# An Infinite Number of Hamiltonian Flows Arising from Hyper-Kähler Metrics

Kanehisa Takasaki
Research Institute for Mathematical Sciences, Kyoto University

One of the major properties of nonlinear integrable systems is the existence of an infinite number of commuting 'flows' that include the original equation, written usually in an evolutionary system, as a special sector. The system of all such flows is called a 'hierarchy' (such as the 'KdV hierarchy', the 'KP hierarchy', *etc.*). Mathematical structures concerning 'integrability' often become more understandable at the level of such a hierarchy rather than the original form with only a finite number of independent variables. The hierarchy structure plays a particularly crucial role in the study of the so called '$\tau$ function' (or 'Hirota's dependent variable). For example, the KP hierarchy can be converted into an infinite system of bilinear equations (in Hirota's form) with just one unknown function, which is nothing other than the $\tau$ function. The introduction of the $\tau$ function and these hidden independent variables is a key to lead us to richer representation-theoretical understandings of nonlinear integrable systems.

Motivated by these developments in soliton theory in the early eighties, I have pursued possibilities of extending the theory of $\tau$ function to higher dimensional cases. The most familiar examples of higher higher dimensional nonlinear integrable systems would be the self-duality equations in both gauge and gravitation theories. These have been known to share several remarkable properties with nonlinear systems in soliton theory. As to the case of self-dual gauge fields I have almost lost my previous hope to find an analogue of the $\tau$ function. On the other hand it turned out recently that self-dual metrics as well as their $4k\,(k = 1, 2, \ldots)$ dimensional extensions ('hyper-Kähler metrics') do have potential functions which are very similar to the $\tau$ function in soliton theory. In fact, these functions have been basically known under the name of '(first and second) key functions' due to Plebanski; besides, the first key function is exactly a Kähler potential in an appropriately chosen coordinate system. The point is that after the introduction of an infinite number of new independent variables (and an associated hierarchy) such a key function becomes a true potential of the whole system, i.e. all unknown functions of the hierarchy are written as first order derivatives of such a key function. Of course this is quite parallel to the role of the key functions in the original setting of Plebanski, but one should note that we are now dealing with an infinite number of variables (both independent and dependent) and differential equations.

Let me show what the hierarchy for hyper-Kähler metrics looks like. There are actually several different (but mutually equivalent) expressions of this hierarchy; the most fundamental one is given by an exterior differential equation of the form

$$\epsilon_{AB}\, du^A(\lambda) \wedge du^B(\lambda) = \epsilon_{AB}\, d\hat{u}^A(\lambda) \wedge d\hat{u}^B(\lambda),$$

where $u^A(\lambda)$ and $\hat{u}^A(\lambda)$ $(A = 1, \ldots, 2k)$ are (formal) Laurent series of the form

$$u^A(\lambda) = \sum_{n=-\infty}^{\infty} u_n^A \lambda^n, \hat{u}^A(\lambda) = \sum_{n=-\infty}^{\infty} \hat{u}_n^A \lambda^n$$

and $\epsilon_{AB}$ denotes the standard symplectic form, $\epsilon_{12} = -\epsilon_{21} = \ldots = \epsilon_{2k-1\,2k} = -\epsilon_{2k\,2k-1} = 1, \epsilon_{AB} = 0$ for the other indices. $\lambda$ is a 'spectral parameter', on which the total diffentiation $d$ acts trivially, $d\lambda = 0$. Further we require, in the above equation, that $u_n^A$ and $\hat{u}_{-n-1}^A$ for $n \geq 0$ be independent variables and the others dependent ones (*i.e.* unknown functions). Among these only $u_0^A$ and $u_1^A$ may be identified with space-time coordinates in the ogiginal geometric setting. In fact, the above separation of all the variables into independent/dependent ones is by no means unique; there are an infinite number of different ways, each of which provides a different interpretation of the same exterior differential equation.

Flows in a Hamiltonian form occur if one rewrites the above exterior differential equation into a system of partial differential equations. Then one obtains the following:

$$\frac{\partial w^B(\lambda)}{\partial u_n^A} + \{(\lambda^n u_A(\lambda))_{\geq 0}, w^B(\lambda)\} = 0 \quad (n \geq 0)$$
$$\frac{\partial w^B(\lambda)}{\partial \hat{u}_n^A} + \{(\lambda^n \hat{u}_A(\lambda))_{\leq -1}, w^B(\lambda)\} = 0 \quad (n \leq -1)$$

for $w^A = u^A$ and $\hat{u}^A$, and

$$\{u^A, u^B\} = \{\hat{u}^A, \hat{u}^B\} = \in^{AB},$$

where $(\quad)_{\geq 0}$ and $(\quad)_{\leq -1}$ respectively denote the operation to extract powers of $\lambda$ for the exponents indicated therein, *i.e.*

$$(\sum_{n=-\infty}^{\infty} a_n \lambda^n)_{\geq 0} = \sum_{n \geq 0} a_n \lambda^n,$$
$$(\sum_{n=-\infty}^{\infty} a_n \lambda^n)_{\leq -1} = \sum_{n \leq -1} a_n \lambda^n;$$

the Poisson bracket of two functions means

$$\{f, g\} = \in^{AB} \frac{\partial f}{\partial u_0^A} \frac{\partial g}{\partial u_0^B};$$

symplectic indices $A, B, \ldots$ are raised and lowered as $\xi_A = \in_{AB} \xi^B, \eta^B = \eta_A \in^{AB}$ $(\in^{AB} = \in_{AB})$.

From the above exterior differential equation one, in particular, finds that the right hand side of the following equation is a closed form:

$$d\Theta = \sum_{n \geq 0} \in_{AB} u_{-n-1}^A du_n^B - \sum_{n \leq -1} \in_{AB} \hat{u}_{-n-1}^A d\hat{u}_n^B$$

therefore such a potential $\Theta$ does exists (at least locally). This is a natural extension of Plebański's second key function. As one sees readily, all the dependent variables of the system are now written as derivatives of $\Theta$. The whole hierarchy can be now rewritten as a system of bilinear differential equations on the new dependent variable $\Theta$. Although this does not take the form of Hirota's bilinear equations, the $\Theta$ thus introduced has several interesting properties which are quite similar to the $\tau$ function.

The first key function can be defined in much the same way, but with a different choice of dependent/independent variables in the funcdamental exterior differential equation. To be more precise, one then chooses $u^A_{n+1}$ and $\hat{u}^A_{-n}$ for $n \geq 0$ as indepenedent variables. (As mentioned above, this is simply a change of the 'reference frame' and is always permitted.) Then one can derive another Hamiltonian form of the same hierarchy. The first key function is now defined as a solution of the following equation:

$$d\Omega = -\sum_{n\geq 0} \in_{AB} u^A_{-n} du^B_{n+1} + \sum_{n\leq -1} \in_{AB} \hat{u}^A_{-n} d\hat{u}^B_{n+1}$$

The key functions thus extended to the hierarchy acquire several interesting properties which were 'hidden' behind the orginary geometric setting. An application lies in the analysis of some classes of special solutions. Another significant result is that hidden symmetries of the hierarchy, which forms a Lie algebra isomorphic to { Laurent series } $\otimes$ { infinitesimal canonical transformations in $2k$ dimensions }, can be naturally lift up to the level of the key functions without any 'anomaly'. This circumstance is quite different from the case of the ordinary $\tau$ function. In the latter case a similar construction is made with the aid of the Riemann-Hilbert problem, but its lift to the level of the $\tau$ function is always suffered with the occurence of 'commutator anomalies' (as a cocycle of a loop algebra), hence one is forced to consider a central extension of a loop algebra. This suggests a substantial difference between the two notions. Such a behavior of the key functions under hidden symmetries can be re-explained from the point of view of symplectic/contact geometry.

For further details, see: K. Takasaki, Kyoto preprints RIMS-521, RIMS-525 (May, June 1988).