SUPERSYMMETRIC YANG-MILLS THEORIES AND TODA EQUATIONS *

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1 Introduction

The so called "Seiberg-Witten solution" gives an exact expression of the low energy effective action of the $N = 2$ supersymmetric SU(2) Yang-Mills theory [N. Seiberg and E. Witten, *Nucl. Phys.* **B462** (1994), 19-52; *Nucl. Phys.* **B431** (1994), 484-550]. Its building blocks are the following:

- A family of elliptic curves
- A meromorphic differential dS on these curves
- Period integrals of dS
- A "prepotential" $F = F(a)$

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The elliptic curves take a rather special form:

$$y^2 = (z^2 - \Lambda^4)(z - u),$$

where Λ and u are parameters. The parameter u is a coordinate in the moduli space of vacua of the $N = 2$ supersymmetric Yang-Mills theory. The differential dS is given by

$$dS = \frac{z - u}{y} dz.$$

Their period integrals

$$a = \oint_\alpha dS, \ a_D = \oint_\beta dS,$$

along the standard α and β cycles turn out to give units of the monopole and dyon mass spectrum. By the change of coordinate $u \to a$ of the moduli space, a_D may be thought of as a function of a. The prepotential $F = F(a)$ is then defined by

$$a_D = \frac{dF}{da}.$$

Its second derivative gives the modulus of the elliptic curve, $\tau = d^2F/da^2$.

Gorsky, Krichever, Marshakov, Mironov and Morozov pointed out that an integrable structure underlies the Seiberg-Witten solution [A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, *Phys. Lett.* **335B** (1995), 466-474]. According to their observation, the elliptic curve is nothing but the spectral curve of an elliptic solution of the KdV equation, and these special family of elliptic curves can be found in the work of Gurevich and Pitaevsky in the early seventies on *modulation* of elliptic solutions of the KdV equation.

A general recipe for the study of modulation of periodic nonlinear waves is the the so called "Whitham averaging method" [G.B. Whitham, Linear and nonlinear waves (Wiley, 1974)]. The averaging usually yields differential equations (called "modulation equation' or "Whitham equations") on parameters, such as the u above, of the periodic wave with respect to *slow variables.*

Modulation equations in integrable systems (soliton equations) possess a universal expression, the "Flaschka-Forest-McLaughlin equations" [H. Flaschka, M.G. Forest and D.W. McLaughlin, *Comm. Pure Appl. Math.* **33** (1980), 739-784]:

$$\frac{\partial}{\partial T_m} d\Omega_n = \frac{\partial}{\partial T_n} d\Omega_m.$$

Here T_n's are slow variables associated with the time variables t_n of a soliton hierarchy, and $d\Omega_n$'s are meromorphic differentials on the spectral curve in the construction of Baker-Akhiezer functions. One can also introduce a generating differential dS for which $d\Omega_n$'s are written

$$\frac{\partial}{\partial T_n} dS = d\Omega_n.$$

(Actually, I have cheated here. The derivatives in T_n should be understood as a connection.)

In fact, the ordinary slow variables are frozen in the Seiberg-Witten solution. The variable a now plays the role of a new slow variable. dS indeed satisfies the differential equation

$$\frac{\partial}{\partial a} dS = d\omega,$$

where $d\omega = dz/w$ is a holomorphic differential. If all the other T_n's are turned on, dS flows into a special solution ("homogeneous solution") of the full Whitham hierarchy.

In the following, I present a generalization of these results to SU(N). In the SU(N) theory, the Seiberg-Witten elliptic curve is replaced by the hyperelliptic spectral curve of an N-periodic Toda chain, and the affine SU(N) Toda field equation emerges as the stage of Whitham averaging . A link with the (generalized) Painlevé III equation is also speculated.

Most part of this work is published in the following papers.

- T. Nakatsu and K. Takasaki, Whitham-Toda hierarchy and $N = 2$ supersymmetric Yang-Mills theory, *Mod. Phys. Lett.* **11** (1996), 157-168; hep-th/9509162.

- K. Takasaki and T. Nakatsu, Isomonodromic Deformations and Supersymmetric Gauge Theories, *Int. J. Mod. Phys.* **A11** (1996), 5505-5518 ; hep-th/9603069.

2 Seiberg-Witten solution generalized to SU(n)

The Seiberg-Witten solution has been generalized in a variety of directions — $N = 2$ supersymmetric Yang-Mills theories with all unitary and orthogonal gauge groups, their $N = 4$ versions, $N = 2$ supersymmetric QCD (i.e., Yang-Mills + matters), etc. (cf. references cited in the above papers [Nakatsu-Takasaki, Takasaki-Nakatsu, loc.cit.]). In the following, we consider the Seiberg-Witten solution for the SU(N) theories.

In the SU(N) case, the aforementioned four fundamental ingredients of the Seiberg-Witten solution are replaced by the following [Argyres and A. Faraggi, *Phys. Rev. Lett.* **73** (1995), 3931-3934; A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, *Phys. Lett.* **344B** (1995), 169-175]:

- A hyperelliptic curve of the form

$$y^2 = P(z)^2 - \Lambda^{2N},\ P(z) = z^N + \sum_{k=0}^{N-2} u_{N-k} z^k.$$

- A meromorphic differential of the form

$$dS = \frac{z dP(z)}{y}.$$

- Period integrals along a symplectic basis α_j, β_j ($j = 1, \ldots, N-1$) of cycles,

$$a_j = \oint_{\alpha_j} dS,\ a_j^D = \oint_{\beta_j} dS.$$

- A prepotential $F = F(a_1, \ldots, a_{N-1})$ for which the two sets of periods are connected as

$$a_j^D = \frac{\partial F}{\partial a_j^D}.$$

3 Relation to Toda chain

As Martinec and Warner pointed out [E. Martinec and N. Warner, *Nucl. Phys.* **B459** (1996), 97-112], these hyperelliptic curves are spectral curves of the affine SU(N) (periodic) Toda chain, and the period integrals a_j and

a_j^D are related to action-angle variables [H. Flaschka and D.W. McLaughlin, *Prog. Theor. Phys.* **55** (1976), 438-456.] Martinec and Warner also considered the case of other gauge groups and observed a similar interpretation of generalized Seiberg-Witten curves as spectral curves of an associated Toda system. The correspondence is however somewhat strange: to the $N = 2$ Yang-Mills theory with gauge group G is associated a Toda chain of the "dual" group $\hat{G}$, the duality being a hypothetical "Langlands duality". Therefore it seems only for the A-D-E series that the correspondence is literally meaningful.

The relation to action-angle variables becomes manifest if one notices that dS can be rewritten

$$dS = z d \log h,$$

where

$$h = \frac{P(z) + y}{\Lambda^N}.$$

This looks different from the ordinary action integrals [Flaschka-McLaughlin, loc. cit.] which are line integrals of $|\log h| dz$, but in fact, period integrals of dS along a cycle encircling a cut can be rewritten into such a line integral.

The function h is a meromorphic function on the Toda spectral curve. Its inverse can be written

$$h^{-1} = \frac{P(z) - y}{\Lambda^N},$$

and from this fact, one can see that h has a pole and a zero of order N at the two points P_∞ and $\bar{P}_\infty$ at infinity (i.e., $z = \infty$), and no other pole or zero. h can be identified with the spectral parameter in the Lax representation $\dot{L} = [A, L]$, of the N-periodic Toda chain: L is the $N \times N$ matrix

$$L(h) = \begin{pmatrix} b_1 & c_1 & & & c_N h^{-1} \\ c_1 & b_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_{N-1} \\ c_N h & & & c_{N-1} & b_N \end{pmatrix},$$

and z and h are related by the eigenvalue equation

$$\det\Big(z - L(h)\Big) = 0.$$

It is this function h rather than z that plays a crucial role in our construction of Whitham equations.

4 Relation to affine Toda fields

Nakatsu and I, meanwhile, considered the generalized Seiberg-Witten solutions in the language of Whitham equations [Nakatsu-Takasaki, loc. cit.]. A clue is the fact that dS satisfies the following equations:

$$\frac{\partial}{\partial a_j} dS \bigg|_{h=const.} = d\omega_j,$$

where $d\omega_j$'s are holomorphic differentials (differentials of the first kind) normalized as $\oint_{\alpha_j} d\omega_k = \delta_{jk}$. "$h = const.$" means that dS is differentiated while h being kept constant; geometrically, this is a connection. (In the Whitham equations of the KdV hierarchy, the T-derivatives are understood to be leaving z constant.) These equations are obviously similar to the Whitham equations in the Flaschka-Forest-McLaughlin form. The only difference is the fact that the a-variables, as opposed to the T-variables in the KdV (and any other ordinary) case, are accompanied with holomorphic differentials. This suggests the existence of underlying Whitham dynamics generated by differentials of the second kind.

Our conclusion is that the hidden Whitham dynamics can be derived from the affine SU(N) Toda field hierarchy. The N-periodic Toda chain can be embedded into the Toda field equation as quasi-periodic (finite-band) solutions. We constructed an associated Whitham hierarchy in the framework developed by Dubrovin [B.A. Dubrovin, *Commun. Math. Phys.* **145** (1992), 195-207; *Nucl. Phys.* **B379** (1992), 627-689] and Krichever [I.M. Krichever, Topological minimal models and soliton equations, Talk at the 1st Sakharov Conference, Moscow 1991; *Commun. Pure and Appl. Math.* **XLVII** (1994), 437-475] .

4.1 Construction of Whitham hierarchy

Besides the a-variables, our Whitham hierarchy has two series of slow variables T_n and $\bar{T}$ and associated meromorphic differentials $d\Omega_n$ and $d\bar{\Omega}_n$. They stem from the two series of commuting flows (t_n and $\bar{t}_n$, $n = 1, 2, \ldots$) in the

Toda field equations. (Actually, the kN-th flows, $k = 1, 2, \ldots$, become trivial.) The extended Whitham hierarchy is derived from the master equations

$$\frac{\partial}{\partial a_j} dS \bigg|_{h=const.} = d\omega_j, \ \frac{\partial}{\partial T_n} dS \bigg|_{h=const.} = d\Omega_n, \ \frac{\partial}{\partial \bar{T}_n} dS \bigg|_{h=const.} = d\bar{\Omega}_n.$$

The meromorphic differentials $d\Omega_n$ and $d\bar{\Omega}_n$ are determined in the following conditions.

- $d\Omega_n$ is holomorphic except at P_∞ ($h = \infty$), and the behavior in a neighborhood of P_∞ is such that

$$d\Omega_n = dh^{n/N} + \text{holomorphic}.$$

  Similarly, $d\bar{\Omega}_n$ is holomorphic except at $\bar{P}_\infty$ ($h = 0$), and in a neighborhood of $\bar{P}_\infty$,

$$d\bar{\Omega}_n = dh^{-n/N} + \text{holomorphic}.$$

- The period integrals along the α-cycles vanish:

$$\oint_{\alpha_j} d\Omega_n = 0, \ \oint_{\alpha_j} d\bar{\Omega}_n = 0.$$

Besides these differentials of the second kind, one can add a differential of the third kind, $d\Omega_0$ and an associated time variable T_0. $d\Omega_0$ is determined by the following conditions.

- $d\Omega_0$ is holomorphic except at P_∞ and $\bar{P}_\infty$, and

$$d\Omega_0 = d \log h^{1/N} + \text{holomorphic}$$

  in a neighborhood of P_∞ and $\bar{P}_\infty$, respectively.

- The period integrals along the α-cycles vanish:

$$\oint_{\alpha_j} d\Omega_0 = 0,$$

4.2 Characterization of Seiberg-Witten solution

Of course the above hierarchy has infinitely many solutions; the Seiberg-Witten solution of the supersymmetric SU(N) Yang-Mills theory corresponds to a special solution. This solution can be characterized as a "homogeneous solution" in the general framework of Dubrovin and Krichever. This is a solution for which dS is written

$$dS = \sum_{j=1}^{N-1} a_j d\omega_j + \sum_{n=0}^{\infty} T_n d\Omega_n + \sum_{n=1}^{\infty} \bar{T}_n d\bar{\Omega}_n.$$

Following Krichever, we require

$$Q := \frac{dS}{d\log h}$$

to have no singularity at the critical points of h (i.e., at the points where $dh = 0$). The ciritial points are given by the zeroes of $dP(z)/dz = 0$. This gives $N-1$ equations to the $N-1$ coefficients u_j of $P(z)$ that parametrize the Toda spectral curve, thus determines them (in a very implicit way) as functions of a, T and $\bar{T}$.

The original meromorphic differential of the generalized Seiberg-Witten solution can be reproduced by setting $T_1 = 1$, $\bar{T}_1 = -1$, and all other T and $\bar{T}$ variables zero. This is very similar to the situation in the interpretation of "topological Landau-Ginzburg models" by Dubrovin and Krichever [loc. cit.].

4.3 Characterization of prepotential F

The notion of prepotential can be defined in a quite general context, as demonstrated by Dubrovin and Krichever [loc. cit.]. In the present setting, the prepotential F is defined, up to an additive constant, by the differential equations

$$\begin{aligned} \frac{\partial F}{\partial a_j} &= \frac{1}{2\pi i} \oint_{\beta_j} dS, \\ \frac{\partial F}{\partial T_n} &= \frac{1}{2\pi i} \oint_{P_\infty} h^{n/N} dS, \end{aligned}$$

$$
\begin{aligned}
\frac{\partial F}{\partial \bar{T}_n} &= \frac{1}{2\pi i}\oint_{\bar{P}_\infty} h^{-n/N} dS, \\
\frac{\partial F}{\partial T_0} &= -\frac{1}{2\pi i}\oint_{P_\infty} \log h dS - \frac{1}{2\pi i}\oint_{\bar{P}_\infty} \log h dS.
\end{aligned}
$$

The Frobenius integrability of these equations is ensured by Riemann's bilinear relations. F emerges as a "quasi-classical part" of the τ function in the Whitham averaging (= nonlinear JWKB-Born-Oppenheimer approximation):

$$\tau = e^{\epsilon^{-2}F} \times \text{combination of theta functions.}$$

The prepotential for the "homogeneous solution" is homogeneous of degree two:

$$2F = \sum_{j=1}^{N-1} a_j \frac{\partial F}{\partial a_j} + \sum_{n=0}^{\infty} T_n \frac{\partial F}{\partial T_n} + \sum_{n=1}^{\infty} \bar{T}_n \frac{\partial F}{\partial \bar{T}_n}.$$

This is also a property shared by the prepotentials of two-dimensional topological field theories [Dubrovin, Krichever, loc. cit.].

5 Interpretation in terms of isomonodromic problems

Whitham equations emerge in adiabatic deformations of an isospectral problem. In the above problem, the isospectral problem is the dynamics of an N-periodic Toda chain (embedded into the Toda field equation). Although we have specified a Whitham hierarchy underlying the (generalized) Seiberg-Witten solution, the origin of the Whitham dynamics still remains mysterious.

Our second paper [Takasaki-Nakatsu, loc. cit.] is an attempt to derive the Whitham dynamics from an isomonodromic problem. Let us first consider this issue in a somewhat general form, then turn to the above setting, and present a direction of further applications.

5.1 General recipe step 1: WKB analysis

Suppose that we are given an isospectral problem written in a Lax form:

$$\frac{\partial Q(\lambda)}{\partial t_n} = [P_n(\lambda), Q(\lambda)], \quad \left[\frac{\partial}{\partial t_n} - P_n(\lambda), \frac{\partial}{\partial t_m} - P_m(\lambda)\right] = 0,$$

where $P_n(\lambda)$ and $Q(\lambda)$ are suitably selected matrix-valued rational curve. The associated linear problem

$$z\psi(\lambda) = Q(\lambda)\psi(\lambda), \quad \frac{\partial \psi(\lambda)}{\partial t_n} = P_n(\lambda)\psi(\lambda)$$

is integrable in the sense of Frobenius, and $\psi(\lambda)$ can be characterized as a "Baker-Akhiezer function" on the spectral curve

$$\det\Big(z - Q(\lambda)\Big) = 0.$$

Our idea is to replace the above isospectral problem by an isomonodromic problem, and to apply an old idea of Flaschka and Newell [H. Flaschka and A.C. Newell, Commun. Math. Phys. **76** (1980), 65-116; Physica **3D** (1981), 203-221], which was later refined by Novikov [S.P. Novikov, Funct. Anal. Appl. **24** (1990), 296-306]. The isomonodromic problem is given by

$$\epsilon \frac{\partial \Psi(\lambda)}{\partial \lambda} = Q(\lambda)\Psi(\lambda), \quad \frac{\partial \Psi(\lambda)}{\partial t_n} = P_n(\lambda)\Psi(\lambda).$$

Here ϵ is a small parameter that plays the role of the Planck constant in the subsequent (formal) "WKB analysis". $\Psi(\lambda)$ is not the same as $\psi(\lambda)$, though connected with $\psi(\lambda)$ by a simple relation as we shall show below.

The idea of Flaschka and Newell and of Novikov is to put $\Psi(\lambda)$ in a WKB form (as $\epsilon \to 0$):

$$\Psi(\lambda) = \Big(\phi(\lambda) + \sum_{n=1}^{\infty} \epsilon^n \phi_n(\lambda)\Big) \exp\Big(\epsilon^{-1} S(\lambda)\Big).$$

In the leading order of this ϵ-expansion, the ODE in z of the above isomonodromic problem gives the algebraic equation

$$\frac{\partial S(\lambda)}{\partial z}\phi(\lambda) = Q(\lambda)\phi(\lambda)$$

for the "amplitude" $\phi(\lambda)$. If we identify

$$z = \frac{\partial S(\lambda)}{\partial \lambda},$$

this algebraic equation gives essentially the same eigenvalue problem as in the previous isospectral problem. This relation can be rewritten

$$dS(\lambda) = z d\lambda.$$

If we identify $\lambda = \log h$, this is nothing but the dS of the generalized Seiberg-Witten differential.

5.2 General recipe step 2: multiscale analysis

The next issue is to derive the Whitham dynamics. This is treated by the concept of "multiscale analysis".

Following the idea of multiscale analysis, we introduce two sets of variables t and T connected by the relation

$$\epsilon t_n = T_n,$$

and assume that that all quantities in this problem are functions of variables t and T. The two sets of time variables represent fast and slow time scales. Derivatives of the fields can be written as a sum of contributions from these two scales:

$$\frac{\partial}{\partial t_n} u_\alpha(t, \epsilon t) = \frac{\partial u_\alpha(t,T)}{\partial t_n} + \epsilon \frac{\partial u_\alpha(t,T)}{\partial T_n}\bigg|_{T=\epsilon t}.$$

Thus the coefficient matrices $P_n(\lambda)$ and $Q(\lambda)$ are assumed to be functions of (t, T),

$$P_n(\lambda) = P_n(t, T, \lambda),\ Q(\lambda) = Q(t, T, \lambda).$$

(To emphasize the roles of t and T, we write all independent variables explicitly.) We now look for $\Psi(\lambda)$ of the form

$$\Psi(\lambda) = \Big(\phi(t, T, \lambda) + \sum_{n=1}^{\infty} \epsilon^n \phi_n(t, T, \lambda)\Big) \exp\Big(\epsilon^{-1} S(T, \lambda)\Big).$$

Note that $S(T, \lambda)$ is assumed to be t-independent. This is an essential ansatz in the following calculations.

Now, from the leading order of ϵ-expansion, $\phi(t,T,\lambda)$ and $S(t,T,\lambda)$ turn out to satisfy the equations

$$\begin{aligned} \frac{\partial S(T,\lambda)}{\partial z}\phi(t,T,\lambda) &= Q(t,T,\lambda)\phi(t,T,\lambda), \\ \frac{\partial \phi(t,T,\lambda)}{\partial t_n} + \frac{\partial S(T,\lambda)}{\partial T_n}\phi(t,T,\lambda) &= P_n(t,T,\lambda)\phi(t,T,\lambda). \end{aligned}$$

These equations can be further converted into a more familiar form

$$\begin{aligned} z\psi(t,T,\lambda) &= Q(t,T,\lambda)\psi(t,T,\lambda), \\ \frac{\partial \psi(t,T,\lambda)}{\partial t_n} &= P_n(t,T,\lambda)\psi(t,T,\lambda). \end{aligned}$$

where we have defined

$$\psi(\lambda) := \phi(\lambda)\exp\left(\sum t_n \frac{\partial S(\lambda)}{\partial T_n}\right).$$

Thus an isospectral problem has been derived from the isomonodromic problem.

In general, the Baker-Akhiezer function of this type of isospectral problem can be written

$$\psi(\lambda) = \phi(\lambda)\exp\Big(\sum t_n \Omega_n(\lambda)\Big),$$

where Ω_n are primitive functions of meromorphic differentials $d\Omega_n$ on the spectral curve,

$$\Omega_n(\lambda) = \int^\lambda d\Omega_n,$$

and $\phi(\lambda)$ is comprised of Riemann theta functions.

Assuming that the two expressions for $\psi(\lambda)$ coincide, one obtains the equations

$$\frac{\partial S(\lambda)}{\partial T_n} = \Omega_n(\lambda),$$

which are the master equations of the Whitham hierarchy. One can thus derive the Whitham equation from the isomonodromic problem.

5.3 Back to Seiberg-Witten solution

Applying the above recipe to the generalized Seiberg-Witten solution is now straightforward. The affine SU(N) Toda field hierarchy can be written

$$\begin{aligned}\left[\frac{\partial}{\partial t_n} - A_n(h), \frac{\partial}{\partial t_m} - A_m(h)\right] &= 0,\\ \left[\frac{\partial}{\partial \bar{t}_n} - \bar{A}_n(h), \frac{\partial}{\partial \bar{t}_m} - \bar{A}_m(h)\right] &= 0,\\ \left[\frac{\partial}{\partial t_n} - A_n(h), \frac{\partial}{\partial \bar{t}_m} - \bar{A}_m(h)\right] &= 0,\end{aligned}$$

where $A_n(h)$ and $\bar{A}_n(h)$ are $N \times N$ matrices. The L matrix of the periodic Toda chain is related to $A_1(h)$ and $\bar{A}_1(h)$ as:

$$L(h) = A_1(h) + \bar{A}_1(h).$$

The associated linear problem is given by

$$\frac{\partial \Psi}{\partial t_n} = A_n(h)\Psi, \quad \frac{\partial \Psi}{\partial \bar{t}_n} = \bar{A}_n(h)\Psi.$$

To define an isomonodromic problem, we follow a well known prescription, namely, we impose the following homogeneity condition to each component of $\Psi = {}^t(\psi_1, \ldots, \psi_N)$:

$$\lambda \frac{\partial \psi_j}{\partial \lambda} = \left(\sum_{n=1}^{\infty} n t_n \frac{\partial}{\partial t_n} + j - \sum_{n=1}^{\infty} n \bar{t}_n \frac{\partial}{\partial \bar{t}_n}\right)\psi_j$$

This forces the matrix elements of $A_n(h)$ and $\bar{A}_n(h)$ to be also homogeneous.

If all t_n and $\bar{t}_n$ other than t_1 and $\bar{t}_1$ are set to zero, this gives a generalization of the third Painlevé equation. An infinite series of isomonodromic problems can be obtained by leaving more nonzero variables, such as $t_1, \ldots, t_r$ and $\bar{t}_1, \ldots, \bar{t}_{\bar{r}}$.

By the aforementioned general recipe, one can reproduce the Whitham hierarchy for the (generalized) Seiberg-Witten solution.

5.4 Further generalizations and issues

Applying this recipe to other isomonodromic problems is now under investigation. This recipe, meanwhile, is still heuristic and requires a more rigorous justification.

The Schlesinger equations will be a nice laboratory for testing the above ideas. (In fact, as Flaschka and Newell [loc. cit.] noted, this issue is related to Garnier's old work early in this century.) To this end, I propose to reformulate the problem in an ϵ-dependent form as:

$$\epsilon\frac{\partial\Psi}{\partial\lambda} = \sum_{j=1}^{N}\frac{A_j}{\lambda - T_j}\Psi, \quad \epsilon\frac{\partial\Psi}{\partial T_j} = -\frac{A_j}{\lambda - T_j}\Psi.$$

Note that I have replaced the ordinary deformation parameters t_j on the right hand side by their "slow" versions. This reformulation of the Schlesinger equation is more suited for applying the previous recipe. (I owe this idea to papers by V. Verecshagin[hep-th/9409131, hep-th/9411211,hep-th/9605092], who treated the six Painlevé equations in this modified form.)

An even more interesting issue is to extend the previous results to isomonodromic problems on Riemann surfaces. The Seiberg-Witten solutions of $N = 4$ supersymmetric Yang-Mills equations are known to be described by a "spectral cover" $C \to E$ over an elliptic curve E [R. Donagi and E. Witten, *Nucl. Phys.* **B460** (1996), 299-334; Martinec, *Phys. Lett.* **B367** (1996), 91-96; H. Itoyama and A. Morozov, hep-th/9511126, hep-th/9512161]. If this case is also related to an isomonodromic problem, the isomonodromic problem should be defined on the elliptic curve E.