EXTENSIONS OF "SOLITON EQUATIONS"

1 • SELF-DUALITY EQUATION
IN YANG-MILLS THEORY

2 • SELF-DUALITY EQUATION
IN KÄHLER GEOMETRY

3 • SUPER KP HIERARCHY

CONCERNS: —

1 AND 2 — "HIGHER DIMENSIONAL"
(OR "MULTI-DIMENSIONAL")
EXTENSION OF "NONLINEAR
INTEGRABLE SYSTEMS"

1, 2, 3 ...
SYMMETRIES
???

3 — EXTENSION OF METHOD OF
"D-MODULE" (M. SATO) ON
KP HIERARCHY

GENERALIZED SELF-DUALITY EQUATION IN YANG-MILLS THEORY

$$(x, p) = (x^1, \cdots, x^{2N}, p^1, \cdots, p^{2N})$$

<u>4N</u>-DIMENSIONAL SPACE-TIME

I. $K = K(x, p) \in \text{Lie } G$

$$\boxed{\frac{\partial^2 K}{\partial x^i \partial p^j} - \frac{\partial^2 K}{\partial x^j \partial p^i} + \left[\frac{\partial K}{\partial x^i}, \frac{\partial K}{\partial x^j}\right] = 0}$$

$i, j = 1, \cdots, 2N$

II. $J = J(x, p) \in G$

$$\boxed{\frac{\partial}{\partial x^i}\left(\frac{\partial J}{\partial p^j} \cdot J^{-1}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial J}{\partial p^i} \cdot J^{-1}\right) = 0}$$

- I AND II ARE <u>EQUIVALENT</u>.

AUXILIARY VARIABLES

$$W = 1 + \sum_{n=-\infty}^{-1} W_n(x,p)\lambda^n$$

$$\hat{W} = \sum_{n=0}^{\infty} \hat{W}_n(x,p)\lambda^n$$ THAT SATISFY:

$$\left(\frac{\partial}{\partial p^i} - \lambda\frac{\partial}{\partial x^i} + A_i\right)\Psi = 0,$$

$$\Psi = W, \hat{W}$$

$$A_i = -\frac{\partial K}{\partial x^i} = -\frac{\partial J}{\partial p^i}\cdot J^{-1}$$

$$W_{-1} = -K$$

$$\hat{W}_0 = J$$

$\rightarrow$ $\boxed{\text{NONLINEAR SYSTEM ON } \{W_n, \hat{W}_n\}}$ ↺ SYMMETRIES

((W,V) - SYSTEM)

ORIGIN OF SYMMETRIES

1-PARAMETER FAMILY OF TRANSFORMATIONS OF SOLUTIONS (TO $(W,\hat{W})$-SYSTEM) :—

ε

$$(W,\hat{W}) \longmapsto (W(\varepsilon), \hat{W}(\varepsilon))$$

DEFINED BY RIEMANN-HILBERT FACTORIZATION

$$\boxed{W(\varepsilon)\, e^{-\varepsilon X} W^{-1} = \hat{W}(\varepsilon)\, e^{-\varepsilon Y} \hat{W}^{-1}}$$

$$X = X(\lambda, \underline{\underline{x^1 + p^1\lambda}}, \cdots, \underline{\underline{x^{2N} + p^{2N}\lambda}})$$

$$Y = Y(\lambda, \underline{\underline{x^1 + p^1\lambda}}, \cdots, \underline{\underline{x^{2N} + p^{2N}\lambda}})$$

Lie G-VALUED DATA

INFINITESIMAL TRANSFORMATIONS (SYMMETRIES) TO $(W,\hat{W})$-SYSTEM

$$\delta_{X,Y} W = \frac{d}{d\varepsilon} W(\varepsilon) \Big|_{\varepsilon \to 0}$$

$$\delta_{X,Y} \hat{W} = \frac{d}{d\varepsilon} \hat{W}(\varepsilon) \Big|_{\varepsilon \to 0} \quad \text{(DEFINITION)}$$

$$\left[\begin{aligned} &\delta_{X,Y}\left(\Phi[W,\hat{W}]\cdot \Psi[W,\hat{W}]\right) \\ &\quad = \delta_{X,Y}\Phi[W,\hat{W}]\cdot\Psi[W,\hat{W}] + \Phi[W,\hat{W}]\cdot \delta_{X,Y}\Psi[W,\hat{W}] \end{aligned} \right]$$

(1)
$$\delta_{X,Y} W \cdot W^{-1} = \left(WXW^{-1} - \hat{W}Y\hat{W}^{-1}\right)_{\leq -1}$$

$$\delta_{X,Y} \hat{W} \cdot \hat{W}^{-1} = \left(\hat{W}Y\hat{W}^{-1} - WXW^{-1}\right)_{\geq 0}$$

(2)
$$\left[\delta_{X_1,Y_1}, \delta_{X_2,Y_2}\right] = \delta_{[X_1,X_2],[Y_1,Y_2]}$$

$(\ )_{\leq -1} : \lambda^n \to \theta(n \leq -1)\lambda^n$, $\quad (\ )_{\geq 0} : \lambda^n \to \theta(n \geq 0)\lambda^n$

loop

extra-variables

$\lambda, u^1, \dots, u^{2N}$ (twistor)

$\delta_{..}$: LOOP ALEBRA $\oplus$ LOOP ALGEBRA

$\rightarrow$ VECTOR FIELDS ON "MANIFOLD" OF (W, V)-SYSTEM

DIFEERENTIAL-ALGEBRAIC MANIFOLD (IN THE SENSE OF KOLCHIN)

TRANSFORMATIONS OF J AND K : —

$$\delta_{X,Y} K = \operatorname*{res}_{\lambda=\infty} W X W^{-1} + \operatorname*{res}_{\lambda=0} \hat{W} Y \hat{W}^{-1}$$

$$\delta_{X,Y} J = \operatorname*{res}_{\lambda=\infty} \frac{W X W^{-1}}{\lambda} + \operatorname*{res}_{\lambda=0} \frac{\hat{W} Y \hat{W}^{-1}}{\lambda}$$

$$\left[\operatorname*{res}_{\lambda=\infty} \lambda^n = -\delta_{n,-1}, \quad \operatorname*{res}_{\lambda=0} \lambda^n = \delta_{n,-1} \right]$$

GENERALIZED SELF-DUALITY IN KÄHLER GEOMETRY (= HYPER-KÄHLER)

TWO EQUIVALENT LOCAL EXPRESSIONS:—
("$\mathcal{H}$-SPACE" DUE TO PLEBANSKI)

I. (x, p, H), $x = (x^1, \dots, x^{2N})$
$p = (p^1, \dots, p^{2N})$
$H = H(x, p)$ (SCALAR)

$$\frac{\partial^2 H}{\partial x^i \partial p^j} - \frac{\partial^2 H}{\partial x^j \partial p^i} + \left\{ \frac{\partial H}{\partial x^i}, \frac{\partial H}{\partial x^j} \right\}_{(x)} = 0$$

"SECOND HEAVENLY EQUATION"

- $\{F, G\}_{(x)} = \epsilon^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}$ Poisson bracket
- $\epsilon^{ij} = \begin{pmatrix} \begin{matrix} & 1 \\ -1 & \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} & 1 \\ -1 & \end{matrix} \end{pmatrix}$
- REALITY IS HARD TO CHARACTERISE

II. $(p, \hat{p}, \Omega)$

$$p = (p^1, \cdots, p^{2N}),$$
$$\hat{p} = (\hat{p}^1, \cdots, \hat{p}^{2N}),$$
$$\Omega = \Omega(p, \hat{p}) \quad (\text{SCALAR})$$

$$\boxed{\epsilon^{ij} \frac{\partial^2 \Omega}{\partial \hat{p}^i \partial p^k} \frac{\partial^2 \Omega}{\partial \hat{p}^j \partial p^l} = \epsilon_{kl}}$$

"FIRST HEAVENLY EQUATION"

$$\Longleftrightarrow \quad \left\{ \frac{\partial \Omega}{\partial p^k}, \frac{\partial \Omega}{\partial p^l} \right\}_{(\hat{p})} = \epsilon_{kl}$$

POISSON BRACKET IN $\underline{\underline{\hat{p}}}$

$$\Longleftrightarrow \quad \left(\frac{\partial^2 \Omega(p, \hat{p})}{\partial \hat{p}_i \partial p^j} \right) \in Sp(N)$$

$$\left[\quad \hat{p}_i = \epsilon_{ij} \hat{p}^j, \ \cdots \quad \right]$$

- $(p^i, \hat{p}_i) \longleftrightarrow (z^i, \bar{z}^i)$] IN GEOMETRIC SITUATION
  Ω = KÄHLER POTENTIAL

AUXILIARY VARIABLES

$$u^i = \sum_{n=-\infty}^{-1} u^i_n \lambda^n + x^i + p^i \lambda$$

$$\hat{u}^i = \hat{p}^i + \sum_{n=1}^{\infty} \hat{u}^i_n \lambda^n \qquad \text{WITH:}$$

$$\epsilon_{ij}\, du^i \wedge du^j = \epsilon_{ij}\, d\hat{u}^i \wedge d\hat{u}^j$$

$$d\Theta = \epsilon_{ij} u^i_{-2} dp^j + \epsilon_{ij} u^i_{-1} dx^j$$

$$d\Omega = -\epsilon_{ij} x^i dp^j + \epsilon_{ij} \hat{u}^i_1 d\hat{p}^j$$

$\rightarrow$ NONLINEAR SYSTEM WITH INDEP. VARIABLES (x,p) OR $(\hat{p},p)$

EQUIVALENT

$((u,\hat{u})$–SYSTEM$)$

SYMMETRIES

ORIGIN OF SYMMETRIES

1-PARAMETER FAMILY OF TRANSFORMATIONS OF SOLUTIONS

$$(u, \hat{u}) \mapsto (u(\varepsilon), \hat{u}(\varepsilon))$$

DEFINED BY RIEMANN-HILBERT FACTORIZATION:

$$\boxed{u(\varepsilon)^{-1} \circ e^{-\varepsilon \xi_F} \circ u = \hat{u}(\varepsilon)^{-1} \circ e^{-\varepsilon \xi_{\hat{F}}} \circ \hat{u}}$$

- "$\circ$" – COMPOSITION OF MAPS

$$u : x \mapsto u(\lambda, x, p)$$
$$\hat{u} : x \mapsto \hat{u}(\lambda, x, p)$$

- $\xi_F = \epsilon^{ij} \frac{\partial F}{\partial u^i} \frac{\partial}{\partial u^j}, \quad \xi_{\hat{F}} = \epsilon^{ij} \frac{\partial \hat{F}}{\partial \hat{u}^i} \frac{\partial}{\partial \hat{u}^j}$

$$F = F(\lambda, u^1, \cdots, u^{2N})$$
$$\hat{F} = \hat{F}(\underset{\text{LOOP}}{\lambda}, \underbrace{\hat{u}^1, \cdots, \hat{u}^{2N}}_{\text{EXTRA } 2N \text{ VARIABLES}})$$

HAMILTONIAN VECTOR FIELDS (= INFINITESIMAL CANONICAL TRANSFORMATION)

INFINITESIMAL TRANSFORMATIONS (Second form)

$$\delta_{F,\hat F} u^i = \frac{d}{d\varepsilon} u^i(\varepsilon) \Big|_{\varepsilon \to 0},$$

$$\delta_{F,\hat F} \hat u^i = \frac{d}{d\varepsilon} \hat u^i(\varepsilon) \Big|_{\varepsilon \to 0}$$ (definition)

$$\boxed{\begin{aligned} \delta_{F,\hat F} u^i &= \left\{ (\mathcal{F} - \hat{\mathcal{F}})_{\leq -1},\ u^i \right\}_{(x)} \\ \delta_{F,\hat F} \hat u^i &= \left\{ (\hat{\mathcal{F}} - \mathcal{F})_{\geq 0},\ \hat u^i \right\}_{(x)} \end{aligned}}$$

WHERE

$$\mathcal{F} = F(\lambda, u(\lambda)),$$

$$\hat{\mathcal{F}} = \hat F(\lambda, \hat u(\lambda)),$$ F and $\hat F$ being arbitrary

CONSISTENT INF'L TRANSFORMATIONS OF Ⓗ ~~AND Ⓤ ARE~~ GIVEN BY:

$$\boxed{\begin{aligned} \delta_{F,\hat F} Ⓗ &= \operatorname*{res}_{\lambda=\infty} \mathcal{F} + \operatorname*{res}_{\lambda=0} \hat{\mathcal{F}} \\ &\cancel{\delta_{F,\hat F} Ⓤ = -\operatorname*{res}_{\lambda=\infty} \frac{\mathcal{F}}{\lambda^2} - \operatorname*{res}_{\lambda=0} \frac{\hat{\mathcal{F}}}{\lambda^2}} \end{aligned}}$$

$\backsim$ $\delta_{x,y} K$

~~$\delta_{x,y} J$~~