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u-plane integrals and integrable systems

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- correlation functions and contact terms in 4d topological gauge theory
- $G = SU(N)$: Coulomb moduli (u-plane), blowup formula, single-time tau function (after Mariño & Moore)
- multi-time tau function
- other gauge groups

correlation functions and contact terms in 4d topological gauge theory

Physical Setup

$\mathcal{N}=2$ SUSY gauge theory $\xrightarrow[\text{twisting}]{\text{topological}}$ topological gauge theory on X

$\downarrow$

correlation functions

$\parallel$

Donaldson - Witten invariants of X

generating function of correlation functions:

$$Z_{DW}(xS+yP) = \langle \exp\big(xI(S) + y\mathcal{O}(P)\big)\rangle$$

$S \in H_2(X,\mathbb{Z})$ $\qquad I(S) \sim \int_S G^2 \mathcal{P}$

$P \in H_0(X,\mathbb{Z})$ $\qquad \mathcal{O}(P) \sim \Sigma\, c_k \operatorname{Tr} \phi(P)^k$

x, y: coupling constants

$$b_2^{\pm} = \dim H^2(X, \mathbb{C})_{\pm} \quad \begin{cases} \text{self-dual} \\ \text{anti-selfdual} \end{cases}$$

<u>If $b_2^+ > 1$</u>, $\quad Z_{DW} = Z_{SW}$

$\uparrow$ contribution of moduli space of Seiberg-Witten monopole eq.

Z_{DW} is topological invariant.

<u>If $b_2^+ \leq 1$</u>, $\quad Z_{DW} = Z_{SW} + Z_u$

$\uparrow$ contribution of "u-plane" (Coulomb moduli)

<u>u-plane integral</u> (Moore-Witten, Losev et al, Mariño-Moore)

$$Z_u = \int_{\mathcal{M}_{coulomb}} d\mu \, A^{\chi} B^{\sigma} \exp(U + x^2 S^2 T) \Psi$$

U: contribution of $\mathcal{O}(P)$ only

Ψ: partition function of (abelian) gauge fields etc. $\Psi = \sum_{\text{lattice}} (\cdots)$

T: contact term induced by S

$\mathcal{M}_{coulomb}$: "u-plane" (quantum moduli space of vacua in S-W effective theory)

"u-plane" in SW effective theory

"u-plane" (Coulomb moduli space) $\mathcal{U}$

= moduli space of solutions of periodic Toda lattice

(Gorsky et al, Martinec-Warner)

Hamiltonian map $X \longrightarrow \mathcal{U}$

X: phase space of Toda lattice

E.g. $G = SU(N)$

$$\frac{dL(z)}{dt} = [A(z), L(z)] \quad (N \times N \text{ matrix})$$

$$\det(L(z) - xI) = 0 \quad (\text{spectral curve})$$

$$\downarrow$$

$$\Lambda (z + z^{-1}) = P(x) = x^N - \sum_{j=2}^{N} u_j x^{N-j}$$

$u_2, \cdots, u_N$: Hamiltonians

$(q_i, p_i) \mapsto (u_2, \cdots, u_N)$: Hamiltonian map

special geometric structure:

- $dS = x \, d\log z$: SW differential

- $a_j = \oint_{A_j} dS\,, \quad a_j^D = \oint_{B_j} dS$

  $$(\quad A_j \cdot B_k = \delta_{jk} \quad)$$

  : special coordinates and their duals

- $\mathcal{F} = \mathcal{F}(a_1, \dots, a_{N-1}) \qquad (N-1 = \text{genus})$

  : prepotential (SW effective potential)

$$\frac{\partial \mathcal{F}}{\partial a_j} = a_j^D\,, \quad \frac{\partial^2 \mathcal{F}}{\partial a_j \partial a_k} = \mathcal{T}_{jk} = \oint_{B_j} d\omega_k$$

Whitham-type equations

- $\left.\frac{\partial}{\partial a_j} dS\right|_{z=\text{const}} = d\omega_j \quad \left(\begin{array}{c}\text{normalized}\\ \text{holomorphic}\\ \text{differentials}\end{array}\right)$

- $\left.\Lambda \frac{\partial}{\partial \Lambda} dS\right|_{z=\text{const}} = d\Omega \quad \left(\begin{array}{c}\text{meromorphic}\\ \text{differential}\end{array}\right)$

  (RGeq) $\qquad (\Lambda = A^{1/N})$

blowup formula

- Connecting u-plane integrals for X and its blowup $\tilde{X}$ at a point $P \in X$:

X : complex algebraic surface
(e.g. $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, etc ...)

$\varphi : \tilde{X} \longrightarrow X$ blowup map

$B = \varphi^{-1}(P) \simeq \mathbb{C}P^1$ (exceptional divisor)

$$Z_u^X(S,p) \longrightarrow Z_u^{\tilde{X}}(\tilde{S},p).$$

$(x = y = 1)$

$$\tilde{S} = S + tB \quad (\text{in } H_2)$$

amounts to substituting

$$e^U \longrightarrow e^U \frac{\alpha}{\beta} \det\left(\frac{\partial u_k}{\partial a_j}\right)^{1/2} \Delta^{-1/8} \cdot \boxed{e^{-t^2 T}\, \Theta_{\gamma,\delta}\left(\frac{t}{2\pi} V \middle| J\right)}$$

Mariño & Moore pointed out (for $G = SU(N)$):

This is a tau function of the Toda lattice

$\boxed{\text{Multi-time tau function}}$

Mariño - Moore : $\tau = e^{-t^2 T} \Theta_{\gamma,\delta}\left(\frac{t}{2\pi} V \middle| J\right)$

T : contact term of $B \simeq S^2$

More precisely, T is induced by the insertion of $\exp\left(\text{const.}\, t \int_B G^2 \mathcal{P}\right)$ $(\mathcal{P} = \operatorname{tr}\phi^2 \sim u_2)$ into

$$\left\langle \exp\left(\int_S G^2 \mathcal{P} + \mathcal{O}(p)\right)\right\rangle_{\tilde{X}}$$

<u>Guess.</u> Inserting more general terms as

$$\left\langle \exp\left(\sum_n t_n \int_B G^2 \mathcal{P}_n + \int_S G^2 \mathcal{P} + \mathcal{O}(p)\right)\right\rangle_{\tilde{X}}$$

will induce a multi-time tau function

$$\tau = \tau(t_1, t_2, \cdots) = e^{\frac{1}{2}\sum q_{mn} t_m t_n} \Theta_{\gamma,\delta}\left(\sum t_n V_n \middle| J\right)$$

and q_{mn} will give contact terms of higher order observables $\int_B G^2 \mathcal{P}_n$.

Evidence (not a proof)

- The correction term in the blowup formula must obey a modular transformation under symplectic transformations of the homology cycles A_j, B_j.

- The correction term $e^{-t^2 T}\Theta_{\gamma\delta}(\frac{t}{2\pi}V|J)$ of Mariño and Moore satisfies this requirement.

- The multi-time extension, too, passes this test.

  — Such a modular transformation property was already discovered in late 80's. of tau functions

- "Slow variables" T_n and a prepotential $\mathcal{F}$ can be introduced in the standard way so that q_{mn} etc. can be written as derivatives of $\mathcal{F}$. Those formulae are consistent with known results on contact terms.

Relations thus derived :

$$\mathcal{P}_n = \frac{1}{2\pi i} \frac{\partial \mathcal{F}}{\partial T_n}$$

$$V_{jn} = \frac{1}{2\pi i} \frac{\partial^2 \mathcal{F}}{\partial a_j \partial T_n} \qquad \left(V_n = (V_{1n} \cdots V_{N-1,n}) \right)$$

$$q_{mn} = \frac{1}{2\pi i} \frac{\partial^2 \mathcal{F}}{\partial T_m \partial T_n}$$

$$C(\mathcal{P}_m, \mathcal{P}_n) = -\frac{1}{2} q_{mn} = -\frac{1}{4\pi i} \frac{\partial^2 \mathcal{F}}{\partial T_m \partial T_n}$$

(contact terms)

$\boxed{\text{Construction of Whitham flows}}$ (Gorsky et al)

$$d\hat{\Omega}_n = (P(x)^{n/N})_+ \, d\log z \qquad \left(d\hat{\Omega}_1 = dS_{SW} \right)$$

$$dS = \sum_{n=1}^{2N-1} T_n \, d\hat{\Omega}_n$$

$$a_j = \oint_{A_j} dS$$

$P(x) = W(x)$

"SUPERPOTENTIAL"

The inverse of the period map $(u_j) \mapsto (a_j)$ gives Whitham deformations.

Other gauge groups (in progress)

1. For a gauge group G, the relevant integrable system is the Toda lattice associated with the dual Lie algebra $\mathfrak{g}^{(1)\vee}$ of the untwisted affine algebra $\mathfrak{g}^{(1)}$. (Martinec & Warner)

- $G = SU(N)$: $\quad \mathfrak{g}^{(1)\vee} = \mathfrak{g}^{(1)} = A_{N-1}^{(1)}$
- $G = SO(2N+1)$: $\quad \mathfrak{g}^{(1)\vee} = A_{2N-1}^{(2)}$
- $G = Sp(N)$: $\quad \mathfrak{g}^{(1)\vee} = D_{N+1}^{(2)}$
- $G = SO(2N)$: $\quad \mathfrak{g}^{(1)\vee} = \mathfrak{g}^{(1)} = D_N^{(1)}$
- $G = E_{6,7,8}$: $\quad \mathfrak{g}^{(1)\vee} = \mathfrak{g}^{(1)} = E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$
- $G = F_4$: $\quad \mathfrak{g}^{(1)\vee} = E_6^{(2)}$
- $G = G_2$: $\quad \mathfrak{g}^{(1)\vee} = D_4^{(3)}$

☆ $A_{2N}^{(2)}$ does not appear.

Spectral curves:

- $G = SU(N)$ $\quad z + z^{-1} = \Lambda^{-1} P(x)$
- $G = SO(2N+1)$ $\quad z + z^{-1} = \Lambda^{-1} x^{-1} P(x^2)$
- $G = Sp(N)$ $\quad z + z^{-1} - 2 = \Lambda^{-1} x^2 P(x^2)$
- $G = SO(2N)$ $\quad z + z^{-1} = \Lambda^{-1} x^{-2} P(x^2)$

⋮

$$P(x) = x^N + \cdots \quad \text{(polynomial)}$$

☆ For $A^{(2)}_{2N}$ Toda lattice,

$$z + z^{-1} = \Lambda^{-1} x P(x^2)$$

2. For $G = SU(N)$, the Toda lattice system is linearized on the Jacobi variety $\mathrm{Jac}(C)$ of the above spectral curve. The theta function $\Theta_{\gamma,\delta}$ lives on $\mathrm{Jac}(C)$. For other groups, however, $\mathrm{Jac}(C)$ is too large, and a natural stage (for both the Toda lattice and the gauge theory) is a Prym variety embedded in $\mathrm{Jac}(C)$. (Martinec & Warner)

The theta function in the blowup formula, too, should be a theta function on the Prym variety.

3. For the classical gauge groups, the Prym variety is almost the same as (more precisely, isogenous to) the Jacobi variety $\mathrm{Jac}(C')$ of a reduced spectral curve $C' = C/\imath$ by an involution $\imath$:

- $G = SO(2N+1)$ $\quad \imath(x, z) = (-x, -z^{-1})$
- $G = Sp(N)$ $\quad \imath(x, z) = (-x, z^{-1})$
- $G = SO(2N)$ $\quad \imath(x, z) = (-x, z^{-1})$

☆ For $A_{2N}^{(2)}$ Toda spectral curve,

$$\imath(x, z) = (-x, -z^{-1})$$

Eg. $G = SO(2N)$

$$C: \quad z + z^{-1} = A^{-1} x^{-2} P(x^2)$$

$$(\Leftrightarrow \quad y^2 = P(x^2)^2 - 4A^2 x^4)$$

$$\Big\downarrow \quad \xi = x^2, \quad \eta = xy$$

$$C': \quad \eta^2 = \xi\,(P(\xi)^2 - 4A^2\xi^2)$$

$$\boxed{\text{C' is a hyperelliptic curve of genus N and branched at $\xi = \infty$}}$$

(This is also the case for other groups)

<u>Cf.</u> <u>$G = SU(2)$</u>

Toda spectral curve

$$C: z + z^{-1} = A^{-1}(x^2 - u)$$

has a similar involution

$$\iota(x, z) = (-x, z^{-1})$$

(This is an accidental symmetry of the spectral curve.) The reduced curve

$$C': \eta^2 = \xi((\xi - u) - A^2)$$

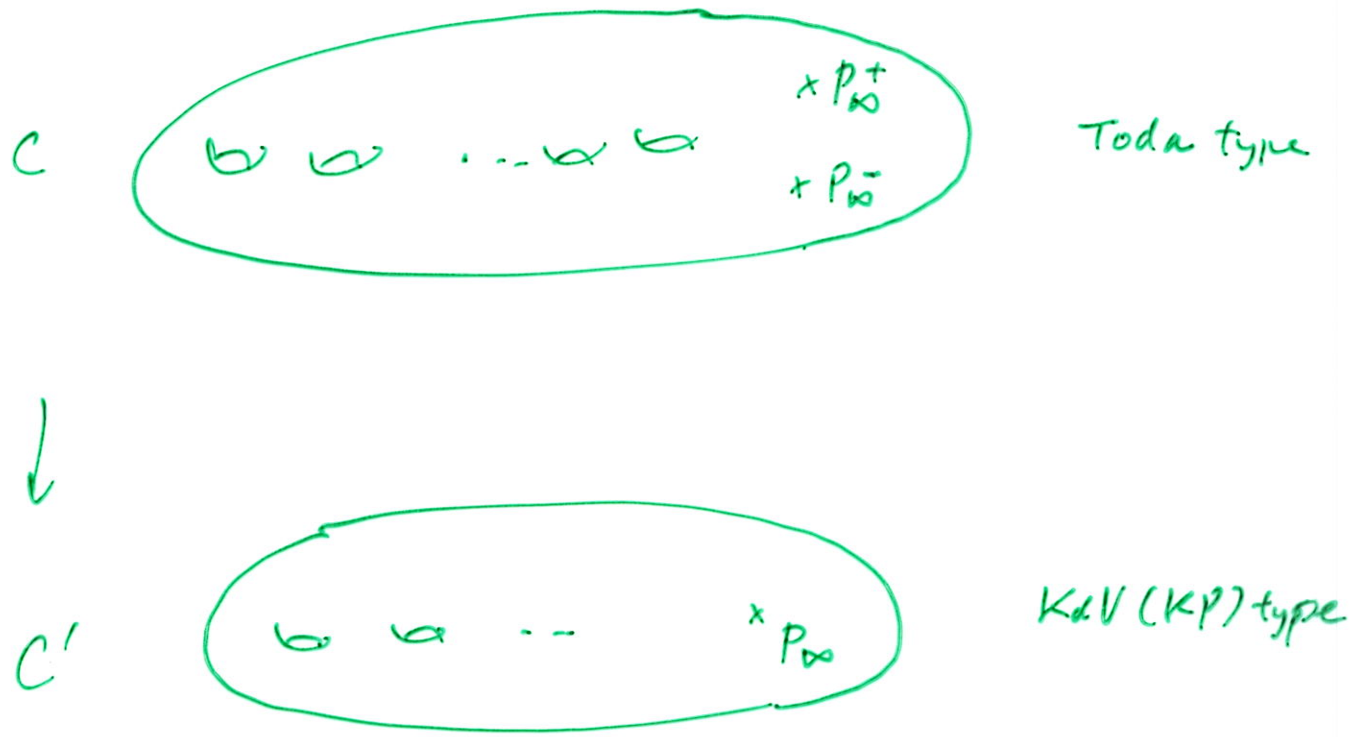
is essentially the original Seiberg-Witten curve

$$\eta^2 = (\xi + u)(\xi - A)(\xi + A)$$

(by replacing $\xi \to \xi + u$). This curve is rather related to the KdV equation

(as pointed out by Gorsky, Krichever, Marshakov, Mironov and Morozov, 1995)

— The reduced spectral curves C' for all classical gauge groups other than $SU(N)$ $(N \geq 3)$, too, are of the KdV (or KP) type.



<u>Conjecture</u> A (multi-time) tau function of the KdV (or KP) type will give the correction factor for the classical gauge groups $G \neq SU(N)$ $(N \geq 3)$.