda(\mathbf{t}) = \det(S_{\lambda_i - i + j}(\mathbf{t}))_{i,j=1}^{\infty},$$
$$\sum_{n=0}^{\infty} S_n(\mathbf{t}) z^n = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right).$$

- $g_\lambda(s)$'s are the diagonal matrix elements $\langle \lambda, s | \hat{g} | \lambda, s \rangle$ of an operator $\hat{g}$ in the Fock space of 2D charged fermions. More explicitly,

$$g_\lambda(s) = \frac{\prod_{i=1}^{\infty} g_{\lambda_i - i + s + 1}}{\prod_{i=1}^{\infty} g_{-i+1}}.$$

Specialized hypergeometric tau function

Specialization of g

$$g = e^{\beta(\Delta - 1/2)^2/2} Q^{\Delta}, \qquad \Delta = \mathrm{diag}(i)_{i=-\infty}^{\infty}$$

- $g_\lambda(s)$ can be expressed as

$$g_\lambda(s) = \exp\left(\frac{\beta}{2}\left(\kappa(\lambda) + 2s|\lambda| + \frac{4s^3 - s}{12}\right)\right) Q^{|\lambda| + s(s+1)/2},$$

  where $|\lambda| = \sum_{i=1}^{\infty} \lambda_i$ and $\kappa(\lambda) = \sum_{i=1}^{\infty} \lambda_i(\lambda_i - 2i + 1)$, and β and Q are constants.
- The tau function $\mathcal{T}(s, \mathbf{t}, \bar{\mathbf{t}})$ is related to the double Hurwitz numbers of $\mathbb{CP}^1$ (A. Okounkov).

Further specialized tau function

$$\mathcal{T}(s, \mathbf{t}) = \mathcal{T}(s, \mathbf{t}, -\mathbf{c}) = \sum_{\lambda \in \mathcal{P}} S_\lambda(\mathbf{t}) g_\lambda(s) S_\lambda(\mathbf{c}).$$

This is a tau function of the lattice KP hierarchy. We are interested in the case where g is the foregoing special diagonal matrix and $\mathbf{c} = (c_k)_{k=1}^{\infty}$ takes the following particular values:

Particular values of c

$$\text{(i)} \quad c_k = \delta_{k,1} \qquad \text{(ii)} \quad c_k = \frac{q^{k/2}}{k(1 - q^k)}$$

Lattice KP hierarchy

Lattice KP hierarchy in Lax form

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad k = 1, 2, \ldots,$$
$$L = \Lambda + u_1 + u_2\Lambda^{-1} + \ldots, \quad \Lambda = e^{\partial_s},$$
$$B_k = (L^k)_{\geq 0} \quad \text{(non-negative powers of } \Lambda)$$

The Lax operator L can be expressed as

$$L = W\Lambda W^{-1}$$

in terms of the dressing operator

$$W = 1 + w_1\Lambda^{-1} + w_2\Lambda^{-2} + \ldots.$$

Logarithm and fractional/irrational powers of L

If the coefficients of L and W are defined for $s \in \mathbb{R}$ (this is indeed the case for our special tau functions), one can interpret Λ and Δ as operators

$$\Lambda = e^{\partial_s}, \quad \Delta = s$$

for functions of s. Accordingly, $\log\Lambda$ and Λ^α can be expressed as

$$\log\Lambda = \partial_s, \quad \Lambda^\alpha = e^{\alpha\partial_s}.$$

One can thereby define $\log L$ and L^α as

$$\log L = W \cdot \log\Lambda \cdot W^{-1}, \quad L^\alpha = W\Lambda^\alpha W^{-1}.$$

Volterra-type hierarchies in lattice KP hierarchy

Continuous Bogoyavlensky-Itoh hierarchy

$$\log L = \log \Lambda + u\Lambda^{-1}, \quad u = h(s, \mathbf{t}).$$

$N + 1$-step Bogoyavlensky-Itoh hierarchy

$$L^{1/(N+1)} = (1 + u\Lambda^{-1})\Lambda^{1/(N+1)}, \quad u = h(s, \mathbf{t}).$$

- These reduced forms of the Lax operator are preserved by the flows of the lattice KP hierarchy.
- The finite-step Bogoyavlensky-Itoh hierarchy is also known as the hungry Lotka-Volterra hierarchy. The $N = 1$ case amounts to the Volterra lattice.

Main results

Let $\mathcal{T}(s, \mathbf{t})$ be the specialized hypergeometric tau function with $g = e^{\beta(\Delta - 1/2)^2/2} Q^{\Delta}$ and $\bar{\mathbf{t}} = -\mathbf{c}$, and L the associated Lax operator of the lattice KP hierarchy.

Theorem 1

Let $\mathbf{c}$ be specialized to $c_k = \delta_{k,0}$. Then L takes the form

$$\log L = \log \Lambda + u\Lambda^{-1}, \quad u = u(s, \mathbf{t}),$$

of reduction to the continuous Bogoyavlensky-Itoh hierarchy.

- This tau function is a generating function of the single Hurwitz numbers of $\mathbb{CP}^1$.

Main results (cont'd)

Theorem 2

Let $\mathbf{c}$ be specialized to $c_k = q^{k/2}/(k(1-q^k))$, and β be expressed as $\beta = (N+1)\log q$, where N is a positive integer. Then L takes the form

$$L^{1/(N+1)} = (1 + u\Lambda^{-1})\Lambda^{1/(N+1)}, \quad u = u(s, \mathbf{t}),$$

of reduction to the $N+1$-step Bogoyavlensky-Itoh hierarchy.

- This tau function is related to the cubic Hodge integrals with $\tau = N$. This result seems to explain the observation of B. Dubrovin, S.-Q. Liu, D. Yang and Y. Zhang (arXiv:1612.02333) from a different point of view.

Outline of proof

- The dressing operator can be characterized by the factorization problem

$$\exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) g \exp\left(\sum_{k=1}^{\infty} c_k \Lambda^{-k}\right) = W^{-1}\bar{W},$$

  where $\bar{W} = \bar{w}_0 + \bar{w}_1\Lambda + \bar{w}_2\Lambda^2 + \cdots$.
- One can thereby find the initial value of W at $\mathbf{t} = \mathbf{0}$:

$$W|_{\mathbf{t}=\mathbf{0}} = e^{\beta(\Delta-1/2)^2/2} Q^{\Delta} \exp\left(-\sum_{k=1}^{\infty} c_k \Lambda^{-k}\right) Q^{-\Delta} e^{-\beta(\Delta-1/2)^2/2}.$$

Outline of proof (cont'd)

- One can compute the the initial values of $\log L$ and $L^{1/(N+1)}$ from the the foregoing initial value of W:
  (i) For $c_k = \delta_{k,1}$,
  $$\log L|_{\mathbf{t}=\mathbf{0}} = \log \Lambda + Qe^{\beta\Delta}\Lambda^{-1}.$$
  (ii) For $c_k = q^{k/2}/(k(1-q^k))$ and $\beta = (N+1)\log q$,
  $$L|_{\mathbf{t}=\mathbf{0}}^{1/(N+1)} = (1 - Qq^{-N-1/2}q^{(N+1)\Delta}\Lambda^{-1})\Lambda^{1/(N+1)}.$$
- These initial values take an anticipated form for reduction. This is enough to deduce that $\log L$ and $L^{1/(N+1)}$ take the reduced form at all times.