Integrable Systems in Gauge Theory, Kähler Geometry and Super KP Hierarchy – Symmetries and Algebraic Point of View

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1 Introduction

The following nonlinear systems all provide valuable material to search for new "nonlinear integrable systems".

- self-duality equation in Yang-Mills theory
- self-duality equation in Kähler geometry
- super Kadomtsev-Petviashvili (KP) hierarchy

From these equations, one will be able to imagine several types of extensions of so called "soliton equations" such as the celebrated Korteweg-de Vries (KdV) equation etc. The first two cases are in a sense "higher dimensional" (or "multi-dimensional") nonlinear integrable systems; the last case will be interesting as an extension of M. Sato's work on the KP hierarchy [SS] and background ideas [S] referred to under the key words "algebraic analysis."

This lecture is a summary of my recent work on these equations, in particular, the self-duality equations, with focus on their symmetry properties. It is nowadays widely recognized that symmetries of soliton equations can be described by representation theory of Kac-Moody Lie algebras [DJKM]. A similar observation to the self-duality equation of Yang-Mills theory has been known for years [UN], [CGW], [D], [T1]. The case of the self-duality equation in Kähler geometry seems to have remained less obscure [BP]. Recently I obtained an explicit description of infinitesimal symmetries, which exhibits a Poisson algebra structure [T2]. Very recently, inspired by work of Leznov et al. [LMS], I noticed that these infinitesimal symmetries can be "exponentiated" by a simple method [T3]. This leads to a kind of "perturbative" construction of a class of general (local) solutions. To stress underlying symplectic structures, I will illustrate these results for a $4N$-dimensional generalization of the self-duality equations rather than in the original form.

The basic standpoint of my work largely relies on the philosophy of "algebraic analysis," which understands differential equations as a differential algebra, i.e., a set of abstract symbols and differential-algebraic relations among them. This language has turned out to be particularly useful [T4] in the case of the super KP hierarchy of Manin and Radul [MR] as well as the original KP hierarchy. For the treatment of the self-duality equations, we shall not specify such a differential-algebraic interpretation; however, its spirit is included therein.

2 Generalized Self-Duality Equations

2.1 The Case of Yang-Mills Theory

We consider a $4N$-dimensional space-time with coordinates

$$(x, p) = (x^1, \ldots, x^{2N}, p^1, \ldots, p^{2N}) \tag{1}$$

and a generalized self-duality equation of Yang-Mills theory on this space-time. This equation, as in the four dimensional case, has two equivalent expressions [C]. As we shall see later on, these two expressions have analogues in Kähler geometry. The first expression is given by the equations

$$\frac{\partial^2 K}{\partial x^i \partial p^j} - \frac{\partial^2 K}{\partial x^j \partial p^i} + \left[\frac{\partial K}{\partial x^i}, \frac{\partial K}{\partial x^j}\right] = 0, \tag{2}$$

where the unknown function $K = K(x, p)$ takes values in the Lie algebra Lie$G$ of the structure group G. The second one is given by

$$\frac{\partial}{\partial x^i}\left(\frac{\partial J}{\partial p^j} J^{-1}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial J}{\partial p^i} J^{-1}\right) = 0, \tag{3}$$

where the unknown function $J = J(x, p)$ now takes values in G.

As well known, these equations are the integrability condition (in the sense of Frobenius) of the linear system

$$\left(\frac{\partial}{\partial p^j} - \lambda \frac{\partial}{\partial x^i} + A_i\right) \Psi(\lambda) = 0. \tag{4}$$

The gauge potentials A_i are combined with the previous unknown functions J and K as:

$$A_i = -\frac{\partial K}{\partial x^i} = -\frac{\partial J}{\partial p^i} J^{-1}. \tag{5}$$

We consider, in particular, a special pair of solutions

$$\begin{aligned} \Psi(\lambda) &= W(\lambda), \quad W(\lambda) = 1 + \sum_{n \le -1} W_n \lambda^n, \\ \Psi(\lambda) &= V(\lambda), \quad V(\lambda) = \sum_{n \ge 0} V_n \lambda^n \end{aligned} \tag{6}$$

connected with J and K by the relation

$$K = -W_{-1}, \quad J = V_0. \tag{7}$$

The linear system, with these expressions inserted, gives rise to a *nonlinear* system with the new unknown functions W_n and V_n. Symmetries are to be constructed for this nonlinear system rather than the original equation.

2.2 The Case of Kähler Geometry

We now turn to Kähler geometry. Our notational conventions are as follows. Let $i, j, \ldots$ be symplectic indices with values in integers $1, \ldots, 2N$. $\in^{ij}$ and $\in_{ij}$ denote the standard symplectic $\in$-symbols normalized as $\in_{2i-1,2i} = - \in_{2i,2i-1} = 1$ and $\in^{2i-1,2i} = - \in^{2i,2i-1} = 1$. The Einstein summation convention is understood only for symplectic indices. Symplectic indices are raised and lowered as $\xi_i = \in_{ij} \xi^j$ and $\eta^j = \eta_i \in_{ij}$.

A $4N$-dimensional generalization of the self-duality equation in Kähler geometry is provided by hyper-Kähler geometry. As pointed out (or re-discovered) by Plebanski [P] in the four dimensional (self-dual) case, hyper-Kähler geometry (also called "$\mathcal{H}$-space") has two equivalent local pictures based upon the first and second "heavenly equations." The "second" picture consists of a $4N$-dimensional coordinate system $(x, p) = (x^1, \ldots, x^{2N}, p^1, \ldots, p^{2N})$, a scalar unknown function $\Theta = \Theta(x, p)$, and the "second heavenly equation"

$$\frac{\partial^2 \Theta}{\partial x^i \partial p^j} - \frac{\partial^2 \Theta}{\partial x^j \partial p^i} + \left\{ \frac{\partial \Theta}{\partial x^i}, \frac{\partial \Theta}{\partial x^j} \right\}_{(x)} = 0, \tag{8}$$

where $\{ \quad , \quad \}_{(x)}$ stands for the Poisson bracket in x,

$$\{F, G\}_{(x)} = \in^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}. \tag{9}$$

In the "first" picture, one has a $4N$-dimensional coordinate system $(p, \hat{p}) = (p^1, \ldots, p^{2N}, \hat{p}^1, \ldots, \hat{p}^{2N})$, a scalar unknown function $\Omega = \Omega(p, \hat{p})$, and the "first heavenly equation"

$$\left\{ \frac{\partial \Omega}{\partial p^i}, \frac{\partial \Omega}{\partial p^j} \right\}_{(\hat{p})} = \in_{ij}, \tag{10}$$

where we use another Poisson bracket,

$$\{F, G\}_{(\hat{p})} = \in^{ij} \frac{\partial F}{\partial \hat{p}^i} \frac{\partial G}{\partial \hat{p}^j}. \tag{11}$$

Geometrically, Ω represents a Kähler potential, and p^i and $\hat{p}_i$ correspond to complex coordinates and their complex conjugate. In the following, however, we understand $(p, \hat{p})$ or (x, p) as $4N$ *independent* complex variables and consider formal aspects of the above differential equations.

The role of $W(\lambda)$ and $V(\lambda)$ is now to be played by two sets of functions (or formal Laurent series)

$$\begin{aligned} u^i(\lambda) &= \sum_{n \le -1} u^i_n \lambda^n + x^i + p^i \lambda \quad (1 \le i \le 2N), \\ \hat{u}^i(\lambda) &= \hat{p}^i + \sum_{n \ge 1} \hat{u}^i_n \lambda^n, \quad (1 \le i \le 2N) \end{aligned} \tag{12}$$

subject to the exterior differential equations

$$\in_{ij}\, du^i(\lambda) \wedge du^j(\lambda) = \in_{ij}\, d\hat{u}^i(\lambda) \wedge d\hat{u}^j(\lambda), \tag{13}$$

and

$$\begin{aligned} d\Theta &= \in_{ij}\, u^i_{-2} dp^j + \in_{ij}\, u^i_{-1} dx^i, \\ d\Omega &= -\in_{ij}\, u^i_0 dp^j + \in_{ij}\, \hat{u}^i_1 d\hat{p}^j. \end{aligned} \tag{14}$$

Here u^i_n and $\hat{u}^i_n$ are understood as unknown functions of (x,p) (in the second heavenly picture) or of $(p,\hat{p})$ (in the first heavenly picture); λ is considered a constant under the total differential d, i.e., $d\lambda = 0$. Symmetries are to be constructed for this nonlinear system.

3 Infinitesimal Symmetries

3.1 The Case of Yang-Mills Theory [T1]

For the $\big(W(\lambda), V(\lambda)\big)$-system, a one-parameter family of transformations

$$\big(W(\lambda), V(\lambda)\big) \mapsto \big(W(\epsilon,\lambda), V(\epsilon,\lambda)\big) \tag{15}$$

of solutions is defined by the Riemann-Hilbert factorization

$$W(\epsilon,\lambda) e^{-\epsilon X(\lambda)} W(\lambda)^{-1} = V(\epsilon,\lambda) e^{-\epsilon Y(\lambda)} V(\lambda)^{-1}. \tag{16}$$

Here $X(\lambda) = X(\lambda, x, p)$ and $Y(\lambda) = Y(\lambda, x, p)$, the data of transformations, are Lie$G$-valued functions of $4N+1$ variables of the form

$$\begin{aligned} X(\lambda) &= \mathbf{X}(\lambda, x^1 + p^1\lambda, \ldots, x^{2N} + p^{2N}\lambda), \\ Y(\lambda) &= \mathbf{Y}(\lambda, x^1 + p^1\lambda, \ldots, x^{2N} + p^{2N}\lambda), \end{aligned} \tag{17}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary Lie$G$-valued functions of $2N+1$ variables with Laurent expansion

$$\mathbf{X}(\lambda, u) = \sum_{n=-\infty}^{\infty} \mathbf{X}_n(u)\lambda^n, \quad \mathbf{Y}(\lambda, u) = \sum_{n=-\infty}^{\infty} \mathbf{Y}_n(u)\lambda^n. \tag{18}$$

[In fact, some restriction on these data is required for the Riemann-Hilbert factorization to work well; a prescription is to put upper and lower bounds to the range of n as $-\infty < n \le n_X$ for $\mathbf{X}(\lambda)$ and $-n_Y \le n < \infty$ for $\mathbf{Y}(\lambda)$. A similar remark also applies to the hyper-Kähler case. This is a somewhat technical issue.] The associated infinitesimal transformations

$$\begin{aligned} \delta_{X,Y} W(\lambda) &= \left.\frac{\partial W(\epsilon,\lambda)}{\partial \epsilon}\right|_{\epsilon=0}, \\ \delta_{X,Y} V(\lambda) &= \left.\frac{\partial V(\epsilon,\lambda)}{\partial \epsilon}\right|_{\epsilon=0} \end{aligned} \tag{19}$$

have the following structure.

Proposition 1. *The infinitesimal symmetries act on* $W(\lambda)$ *and* $V(\lambda)$ *as follows.*

$$\begin{aligned} \delta_{X,Y} W(\lambda) \cdot W(\lambda)^{-1} &= \big(W(\lambda)X(\lambda)W(\lambda)^{-1} - V(\lambda)Y(\lambda)V(\lambda)^{-1}\big)_{\le -1}, \\ \delta_{X,Y} V(\lambda) \cdot V(\lambda)^{-1} &= \big(V(\lambda)Y(\lambda)V(\lambda)^{-1} - W(\lambda)X(\lambda)W(\lambda)^{-1}\big)_{\ge 0}, \end{aligned} \tag{20}$$

where $(\quad)_{\ge 0}$ *and* $(\quad)_{\le -1}$ *are linear maps on the space of Laurent series of* λ *defined by*

$$\begin{aligned} (\quad)_{\ge 0} :\ & \lambda^n \mapsto \theta(n \ge 0)\lambda^n, \\ (\quad)_{\le -1} :\ & \lambda^n \mapsto \theta(n \le -1)\lambda^n. \end{aligned} \tag{21}$$

Further, these infinitesimal symmetries obey the commutation relations

$$[\delta_{X_1,Y_1}, \delta_{X_2,Y_2}] = \delta_{[X_1,X_2],[Y_1,Y_2]}. \tag{22}$$

Thus, in particular, the infinitesimal symmetries give rise to a nonlinear realization of a direct sum of two loop algebras (with extra $2N$ variables $u^1, \ldots, u^{2N}$). The associated infinitesimal transformations of $J = V_0$ and $K = -W_{-1}$ can be readily derived from the above result.

3.2 The Case of Kähler Geometry [T2]

The case of $\big(u(\lambda), \hat{u}(\lambda)\big)$-system requires a more involved factorization, i.e., a factorization with respect to composition of maps. Let us consider this issue within the (x,p)-coordinate system. [A fully parallel treatment is possible with the $(p,\hat{p})$-coordinate system.] $u(\lambda)$ and $\hat{u}(\lambda)$ are now interpreted as maps

$$\begin{aligned} u(\lambda) :\ & x \mapsto u(\lambda, x, p), \\ \hat{u}(\lambda) :\ & x \mapsto \hat{u}(\lambda, x, p) \end{aligned} \tag{23}$$

from the x-space into the u-space or $\hat{u}$-space respectively. A one-parameter family of transformations

$$\big(u(\lambda), \hat{u}(\lambda)\big) \mapsto \big(u(\epsilon,\lambda), \hat{u}(\epsilon,\lambda)\big) \tag{24}$$

of solutions can be defined by the Riemann-Hilbert factorization

$$u(\epsilon,\lambda)^{-1} \circ e^{-\epsilon \xi_F(\lambda)} \circ u(\lambda) = \hat{u}(\epsilon,\lambda)^{-1} \circ e^{-\epsilon \xi_{\hat{F}}(\lambda)} \circ \hat{u}(\lambda), \tag{25}$$

where $\xi_F(\lambda)$ and $\xi_{\hat{F}}(\lambda)$ are Hamiltonian vector fields of the form

$$\begin{aligned} \xi_F(\lambda) &= \in^{ij} \frac{\partial F(\lambda)}{\partial u^i} \frac{\partial}{\partial u^j}, \\ \xi_{\hat{F}}(\lambda) &= \in^{ij} \frac{\partial \hat{F}(\lambda)}{\partial \hat{u}^i} \frac{\partial}{\partial \hat{u}^j}. \end{aligned} \tag{26}$$

The generating functions $F(\lambda) = F(\lambda, u)$ and $\hat{F}(\lambda) = \hat{F}(\lambda, \hat{u})$ are arbitrary functions of $2N+1$ variables with Laurent expansion

$$F(\lambda) = \sum_{n=-\infty}^{\infty} F_n(u)\lambda^n, \quad \hat{F}(\lambda) = \sum_{n=-\infty}^{\infty} \hat{F}(\hat{u})\lambda^n. \tag{27}$$

The infinitesimal transformations

$$\begin{aligned} \delta_{F,\hat{F}} u^i(\lambda) &= \left.\frac{\partial u^i(\epsilon,\lambda)}{\partial \epsilon}\right|_{\epsilon=0}, \\ \delta_{F,\hat{F}} \hat{u}^i(\lambda) &= \left.\frac{\partial \hat{u}^i(\epsilon,\lambda)}{\partial \epsilon}\right|_{\epsilon=0} \end{aligned} \tag{28}$$

have the following structure.

Proposition 2. *The infinitesimal symmetries act on $u(\lambda)$ and $\hat{u}(\lambda)$ as:*

$$\begin{aligned} \delta_{F,\hat{F}} u^i(\lambda) &= \left\{ \left[F\big(\lambda, u(\lambda)\big) - \hat{F}\big(\lambda, \hat{u}(\lambda)\big) \right]_{\le -1}, u^i(\lambda) \right\}_{(x)}, \\ \delta_{F,\hat{F}} \hat{u}^i(\lambda) &= \left\{ \left[\hat{F}\big(\lambda, \hat{u}(\lambda)\big) - F\big(\lambda, u(\lambda)\big) \right]_{\ge 0}, \hat{u}^i(\lambda) \right\}_{(x)}. \end{aligned} \tag{29}$$

Further, the infinitesimal symmetries obey the commutation relations

$$\left[\delta_{F_1,\hat{F}_1}, \delta_{F_2,\hat{F}_2} \right] = \delta_{\{F_1,F_2\}_{(u)},\{\hat{F}_1,\hat{F}_2\}_{(\hat{u})}}. \tag{30}$$

Thus the infinitesimal symmetries give a nonlinear realization of a direct sum of two Poisson (loop) algebras.

Remarkably, the above infinitesimal symmetries can be extended to Θ and Ω without modifying the Poisson algebra structure.

Proposition 3. *The infinitesimal symmetries can be consistently extended to Θ and Ω by the following rule.*

$$\begin{aligned} \delta_{F,\hat{F}} \Theta &= \operatorname*{res}_{\lambda=\infty} F\big(\lambda, u(\lambda)\big) + \operatorname*{res}_{\lambda=0} \hat{F}\big(\lambda, \hat{u}(\lambda)\big), \\ \delta_{F,\hat{F}} \Omega &= -\operatorname*{res}_{\lambda=\infty} \lambda^{-2} F\big(\lambda, u(\lambda)\big) - \operatorname*{res}_{\lambda=0} \lambda^{-2} \hat{F}\big(\lambda, \hat{u}(\lambda)\big), \end{aligned} \tag{31}$$

where the residues are normalized as

$$\operatorname*{res}_{\lambda=\infty} \lambda^n = -\delta_{n,-1}, \quad \operatorname*{res}_{\lambda=0} \lambda^n = \delta_{n,-1}. \tag{32}$$

These extended infinitesimal symmetries obey the same commutation relations as in Proposition 2.

4 Perturbative Method [T3]

The infinitesimal symmetries, as we have seen, have a considerably simple and beautiful structure. The Riemann-Hilbert factorization problems in general are hard to solve explicitly. For the case of Yang-Mills fields, several solution methods are developed; for the hyper-Kähler case, only existence theorems are known (except for a few very special families of solutions). The method presented here, so to speak, "exponentiate" the infinitesimal symmetries by expanding everything in powers of ϵ. As Leznov et al. [LMS] pointed out, the parameter ϵ plays the role of "coupling constants" in field theory; therefore we call the following method "perturbative."

4.1 The Case of Yang-Mills Theory

Let us consider the previous Riemann-Hilbert factorization in case where

$$W(\lambda) = V(\lambda) = 1 \text{ (trivial solution)}, \quad Y(\lambda) = 0. \tag{33}$$

Let us define

$$\mathcal{X}(\epsilon, \lambda) = W(\epsilon, \lambda) X(\lambda) W(\epsilon, \lambda)^{-1} \tag{34}$$

Since $\partial/\partial\epsilon$ corresponds to the action of $\delta_{X,0}$ on $\big(W(\epsilon,\lambda), V(\epsilon,\lambda)\big)$, one can readily find a closed differential equation satisfied by $\mathcal{X}(\epsilon,\lambda)$ with respect to ϵ.

Proposition 4. *$\mathcal{X}(\epsilon,\lambda)$ satisfies the differential equation*

$$\frac{\partial \mathcal{X}(\epsilon,\lambda)}{\partial \epsilon} = \Big[\big(\mathcal{X}(\epsilon,\lambda)\big)_{\le -1}, \mathcal{X}(\epsilon,\lambda)\Big] \tag{35}$$

and the initial condition

$$\mathcal{X}(\epsilon = 0, \lambda) = \mathbf{X}(\lambda, x + p\lambda). \tag{36}$$

Further,

Proposition 5. *$K(\epsilon) = -W_{-1}(\epsilon)$ and $J(\epsilon) = V_0(\epsilon)$ obey the differential equations*

$$\begin{aligned} \frac{\partial K(\epsilon)}{\partial \epsilon} &= \operatorname*{res}_{\lambda=\infty} \mathcal{X}(\epsilon,\lambda), \\ \frac{\partial J(\epsilon)}{\partial \epsilon} J(\epsilon)^{-1} &= \operatorname*{res}_{\lambda=\infty} \lambda^{-1} \mathcal{X}(\epsilon,\lambda). \end{aligned} \tag{37}$$

Substitution of the Taylor expansion ("perturbation series")

$$\mathcal{X}(\epsilon,\lambda) = \sum_{k=0}^{\infty} \mathcal{X}^{(k)}(\lambda) \epsilon^k / k! \tag{38}$$

into the above equation yields a set of recursive relations

$$\mathcal{X}^{(0)}(\lambda) = X(\lambda) = \mathbf{X}(\lambda, x + p\lambda),$$
$$\mathcal{X}^{(k+1)} = \sum_{\ell=0}^{k} \binom{k}{\ell} \left[\left(\mathcal{X}^{(k-\ell)}(\lambda)\right)_{\leq -1}, \mathcal{X}^{(\ell)}(\lambda) \right]. \tag{39}$$

The unknown functions $K(\epsilon)$ and $J(\epsilon)$ of the generalized self-duality equations, too, can be determined by expansion into powers of ϵ.

In the original formulation of Leznov et al. [LMS], the projection $(\quad)_{\leq -1}$ is represented by an integral operator; they exploit its algebraic properties to check, by brute force, the validity of their formula.

3.2 The Case of Kähler Geometry

We now start from the Riemann-Hilbert factorization with

$$u^i(\lambda) = \hat{u}^i(\lambda) = x^i + p^i\lambda \text{ (trivial solution)}, \quad \hat{F}(\lambda) = 0, \tag{40}$$

and derive differential equations satisfied by

$$\mathcal{F}(\epsilon, \lambda) = F\bigl(\lambda, u(\epsilon, \lambda)\bigr) \tag{41}$$

and $\Theta(\epsilon)$ with respect to ϵ.

Proposition 6. *$\mathcal{F}(\epsilon, \lambda)$ satisfies the differential equation*

$$\frac{\partial \mathcal{F}(\epsilon, \lambda)}{\partial \epsilon} = \Bigl\{ [\mathcal{F}(\epsilon, \lambda)]_{\leq -1}, \mathcal{F}(\epsilon, \lambda) \Bigr\}_{(x)} \tag{42}$$

and the initial condition

$$\mathcal{F}(\epsilon = 0, \lambda) = F(\lambda, x + p\lambda). \tag{43}$$

Proposition 7. *One can obtain $\Theta(\epsilon)$ by solving the differential equation*

$$\frac{\partial \Theta(\epsilon)}{\partial \epsilon} = \operatorname*{res}_{\lambda=\infty} \mathcal{F}(\epsilon, \lambda) \tag{44}$$

under the initial condition

$$\Theta(\epsilon = 0) = 0. \tag{45}$$

These equations can be solved by the same "perturbative method" as illustrated in the case of Yang-Mills fields.

The above construction is not suited for the first heavenly picture based upon $(p, \hat{p}, \Omega)$. To give a similar construction for the first heavenly picture, we just have to restart from the situation where

$$u^i(\lambda) = \hat{u}^i(\lambda) = \hat{p}^i + p^i\lambda, \quad F(\lambda) = 0, \tag{46}$$

and consider equations satisfied by $\Omega(\epsilon)$ and

$$\hat{\mathcal{F}}(\epsilon, \lambda) = \hat{F}\bigl(\lambda, \hat{u}(\epsilon, \lambda)\bigr). \tag{47}$$

5 KP and Super KP Hierarchies [T4]

In the differential-algebraic approach mentioned in the introduction, a nonlinear system is represented by a commutative algebra $\mathcal{A}$ with a set of derivations $\partial_1, \partial_2, \ldots$. If one is not interested in a particular choice of such derivations, it is convenient to understand a differential algebra as a pair $(\mathcal{A}, \varDelta)$ of a commutative algebra and an $\mathcal{A}$-module $\varDelta$ of derivations in $\mathcal{A}$. Infinitesimal symmetries are then, by definition, derivations $\delta : \mathcal{A} \to \mathcal{A}$ that satisfy the condition

$$[\delta, \partial] \in \varDelta \quad (\forall \partial \in \varDelta). \tag{48}$$

In most applications, the derivations $\partial_1, \partial_2, \ldots$, are chosen to be commutative, and symmetries are characterized as extra derivations of $\mathcal{A}$ that commute with these derivations. (The super KP hierarchy is somewhat distinct; not only $\mathcal{A}$ being a supercommutative algebra, the set of derivations are neither commutative nor supercommutative. One can however see that its basic structure is almost parallel to the case of the KP hierarchy.)

The generalized self-duality equations, too, can be formulated as such an abstract differential algebra. Its algebraic part $\mathcal{A}$ should be a commutative algebra (over a suitable differential subalgebra that specifies in which domain to seek for solutions) generated by the Laurent coefficients of $W(\lambda)$ and $V(\lambda)$, or of $u(\lambda)$ and $\hat{u}(\lambda)$. In the latter case, one may also add $\varTheta$ or $\varOmega$. This certainly provides an umbiguous framework for the notion of infinitesimal symmetries; however, one will gain nothing practically new from this reinterpretation.

The situation is considerably different for the case of the KP and super KP hierarchies. For these equations, the differential-algebraic language seems to have a substantial meaning. Of articular importance is a $\mathcal{D}$-module structure hidden in the formulation of the KP and super KP hierarchy. (This observation for the case of the KP hierarchy is due to Sato, who stresses the relevance of the notion of $\mathcal{D}$-modules even in more general perspectives [S].) With the aid of this $\mathcal{D}$-module structure, one can find a new set of generators w_{ij} $(i \geq 0, j \leq -1)$ in $\mathcal{A}$. This is the most direct way to see a connection with the geometry of infinite dimensional (super) Grassmannian manifolds; w_{ij}'s can be identified with affine coordinates on an open subset therein. This also leads to an explicit description of infinitesimal symmetries δ_A parametrized by elements A of an infinite matrix Lie algebra $gl(\infty)$ (for the KP hierarchy) or of its super-version $gl(\infty|\infty)$ (for the super KP hierarchy), a differential-algebraic characterization of the τ function, its symmetry contents related to central extensions of $gl(\infty)$ and $gl(\infty|\infty)$, etc.

In fact, $\varTheta$ and $\varOmega$, may be in a sense considered an analogue of the τ function. This analogy becomes quite reasonable if we consider a hierarchy of the generalized self-duality equations discussed here. Their representation-theoretic and geometric properties are however considerably different from the τ function.

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